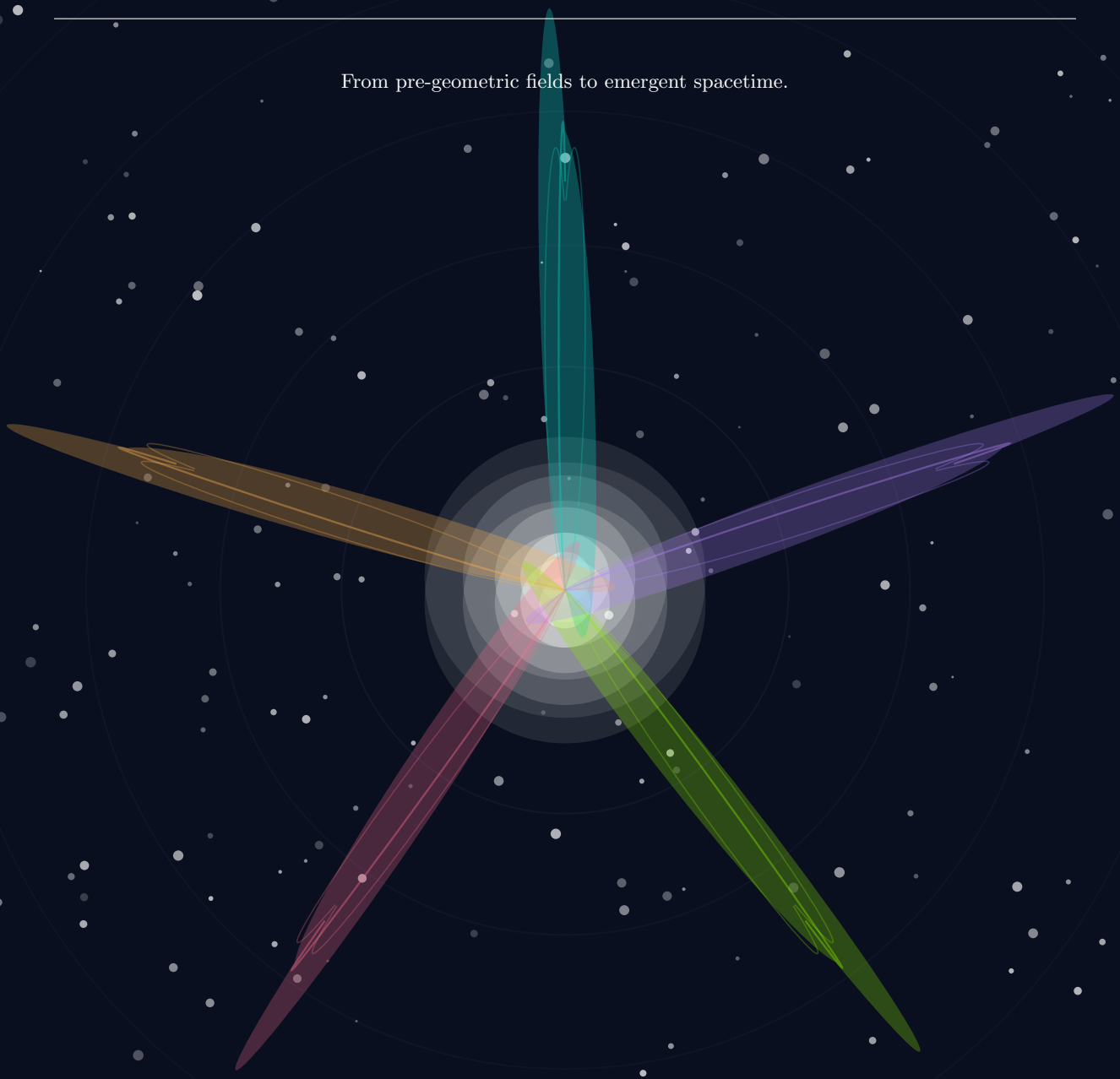
THE QUANTUM AETHER MODEL

Volume I — Foundations of a Spectral Reality

From pre-geometric fields to emergent spacetime.



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THE QUANTUM AETHER MODEL . Volume 1

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The Quantum Aether Model: Rebuilding Physics from First Principles

ADAM

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Part I

Introductions

0.1 The Crisis of Modern Physics

Contemporary physics is experiencing a state of deep conceptual duality, manifested in the fundamental contradiction between the two greatest theories of the twentieth century: **General Relativity** and **Quantum Mechanics**. While relativity describes spacetime as a smooth geometric structure determined by matter and energy through Einstein's equations, quantum mechanics relies on concepts such as the wavefunction, superposition, and entanglement within a probabilistic framework that appears incompatible with the geometric conception of reality.

This division has never been merely a technical dispute, but rather an expression of a deeper crisis: **the absence of a unified conceptual framework for physical reality**. Relativity assumes that spacetime truly exists, independent of the observer, and follows well-defined laws. In contrast, quantum mechanics treats reality as a mysterious entity that becomes defined only upon measurement, placing the observer at the center of the physical process itself.

Caught between these two poles, the **Standard Models of particles** remain unable to explain the origin of the fundamental constants that populate the tables of physics, or to incorporate gravity into their framework. They introduce mass, charge, and spin as experimentally measured constants without offering a physical explanation for their origin.

From here arises a set of profound questions:

- What is the nature of time? Is it a real dimension or an emergent rhythm from a deeper system?
- What is the origin of mass, and why do particles possess specific properties and not others?
- Are the physical constants (such as c, $\hbar$, G, and α) truly constant and universal, or are they reflections of a variable state of a deeper physical medium?
- What is the origin of spacetime itself? Is it a fundamental entity, or an emergent phenomenon?

Such questions remain unanswered within current theoretical frameworks, despite the enormous technological progress built upon them. In this conceptual void, the need arises for **reconstructing physics from its foundations**, on new principles that enable the unification of concepts, the explanation of constants, the unification of forces, and the understanding of emergent phenomena—without resorting to separate hypotheses for each phenomenon.

0.2 What Are We Looking for in a New Theory?

If the current models of physics suffer from contradictions or ambiguity, then any new theory aspiring to resolve these issues must be built upon rigorous standards that go beyond merely offering a "new explanation." The theoretical history of physics is full of alternative attempts that were ultimately dismissed because they failed to meet basic scientific or logical requirements.

But what are the features that a **new fundamental physical theory** must possess? What makes it worthy of consideration as a comprehensive framework for understanding nature? In this chapter, we lay out a set of criteria that we consider foundational for evaluating the model we will present later:

0.2.1 Conceptual Universality

The theory must be capable of describing the widest possible range of natural phenomena—from elementary particles, through fundamental forces, to time, spacetime, and possibly even biological and mental phenomena. It is not required to explain every detail at once, but it should contain the potential to do so within its conceptual structure.

0.2.2 Physical Interpretation

It is not enough for the theory to offer precise equations or numerical models. True physics reconnects mathematics to physical reality and offers a clear picture of the entities being discussed. What is the wavefunction? What is spacetime? What is "force"? The new theory must answer such questions clearly and boldly.

0.2.3 Derivation Instead of Assumption

The fundamental constants in physics—such as the speed of light c, Planck's constant $\hbar$, the gravitational constant G, and the fine-structure constant α—are usually taken as empirical givens. However, a deep theory should aim to derive them from first principles. Why do these values exist as they are? Are they truly constant? Explanatory theories do not merely accept what is measured—they explain why It is measured.

0.2.4 Mathematical and Physical Consistency

The theory must be **internally consistent** mathematically—free from contradictions, and capable of preserving fundamental principles such as energy conservation and causality (or a modified form thereof). It must also be **externally consistent** with well-established experiments, or at least be able to interpret them as approximations or limiting cases.

0.2.5 Predictive Power

Physics is not merely philosophical reflection; it Is an empirical science. The new theory must produce **new, testable predictions**—results that differ from current theories and can be measured experimentally to either confirm or falsify the model. In fact, the ability to risk offering unexpected predictions is a hallmark of a mature theory.

0.2.6 Conceptual Simplicity

By simplicity, we do not mean naïveté or simplistic abstraction, but rather **economy in principles and entities**. The more a theory can explain using fewer and more fundamental concepts, the closer it is to truth. The conceptual structure of the theory should be deep, but not burdensome, inviting understanding rather than confusion.

In short, what we are seeking is not just a "new" theory, but a **foundational theory** that redefines the rules, opens the door to unifying fragmented concepts, and embodies both scientific boldness and methodological rigor. In the following pages, we will see how the *Quantum Aether Model* aspires to fulfill these criteria.

0.3 Why the Quantum Aether Model? Visions of Unification, Interpretation, and Transcendence

The **Quantum Aether Model** emerges as a radical attempt to reconstruct physics on an entirely new foundation. It proposes that the universe is not merely an assemblage of particles and separate laws, but rather a *dynamic manifestation of a continuous, physical quantum medium* known as the aether.

In this model, particles, forces, and even spacetime are not regarded as fundamental entities in themselves, but as **emergent patterns** (excitations or modes) within this quantum aether. This is not a revival of the classical notion of the aether, but a deeper vision: a *quantum aether possessing dynamical, quantum properties, capable of self-organization and self-interaction.*

Below are the key advantages and principles that make this model a promising candidate for offering a unified foundational vision:

0.3.1 Fundamental Unification of Matter, Forces, and Spacetime

The model posits that all physical phenomena—from subatomic particles to spacetime curvature—can be understood as different spectral modes within a single foundational medium. This medium is described by several interrelated fundamental fields:

- **Nomon Field** $\Phi(x,t)$: Represents the spectral structure or dynamical phase of the aether.
- **Density Field** $\rho(x,t)$: Reflects the distribution of energy or local matter within the aether.

- **Polarization Field** $\chi(x,t)$: Represents the internal directional structure of the medium, possibly responsible for phenomena like spin.
- **Quantum Viscosity Field** $\eta(x,t)$: Expresses the aether's resistance to deformation or spectral propagation.
- **Spectral Organization Field** $T(x,t)$: Encodes the internal temporal ordering of modal states within the aether.

Through these fields, the model seeks to explain the diversity of physical phenomena as different solutions to the same general field equations.

0.3.2 Derivation of Physical Constants from Aether Properties

In the traditional framework, physical constants such as the speed of light c, Planck's constant $\hbar$, and the gravitational constant G are taken as externally measured inputs. In the quantum aether model, these constants are not assumed, but are Instead **derived from the dynamical properties of the aether**—such as the background density ρ_0, viscosity η_0, or the structure of the spectral organization T.

This opens the door to explaining why these constants have the values we observe, and whether they might vary under different physical conditions (e.g., in the early universe or within highly organized biological environments).

0.3.3 Concrete Quantum Interpretation

The model offers tangible, physical interpretations of quantum concepts that have long appeared abstract or mysterious:

- **The wavefunction** is not merely a probabilistic tool but a spectral description of a real state of the aether.
- **Quantum superposition** becomes an actual superposition of wave patterns in the aether medium.
- **Entanglement** is reinterpreted as a nonlocal spectral linkage between distant points within the same structure.
- **Quantum measurement** is not an instantaneous or "mysterious collapse," but a *dynamic interaction* between the measured system, the measuring device, and the surrounding aether environment.

0.3.4 Spacetime as an Emergent Phenomenon

Rather than treating spacetime as an absolute framework or purely geometrical structure, the model regards it as a phenomenon emergent from aether dynamics. Time is redefined as the rate of internal spectral organization (through $T(x,t)$), while space is understood as a domain of spectral correlation between points in the quantum medium. Gravity itself becomes a result of changes in aether density or gradients in the fields—paving the way for a new derivation of Einstein's equations from aether dynamics.

0.3.5 Extension Toward Life and Consciousness

While traditional physics avoids engagement with questions of life and consciousness, the quantum aether model—at least in principle—allows for a conceptual extension into these domains. Living systems are seen as structures with high spectral organization within the aether, distinguished by unique quantum-biological spectral signatures that could potentially be detected or analyzed. Perception and consciousness are proposed as *complex spectral synchronizations* arising from the interaction between the aether and biologically organized nodes within the nervous system.

0.3.6 Predictive and Testable Power

The model does not stop at interpretation but offers a set of testable predictions:

- Possible variations in certain physical constants in different environments.
- New spectral signatures that could be observed in the cosmic microwave background or gravitational waves.
- Measurable biological effects corresponding to the distribution of quantum-biological spectral fields.

This ability to propose new experiments makes the model falsifiable or confirmable—one of the key requirements of any mature scientific theory.

In summary, the quantum aether model represents a bold attempt to reconstruct physics from an alternative foundation—one that unifies phenomena, explains constants, and restores deep physical meaning to abstract concepts—all while remaining open to fields beyond the conventional, without compromising methodological rigor.

0.4 Challenges and Methodological Candor

It Is relatively easy to propose new theories, but it is exceedingly difficult to offer a model that establishes a coherent and scientifically evaluable physical framework. Within this context, it is essential for the Quantum Aether Model—and for all who work to develop it—to maintain a high level of **methodological candor**, and the ability to distinguish clearly between what has been actually achieved and what remains within the realm of aspiration or conceptual exploration.

Therefore, from the outset, we wish to explicitly state a set of points we consider essential for understanding the position of this work, Its current limitations, and its future prospects.

0.4.1 The Model Is Still Under Construction

The Quantum Aether Model does not claim to be complete. Rather, it Is a **theoretical framework under development**, containing preliminary formulations, foundational concepts, and philosophical and mathematical orientations. However, many of its parts are still in need of:

- Completing the full mathematical derivations of the field equations.
- Constructing precise quantitative models to compute physical constants from aether properties.
- Analyzing the dynamics of measurement, entanglement, and time in greater detail.

Nonetheless, we believe that the overall conceptual structure of the model Is **coherent and promising** enough to justify its development and to share it with the scientific and philosophical community.

0.4.2 Interpretation Does Not Mean Full Validation

Many chapters of this book present *novel conceptual interpretations* of well-known phenomena (such as entanglement, spacetime, dark matter, or even consciousness). However, we make a clear distinction between:

- **Results derived directly from the model's mathematical structure**.
- **Interpretations inferred as emergent concepts or philosophical extensions**.
- **Speculative ideas presented as future research directions or preliminary models**.

This distinction will become clear within the context of each chapter and reflects the scientific nature of the work, avoiding any confusion between predictions, interpretations, and visionary outlooks.

0.4.3 Bridging Physics and Life: An Ambitious Project, Not a Definitive Claim

One of the most intriguing aspects of the model is its extension toward **living systems** and consciousness. Yet, we clearly emphasize that this extension remains in its early stages and is presented here as an exploratory direction based on:

- Observing non-random spectral organization in living organisms.
- The potential to formulate a "quantum-biological spectral signature" relatable to physical fields.
- Preliminary experimental correlations (such as the bamboo experiment) that encourage further inquiry.

Nevertheless, these topics are offered in the spirit of investigation, not as definitive assertions.

0.4.4 Openness to Critique and Collaborative Development

This book is not presented as a final or closed formulation. Rather, it is an **open invitation to thought and dialogue**. We call upon researchers, thinkers, and those interested in the theoretical foundations of physics to contribute through critique, refinement, expansion, and constructive development.

We hope that the formulation we present is clear enough to be evaluated fairly, bold enough to open unprecedented horizons, and balanced enough to respect the principles of science and its methodology.

0.5 Reviving the Aether: From Myth to Physical Framework

The idea of the “aether” has long captivated human imagination since antiquity, where early philosophers envisioned it as a substance finer than the four classical elements, filling the heavens and carrying light and stars. Over time, this concept gradually evolved into a scientific one, becoming by the 19^{th} century a cornerstone of classical physics, especially in explaining the propagation of electromagnetic waves.

However, the downfall of the aether at the dawn of the 20^{th} century—brought about by the rise of relativity and the success of the Michelson–Morley experiment—led to its being regarded as a relic of outdated thinking or even scientific ignorance. Nevertheless, the concept of a “physical medium” has returned, not as a classical aether, but as a dynamic vacuum teeming with quantum fields, latent energy, and virtual fluctuations.

In this chapter, we reopen the case of the aether—not to revive a dead classical idea, but to propose an entirely different vision: the **quantum aether**, conceived as a dynamic, quantum, and fundamental medium from which everything we observe in nature emerges—from particles and forces, to spacetime and life.

0.5.1 The Classical Aether and Its Fall

In classical physics, the aether was postulated as a stationary medium that filled space and allowed light to propagate as a mechanical wave. Just as sound requires an air medium, physicists believed that light must also need a medium to carry it. Based on this notion, various models were developed in the 19^{th} century, most notably Maxwell’s model of electromagnetic waves.

However, the famous Michelson–Morley experiment (1887) in the early 20^{th} century undermined this idea entirely. It failed to detect any difference in the speed of light attributable to Earth’s motion through the aether—implying that the “classical aether” did not exist in the physical sense envisioned by earlier thinkers.

Then came Einstein’s special relativity in 1905, which eliminated the need for a carrier medium and treated the speed of light as a constant in all reference frames, redefining spacetime itself as an absolute geometric entity that required no medium.

At that point, it appeared the aether had been buried for good.

0.5.2 The Quantum Vacuum in Modern Physics: An Unspoken Return of the Aether

Although special relativity eliminated the need for a classical aether, the development of 20^{th}-century physics—especially through **quantum field theory**—brought back a more complex notion of the vacuum, one that shares many features with the old idea of aether.

In classical theory, the vacuum was simply defined as "the complete absence of matter and energy." However, this definition could not survive the advances of quantum physics. In quantum mechanics, no state can be entirely static due to Heisenberg's uncertainty principle, which forbids perfect simultaneous knowledge of energy and time. Thus, in the quantum context, the "vacuum" is not empty, but the **lowest energy state of a quantum system**, brimming with fluctuations and momentary excitations.

Quantum field theory reveals that the vacuum is a complex dynamical state filled with virtual particles that appear and disappear over extremely short timescales, contributing to observable effects such as:

- **The Casimir Effect**: where an attractive force appears between two metal plates in vacuum due to quantum field fluctuations between them.
- **The Lamb Shift** in the hydrogen spectrum: a small energy level shift caused by the electron's interaction with the quantum vacuum.
- **Quantum Tunneling**: which allows particles to penetrate energy barriers due to the non-zero probability of existence In vacuum.

What's striking is that this "quantum vacuum" behaves as if it were a real physical medium: it has energy, structure, interacts with particles and fields, and can even give rise to new particles under extreme conditions (such as electron–positron pair creation in strong fields).

Thus, modern physics—without explicitly admitting it—has revived the idea of the aether, though in a quantum form, more complex and without the absoluteness once attributed to the classical aether.

It can therefore be said that the **quantum vacuum in field theory** is not just a "silent background," but a *core actor on the physical stage*. While most physicists refrain from using the term "aether," many features of the vacuum closely resemble modern conceptions of a quantum aether.

0.5.3 The Return of the Aether with a New Quantum Concept

If modern physics has, implicitly, restored some of the aether's characteristics within the concept of the "quantum vacuum," then the **Quantum Aether Model** explicitly calls for reframing this notion and placing it at the center of the physical structure of the world, not at its Interpretive margins.

But this return is not an attempt to resurrect the "classical aether," which was conceived as a static absolute medium carrying electromagnetic waves. Rather, it seeks to propose a **quantum and dynamic vision of the aether**, fundamentally different in its nature, structure, and function.

In this model, it is postulated that there exists a continuous quantum physical medium — the quantum aether — characterized by the following properties:

- **Non-static and evolving:** The aether is not an absolute reference frame but a dynamic medium undergoing internal fluctuations and interactions.
- **Multifield structure:** The aether is described through a set of fundamental fields: density $\rho(x,t)$, the nomon $\Phi(x,t)$, polarization $\chi(x,t)$, viscosity $\eta(x,t)$, and spectral organization $T(x,t)$.
- **Spacetime and matter emerge from it:** Particles and spacetime are not fundamental entities, but rather *emergent patterns from the dynamics of the aether.*
- **Self-interactive:** The properties of the aether evolve due to its own internal structure, allowing new interpretations of mass, charge, gravity, and time.

The goal of this vision Is not merely to explain the behavior of particles or forces, but to reformulate our understanding of the very *nature of physical reality.* If everything we observe—from photons to gravity, from atoms to galaxies—is merely a *manifestation of different phases of this quantum medium*,

then we are witnessing a paradigm shift from the particle/geometric conception to a fundamentally **field–spectrum–organization** perspective.

In this context, the "quantum aether" becomes **the universal carrier of reality**, not as an absolute frame, but as a dynamic, evolving entity that contains within it the conditions of its emergence and interactions. Thus, the path is opened for unifying seemingly disparate concepts (mass, spacetime, consciousness...) within a single framework from which these phenomena arise as complex manifestations.

0.5.4 Distinguishing the Quantum Aether from Previous Concepts

One may ask: what differentiates this new quantum aether from the classical aether that was discarded, or the quantum vacuum treated by modern physics? To answer this, it is important to distinguish between these three concepts in terms of essence and physical role:

0.5.4.1 First: Classical Aether

The classical aether was thought to be an absolute mechanical medium that filled space and carried light, just as air carries sound. It was assumed to be fixed, stationary, and measurable through our relative motion within it (as attempted In the Michelson–Morley experiment).

This concept faced fundamental issues:

- The assumption of an absolute reference frame conflicted with the principle of relativity.
- Experiments failed to detect any direct effects of its presence.
- It was limited in explaining quantum phenomena or gravity.

0.5.4.2 Second: Quantum Vacuum

The quantum vacuum is not empty of energy or structure, but rather the lowest energy state of fields, filled with fluctuations and undulations. Although it contributes to real physical phenomena (such as the Casimir effect), It is not presented as an independent system with its own dynamics or internal multifield structure.

Its conceptual weaknesses include:

- Lack of independent dynamic description: no equations describe the evolution of the vacuum as an independent entity.
- It does not explain the origin of spacetime, mass, or fundamental constants.
- Its role is limited: it serves as a correction in existing calculations, not as a foundation of physics.

0.5.4.3 Third: Quantum Aether

In contrast, the quantum aether proposed in this model is an actual dynamic medium composed of a set of fundamental fields that interact and evolve, from which all known phenomena arise:

- It is not an absolute reference, but a quantum medium capable of change, compression, and self-organization.
- It does not merely modify; it is **the essential source of time, space, mass, energy, and matter**.
- It is self-interacting and gives rise to patterns (solitons, spectral disturbances) that explain particles and forces.
- It contains within itself the key to explaining physical constants as derivatives of its structure rather than externally imposed values.
- It allows the emergence of spacetime itself, opening the door to interpreting gravity as an emergent pattern rather than a fundamental force.

In short, the quantum aether proposed here does not resurrect an old idea but rather **redefines the concept of the "physical medium"** entirely and grants it a central role In producing the physical structure of reality. It is not merely a "background," but the entity from which everything arises, into which everything transforms, and which remains ever-present behind all phenomena.

Part II

Conceptual and Mathematical Foundations

Chapter 1

Chapter 1: The Fundamental Fields of the Quantum Ether and Their Mathematical Structure

1.1 Introduction: Toward a Deeper Understanding of the Fundamental Components of Reality

When we claim that the quantum ether is the fundamental entity from which everything in nature emerges — from spacetime to particles to forces — we are not speaking metaphorically or philosophically, but proposing a concrete physical hypothesis. This ether is described by a set of interrelated spectral fields that constitute its internal structure, and upon which all known physical phenomena — and the laws and constants that govern them — are built.

In this model, particles, space, and even time are not considered fundamental, but rather as emergent patterns or dynamical behaviors of these fields. This conceptual inversion — from outcome to origin — marks a sharp departure from prevailing physical models.

The aim of this chapter is to introduce the five fundamental fields that make up the quantum ether, not only in terms of their mathematical form but also their physical interpretation and the conceptual motivations that render each of them indispensable within the structure of the theory.

This chapter may be likened to the introduction of the main characters in a narrative before the story unfolds. Each of these fields is a key "player" on the spectral stage, possessing distinct properties and playing a well-defined role. They will later interact with one another according to laws we shall derive — particularly in the upcoming chapter where we formulate the *Quantum Ether Lagrangian*, unifying these fields into a single dynamical entity.

The reader will notice that the fields described here are fundamentally different from those of the Standard Model of particle physics. Instead of positing a large variety of fields tied to specific particles and forces, we return here to the minimum structural essence: only five fields, forming a continuous medium that permeates what we call spacetime — or more precisely, gives rise to spacetime itself — from which complex phenomena emerge.

For each field presented in this chapter, we will address the following questions:

- What is its mathematical nature? Is it a scalar, a vector, or a complex field?
- What is its physical interpretation within the context of the ether?
- Why is it necessary? What role does it play that no other field can?
- What are the fundamental parameters associated with it? Can these parameters be related to known physical constants such as the speed of light or Planck's constant?
- Does it exhibit any symmetries? If so, can those be spontaneously broken? What consequences follow from such breaking?

We will conclude the chapter with a conceptual outline of the possible interactions among these fields, which will prepare us for the full construction of the quantum ether's Lagrangian dynamics in the next chapter.

Thus, this chapter marks the real transition from conceptual preliminaries to the actual physical architecture of the model — where the theory's principles take shape through well-defined mathematical fields that form the solid foundation for everything that follows.

Why Five Fields? Because our aim is to select the smallest *effective* set capable of generating the observed universe from a pre–geometric substrate. We need: (1) a carrier of matter and quantum phase, Φ, so that charges and solitons can exist; (2) an internal directional structure that produces spectral curvature and emergent gauge responses via χ^μ (through which the effective metric g_{eff} and transport relations are built); (3) an internal temporal ordering supplied by the time field T, while *irreversibility* and entropy production arise when $\eta > 0$ through dissipative couplings; (4) a regulator of scales, stiffness, and of the local effective gravitational strength provided by the density field ρ; (5) a dynamical origin of mass and nonconservative behavior through the viscosity/nonlinearity field η. These five roles are nonoverlapping: dropping any one removes an essential pillar (matter, gauge/geometry response, internal causality, scale control and gravitational strength, or mass and irreversibility), whereas adding more fields does not increase explanatory power within the present framework. With this minimal set we also secure *recovery*: General Relativity appears in the limit of geometric curvature of the emergent connection, the gauge sector is recovered as a low–energy spectral response, and the five–field structure offers a structural account of the origin of masses, parameters, and the arrow of time.

1.2 Overview of the Fundamental Spectral Fields

In this section we introduce a high–level overview of the five spectral fields that constitute the deep internal structure of the quantum aether. These are not independent particles or sectors in the Standard Model sense. Rather, they are complementary facets of a *single, continuous physical medium* whose collective organization gives rise to spacetime, matter, and interactions. Each field highlights a distinct aspect of the medium's internal structure:

- a field that carries phase and underlies quantization (the "spectral phase" of matter),
- a field that sets spectral density and local stiffness (hence local physical scales),
- a field that controls dissipation and nonlinearity,
- a field that establishes internal directionality and seeds emergent gauge responses,
- a field that organizes the internal temporal ordering of the medium.

While detailed subsections will develop each field rigorously, Table 1.1 offers a global view: symbol, name, mathematical type, and primary physical role. The five roles are intentionally nonoverlapping so that, taken together, they provide the smallest effective set capable of supporting emergent geometry, matter content, interaction structure, and the arrow of time.

If you only read one paragraph The five spectral fields are not independent particles or sectors, but complementary facets of a single medium. Φ carries phase and matter (solitons and charges); χ^μ sets internal directionality and seeds emergent gauge responses (contributing to the effective geometry); ρ sets local stiffness and scales (including the effective gravitational strength); η opens dissipative/nonlinear channels and provides a dynamical origin of mass; T organizes internal temporal ordering. The arrow of time is *not* imposed by T alone: it emerges when $\eta > 0$ activates dissipative couplings. The next subsections give precise definitions; here you only need the roles.

Two clarifications will guide the remainder of this chapter. First, the fields above are *facets of one medium*: their interplay, not their isolation, is physically meaningful. Second, the arrow of time is not imposed by T alone; it emerges when dissipative channels are active (e.g. via η), a point we shall make precise once we write the effective action and its couplings. In the rest of the chapter we analyze each field in turn—its mathematical structure, physical role, essential parameters, and symmetries— before assembling the unified variational framework from which dynamics and conservation laws follow.

Table 1.1: Fundamental spectral fields: symbols, types, and primary roles.

Symbol	Name	Mathematical Type	Primary Physical Role
Φ	Spectral Phase Field	Complex scalar	Carrier of phase and matter; supports solitons and Noether charges; underlies quantization.
ρ	Density Field	Real scalar	Sets local spectral density and stiffness; regulates characteristic scales and the effective gravitational strength.
η	Viscosity/Nonlinearity Field	Real scalar	Governs dissipative/nonlinear response and provides a dynamical origin of inertial mass (via couplings to Φ).
χ^μ	Polarization/Directional Field	4D vector (time-like–normalized when needed)	Establishes internal directionality; contributes to the construction of the emergent effective metric g_{eff}; seeds emergent gauge responses through spectral curvature.
T	Internal Time Field	Real scalar	Organizes internal temporal ordering (internal "clock"); physical irreversibility and entropy production arise when $\eta > 0$ through dissipative couplings.

What to remember

- The five fields are *facets of one medium*, not independent entities; their interplay—rather than isolation—is physically meaningful.
- The set is intentionally minimal: removing any field drops an essential pillar (matter, gauge/geometry response, internal causality, scale control, or mass/irreversibility).
- Detailed definitions and couplings follow immediately in the next sections; the unified variational framework will make these roles precise.

1.3 The Spectral Phase Field (Φ)

Gentle Introduction We use the spectral phase field Φ to represent the matter/phase facet of the quantum aether. At this stage, Φ should be read conceptually: it organizes phase relations in the medium and supports localized, particle–like excitations without fixing any specific equations yet. Full mathematical definitions and couplings will be given later.

Mathematical Nature (non-technical) Φ is treated as a complex scalar *degree of freedom* of the medium (amplitude and phase). In this chapter we refrain from formulas; transport structures and covariant operations are introduced in the mathematical part that follows.

Physical Concept Φ encodes how the medium stores and coheres phase information. Coherent domains of Φ behave like ordered phases, while localized configurations can model stable, particle–like entities within the aether.

Physical Role (in the unified picture) The field Φ supplies the observable matter content and the phase structure upon which conservation principles are later formulated. It interacts with the medium's directional and environmental facets, but *does not by itself* set the arrow of time nor the geometric law; those emerge at the level of the full theory once the remaining sectors are active.

Essential Parameters (qualitative) Only qualitatively at this point: a characteristic amplitude/scale for Φ, a notion of self–interaction (stiffness) controlling localization vs. coherence, and environmental sensitivities to the medium's density, viscous/nonlinear response, and internal directionality. Concrete parameterizations are deferred.

Symmetries and Properties (qualitative) The physics associated with Φ respects a global phase invariance at the conceptual level, and can exhibit distinct organizational patterns (including topological sectors) due to the angular nature of the phase. Precise statements and conserved quantities are presented later, within the unified variational framework.

Summary For now, remember: Φ is the phase/matter facet of a *single* medium. It carries the content that becomes observable as particles and coherent phases. The details of transport, interactions, conservation laws, and emergent geometry are established in the subsequent sections, where the five fields act together.

Extended Notes on the Spectral Phase Field (Φ) *Motivation and guiding picture.* The spectral phase field Φ is the facet of the quantum aether that carries phase information and, with it, the very possibility of coherent organization. When regions of the medium share a persistent phase relation, they behave as ordered domains; when that order localizes, it can support particle-like entities. At this level, no equations are assumed—only the organizing role of phase.

A heuristic vignette. Imagine the aether as a vast mosaic whose tiles can lock into a common orientation. The pattern itself is not imposed from outside; it arises from the medium's self-organization. In such a mosaic, "defects" or localized twists of orientation can be stable objects. In our language, Φ provides the orientation (phase), while other fields set the conditions under which patterns persist, propagate, or dissolve.

What Φ does—and does not—do on its own. Φ supplies matter/phase content and the possibility of coherence. It does *not* by itself prescribe an arrow of time, nor does it fix the effective geometry. Those features appear when the remaining facets of the medium are active: directionality and transport come with the polarization field, internal ordering with the time field, scale and stiffness with the density field, and irreversibility with the viscosity/nonlinearity field.

Interplay with the other facets (in words, not formulas). The density facet modulates how costly it is to distort the phase, setting local scales for coherence and localization. The viscosity/nonlinearity facet determines whether phase patterns persist conservatively or relax dissipatively; it also seeds an effective origin of mass once couplings are present. The polarization facet endows the medium with an internal sense of direction, which later translates into transport and emergent gauge responses. The internal time facet provides a global ordering against which histories of phase organization can be meaningfully compared.

Common pitfalls we avoid here. We do not identify Φ with any specific Standard-Model field; nor do we treat it as an additional "substance" layered on top of spacetime. Instead, Φ is a facet of a single medium from which spacetime itself will emerge effectively. Likewise, coherence here is a physical organization of the medium, not a mere metaphor.

If you read just this paragraph Φ is the phase/matter facet of one medium. It carries the content that can become particles and coherent phases, without by itself fixing geometry or the arrow of time. Those emerge only when the other facets are active. Detailed definitions, couplings, and conservation statements follow in the next sections.

1.4 The Density Field (ρ)

Gentle Introduction The density field ρ captures how "much of the medium" is present locally and how stiff or compliant it is to deformations. In the quantum aether picture, ρ is not an independent substance but the *local state variable* that sets characteristic scales for organization, transport, and the response to excitations. In qualitative terms: higher ρ means a stiffer, more tightly organized medium; lower ρ means a more compliant one.

Mathematical Nature (non-technical) We treat ρ as a real, positive scalar degree of freedom of the medium. No detailed equations are assumed here. Later, in the variational framework, ρ will appear in scale-setting functions (e.g. stiffness moduli or effective couplings) and may enter the emergent geometric sector through state-dependent weights. For the present, it suffices to regard $\rho(x)$ as a slowly or rapidly varying background that can modulate the behavior of other facets of the medium.

Physical Concept Physically, ρ plays the role of *spectral density and stiffness*: it sets the local cost of distorting phase patterns, and thus the coherence/localization balance. Regions with large ρ resist rapid rearrangements and support sharper, more robust structures; regions with small ρ are easier to reconfigure and tend to smear structures unless other sectors intervene. This viewpoint is compatible with the idea that the effective gravitational strength and other low-energy scales are not fixed universals but may be *environment-dependent* via ρ.

Physical Role (in the unified picture) Within the five-facet architecture, ρ provides the *scale-setting backbone*. It influences: (i) the typical size/energy of localized excitations (the "solitonic" sector carried by Φ); (ii) propagation characteristics once directional organization is present (the χ^μ facet); (iii) the susceptibility of the medium to dissipative vs. conservative behavior when the viscosity/nonlinearity facet η is active. In addition, an important *phenomenological* possibility—developed later—is that regions of high ρ with negligible direct electromagnetic activity can source observable gravitational effects without optical luminosity, thus offering an effective route to dark-matter-like phenomenology. We stress that this is a working hypothesis to be quantified in the dedicated chapters.

Essential Parameters (qualitative) At this introductory stage we name only qualitative controls:

- a background level ρ_0 setting global scales in a given scenario;
- a stiffness/compressibility profile $\mathcal{K}(\rho)$ that governs how costly it is to deform the medium;
- state-dependent effective couplings where ρ modulates interaction strengths or characteristic speeds;
- sensitivity to inhomogeneity through gradients $\nabla\rho$ (domains, interfaces).

Concrete parameterizations and their appearance in the action are deferred to later sections.

Symmetries and Properties (qualitative) As a real scalar, ρ carries no phase symmetry of its own and transforms trivially under internal phase rotations associated with Φ. It is constrained to $\rho > 0$ and can support qualitatively different *organizational regimes* (e.g. high-stiffness vs. low-stiffness phases) depending on its background value. Homogeneous ρ leads to effectively uniform scales; inhomogeneous ρ creates interfaces that can guide or trap excitations. While ρ can influence characteristic timescales, it does *not* by itself impose an arrow of time; irreversibility requires active dissipative channels (through $\eta > 0$), as will be shown in the full framework.

Summary The density field ρ is the scale- and stiffness-setting facet of a single medium. It regulates how costly it is to organize and localize phase patterns and thereby *sets the stage* on which the other facets act: Φ supplies matter/phase content, χ^μ provides directionality and seeds gauge-like responses, T contributes internal ordering, and η opens dissipative/nonlinear channels and supplies a dynamical origin of mass. Later chapters will make precise how ρ enters the effective geometry and why environment-dependent couplings can arise in this framework.

1.5 The Viscosity/Nonlinearity Field (η)

Gentle Introduction The viscosity/nonlinearity field η captures how the quantum aether departs from perfectly conservative behavior. Qualitatively, it opens dissipative channels and controls nonlinear response: when η is inactive, phase patterns behave nearly conservatively; when η is active, patterns can relax, coarsen, or lock into massive, inertia–bearing configurations.

Mathematical Nature (non-technical) We treat η as a real scalar degree of freedom of the medium. No equations are fixed at this stage. Later, in the unified variational framework, η will enter through effective couplings that control dissipation, relaxation rates, and nonlinear susceptibilities. The only guiding idea here is that η modulates whether the dynamics is nearly lossless or endowed with controlled irreversibility.

Physical Concept Physically, η governs *how* the medium responds to distortions beyond the linear regime. Small η corresponds to weak damping and nearly conservative propagation of phase structures; large η corresponds to stronger damping, nonlinear mixing, and the tendency of patterns to select preferred amplitudes or profiles. In this sense, η is the "gate" that turns on irreversible channels and shapes how coherence degrades or stabilizes.

Physical Role (in the unified picture) Within the five–facet architecture, η provides two key functions: (i) it *generates a physical arrow of time* once dissipative couplings are active, so that histories acquire directionality through entropy production; (ii) it *supplies a dynamical origin of inertial mass* for localized excitations of the phase field, once the appropriate couplings are present. Crucially, η does not replace the internal time facet T: rather, T organizes temporal ordering, while *irreversibility* emerges when $\eta > 0$ activates dissipative channels. Likewise, η does not impose geometry; instead it shapes how structures carried by Φ and guided by χ^μ evolve, stabilize, or decay in realistic environments set by ρ.

Essential Parameters (qualitative) At this introductory level we only name qualitative controls:

- a background activity level η_0 distinguishing nearly conservative from dissipative regimes;
- a nonlinear response profile $\mathcal{N}(\eta)$ that sets how strongly amplitudes and gradients are penalized;
- couplings to the phase/matter facet, symbolically "mass–seeding" interactions that let localized excitations acquire inertial response when η is active;
- dissipative links to the internal time organization (qualitatively: η must be active for entropy to be produced);
- characteristic relaxation timescales and thresholds that separate coherent transport from dissipative coarsening.

Specific parameterizations are deferred to later sections where the action and constitutive relations are stated.

Symmetries and Properties (qualitative) As a real scalar, η carries no independent phase symmetry. Conceptually, admissible choices of η and its couplings must respect *positivity of entropy production* in the effective theory and preserve causality at the level of coarse–grained dynamics. Homogeneous η yields uniform dissipative behavior; inhomogeneous η can produce interfaces between conservative–like and dissipative–like domains, shaping where patterns stabilize or decay. The activation of irreversibility requires $\eta > 0$ in the relevant channels; when $\eta \to 0$, the dynamics tends to the conservative limit.

Summary The field η is the medium's knob for dissipation and nonlinearity. It is responsible for turning temporal ordering (provided by T) into a *physical arrow of time* via entropy production, and it offers a structural path to inertial mass for localized excitations of the phase facet Φ. In tandem with the density facet ρ and the directional facet χ^μ, it determines whether organized patterns propagate conservatively, relax dissipatively, or lock into stable, massive structures. Precise forms of the couplings and their constraints are presented in the subsequent chapters.

1.6 The Polarization/Directional Field (χ^μ)

Gentle Introduction The polarization (directional) field χ^μ is the facet of the quantum aether that endows the medium with an internal sense of direction. Conceptually it plays a role analogous to a "director" in ordered media: it organizes transport, distinguishes longitudinal from transverse responses, and sets the stage on which anisotropic—but still causal—propagation can occur. Importantly, a local direction is *not* a global preferred frame: the theory remains compatible with relativistic principles once the emergent geometry is taken into account.

Mathematical Nature (non-technical) We treat χ^μ as a four–dimensional vector degree of freedom of the medium, timelike– normalized when needed to define a consistent local time–direction for transport. At this stage we avoid formulas; qualitatively, spatial and temporal variations of χ^μ organize how "parallel transport" is understood inside the medium, and will later be related to the emergent effective geometry and to internal (spectral) curvature. Only relative orientation and smoothness matter here.

Physical Concept Physically, domains where χ^μ aligns define preferred channels for the flow of phase/matter distortions; misalignment and shear encode how patterns bend, twist, or splay. In this sense, χ^μ acts as the organizing field for direction–dependent phenomena: guidance of waves, anisotropic diffusion vs. ballistic transport, and the possibility of stable textures and defects (e.g. disclinations and domain walls) when the directional order is frustrated or winds nontrivially.

Physical Role (in the unified picture) Within the five–facet architecture, χ^μ provides: (i) *directionality for transport* (how excitations carried by Φ prefer to propagate), (ii) a *geometric input* to the construction of the emergent effective metric g_{eff} and therefore to kinematics at large scales, (iii) the *seed for emergent gauge–like responses* via internal spectral curvature tied to directional organization, and (iv) an *interface* between the phase facet Φ and environmental facets (ρ, η), controlling whether propagation is coherent, birefringent–like, or relaxational in a state–dependent way. None of this imposes a universal preferred frame: directional order is *state–dependent* and can be homogeneous, textured, or vanish in regimes where isotropy is restored.

Essential Parameters (qualitative) We name only qualitative controls at this stage:

- a normalization choice for χ^μ (timelike–normalized when used as a transport director);
- gradient penalties ("stiffness") for splay/twist/bend–like distortions of the directional order;
- alignment couplings to the phase/matter current (qualitatively: transport is easier along χ^μ);
- environmental sensitivities whereby density ρ sets characteristic speeds and anisotropy, while viscosity/nonlinearity η determines whether directional transport is conservative or relaxational;
- a domain/texture scale controlling the width of interfaces across which χ^μ reorients.

Concrete parameterizations and the precise way χ^μ contributes to the emergent connection and spectral curvature are deferred to the mathematical sections.

Symmetries and Properties (qualitative) As a directional order parameter, χ^μ may enjoy redundancies such as insensitivity to an overall sign (director–like $\chi^\mu \sim -\chi^\mu$) or to normalization choices once a constraint is imposed. Its homogeneous phases can spontaneously restore or break internal isotropy; textured phases can carry nontrivial topology depending on the relevant target space (to be specified later). Admissible dynamics must preserve causality at the effective level: when χ^μ is timelike–normalized and the geometry is consistently built, signal propagation remains subluminal in the emergent metric. Discrete symmetries (parity/time reversal) can be either respected or effectively broken by the state of the medium, leading to characteristic transport signatures.

Summary The field χ^μ is the directional organizer of the medium. It guides how excitations of the phase facet Φ move, contributes structurally to the emergent effective geometry g_{eff}, and seeds gauge–like responses through internal curvature associated with directional order. In concert with the density facet ρ (which sets scales) and the viscosity/nonlinearity facet η (which decides whether transport is conservative or dissipative), χ^μ determines whether the medium behaves as an isotropic background, an anisotropic waveguide, or a textured landscape supporting robust defects. Precise definitions and couplings follow in the subsequent sections.

1.7 The Internal Time Field (T)

Gentle Introduction The internal time field T is the facet of the quantum aether that provides *temporal ordering*: it allows the medium to distinguish "before" from "after" at the level of its own organization. In our framework, T is not an external clock added to physics, nor a universal background time. It is an *intrinsic* state variable of a single medium, used to compare histories of organization. Crucially, T by itself does *not* enforce irreversibility; the arrow of time appears only when dissipative channels are active in the viscosity/nonlinearity facet η.

Mathematical Nature (non-technical) We treat T as a real scalar degree of freedom of the medium. Conceptually, its level-sets define "instants" (foliations) of internal organization, while its gradient indicates a direction along which processes are parametrized. No specific differential equations are assumed here. The precise role of T in transport and geometry is deferred to the mathematical sections, where the emergent connection and effective metric are constructed. What matters at this stage is that T provides a coherent *ordering parameter* with respect to which evolution can be described.

Physical Concept Physically, T makes histories comparable: it is the common parameter against which the medium records and correlates changes of phase, directionality, and density. Domains where T varies smoothly admit a clear sense of progression; domains where T is nearly uniform behave as temporally quiescent regions. In heterogeneous situations, the pattern of T can synchronize or desynchronize distinct parts of the medium, enabling rich spatiotemporal organization without postulating an external clock.

Physical Role (in the unified picture) Within the five–facet architecture, T contributes: (i) a *shared temporal ordering* for transport and interaction processes; (ii) a *reference for rates* with which directional organization (χ^μ), phase/matter content (Φ), and scale-setting background (ρ) are compared; and (iii) the *substrate* upon which dissipative channels (when $\eta > 0$) generate a *physical arrow of time* via entropy production. Thus T orders, while η decides whether that ordering is time-symmetric (conservative limit) or time-oriented (dissipative regime). T does not impose a global preferred frame: any effective large-scale kinematics remains consistent with relativity once the emergent geometry is properly assembled from the medium's state.

Essential Parameters (qualitative) At this introductory level we list only qualitative controls:

- a background drift/pace for T setting the nominal rate at which organization is tracked;
- a coherence scale for T (how smoothly its ordering extends across the medium);
- alignment tendencies between the transport director χ^μ and the local progression of T (channels aligned with T vs. transverse reorganization);
- environmental sensitivity whereby the density facet ρ can slow down or speed up the effective progression of processes as recorded by T;
- compatibility with dissipative activity: when $\eta > 0$, histories acquire directionality relative to T.

Concrete parameterizations and their appearance in the action are postponed to the formal sections.

Symmetries and Properties (qualitative) As a real scalar, T admits trivial shifts of origin ($T \mapsto T + \text{const}$) at the conceptual level: only *differences* of T and its local progression matter. Homogeneous patterns of T support synchronized evolution; inhomogeneous patterns define temporal interfaces that can advance, stall, or retract depending on the state of the other facets. Admissible dynamics must preserve effective causality: when the emergent geometry is consistently built, signal propagation remains subluminal with respect to the effective metric. Reversal of T by itself does not create a physical arrow of time; irreversibility requires active dissipative channels and positive entropy production in the η sector.

Summary The field T is the medium's intrinsic organizer of temporal order. It supplies the parameter with which histories are compared, but it does not alone endow the world with an arrow: that emerges only when dissipative activity is present ($\eta > 0$). In concert with Φ (phase/matter), χ^μ (directionality), and ρ (scale/stiffness), T lets the quantum aether record, compare, and coordinate its own evolution. Subsequent chapters specify how T enters the unified variational framework and how effective notions of time, causality, and irreversibility arise from the interplay of all five facets.

1.8 Conceptual Interactions Among the Spectral Fields

This section provides an equation–free map of how influence flows among the five facets of the medium. It is intentionally conceptual: precise operators and couplings will be introduced in the mathematical framework that follows. The goal here is to state *what must happen* and *what must not happen* so that later formulas are guided by clear physical principles.

Interaction principles (no formulas yet).

- **Phase invariance of Φ:** physics is unchanged by a global shift of the phase of Φ.
- **Shift invariance of T:** only differences and local progression of T matter; no term should depend on T itself linearly.
- **Direction without a preferred global frame:** χ^μ organizes transport locally, but does not impose a universal preferred frame.
- **Density sets scales:** ρ modulates stiffness and characteristic speeds; effective couplings can be environment–dependent.
- **Irreversibility requires activity:** the arrow of time appears only when dissipative/nonlinear channels are active (i.e., when $\eta > 0$).

Φ–ρ: phase content on a scale–setting background. Increasing ρ stiffens the medium and lowers the typical propagation speed of phase patterns. As a result, the size, energy, and stability of localized, particle–like excitations carried by Φ depend on the local density. Homogeneous ρ yields uniform scales; inhomogeneous ρ creates interfaces that can guide or trap phase patterns.

Φ–η: coherence vs. dissipative stabilization. When η is weak, transport is nearly conservative and coherent superposition is robust. When η is strong, phase patterns relax, coarsen, or lock into stable profiles, providing an *effective* mechanism for decoherence and measurement–like stabilization at the coarse–grained level, without assuming fundamental non–unitarity.

Φ–χ^μ: directional transport and gauge–like responses. Directional order organizes preferred channels for the flow of phase/matter distortions. Aligned domains act as waveguides; misalignment produces anisotropy (e.g., birefringence–like effects). Later, this directional organization will be tied to internal spectral curvature and gauge–like responses in the effective description.

ρ–T: rate modulation and geometric reading. The rate at which organizational change is recorded can depend on the density background. Conceptually, denser regions progress "more slowly" as registered by T. In the formalism, this dependence will admit a geometric interpretation through the emergent effective metric.

χ^μ–T: alignment of transport and ordering. Transport is easier when directional organization and temporal ordering are aligned. Domains where the local "direction" of the medium and the progression recorded by T cooperate exhibit efficient propagation; where they shear or oppose, transport becomes diffusive or intermittent. This alignment will later be encoded through *derivative* couplings that respect the shift symmetry of T.

Higher–order composites (templates, still conceptual). Triads such as (ρ, Φ, χ^μ) or (η, Φ, T) combine the effects above: density sets the scale, directionality guides the flow, viscosity/nonlinearity decides whether evolution is conservative–like or dissipative–like, and the internal time field records the ordering of histories. All such composites in the formal action will preserve the phase symmetry of Φ and the shift symmetry of T.

Summary. The five facets act in concert as follows: Φ supplies phase/matter content; ρ sets scales and stiffness; χ^μ organizes direction–dependent transport and seeds gauge–like responses; T provides temporal ordering; and η turns ordering into a *physical arrow of time* once dissipative channels are active. Subsequent chapters will translate these principles into a unified variational framework with precise operators and symmetry–respecting couplings.

1.9 Guiding Principles and Symmetry Map

This section closes Chapter 1 by fixing the conceptual constraints that will guide the mathematical construction in Chapter 2. No formulas are introduced here; we only state what the later equations *must* respect.

Foundational constraints.

- **Single-medium ontology:** the five fields are facets of one continuous medium; physical meaning resides in their interplay, not in isolation.
- **Phase invariance of Φ:** physics is unchanged by a global phase shift of Φ.
- **Shift invariance of T:** only differences and local progression of T matter; no term should depend linearly on T itself.
- **No universal preferred frame:** directional order encoded by χ^μ is state-dependent and compatible with relativistic kinematics once emergent geometry is assembled.
- **Density sets scales:** ρ controls stiffness and characteristic speeds; effective couplings may be environment-dependent.
- **Arrow of time requires activity:** temporal ordering is provided by T, but a physical arrow emerges only when dissipative/nonlinear channels are active (i.e., when $\eta > 0$).

Operational reading.

- Φ supplies phase/matter content and supports localized, particle–like excitations.
- χ^μ organizes direction-dependent transport and seeds gauge–like responses.
- ρ sets the scale/stiffness backdrop against which organization and transport occur.
- η opens dissipative channels and provides a dynamical route to inertial mass for localized excitations.
- T provides intrinsic temporal ordering with respect to which histories are compared.

What this implies for Chapter 2. All kinematic and interaction terms will be chosen so as to respect the above symmetries and constraints. Derivatives will be introduced in a symmetry–compatible manner; emergent geometry will be built from state variables rather than postulated; and any dependence on T will enter through its local progression rather than through T itself.

Bridge to Chapter II — From Concepts to Mathematics Having fixed the roles and constraints of the five facets, Chapter 2 introduces the mathematical toolkit that turns these ideas into equations: the relevant derivatives and connections, the effective action and variational principles, and the decomposition between spectral organization and emergent geometry. The guiding rules stated above (phase invariance of Φ, shift invariance of T, no universal preferred frame, density–set scales, and activity–dependent irreversibility) will be enforced explicitly in the formal construction.

1.10 Symmetry and Conceptual Constraints (Pre-Mathematical Map)

This short section closes Chapter 1 by fixing the conceptual constraints that the formal construction in Chapter 2 must respect. No formulas are introduced here.

Core symmetries and constraints

- **Single-medium ontology:** the five fields are facets of one continuous medium; physical meaning resides in their interplay rather than in isolation.
- **Global phase invariance of Φ:** physics is unchanged under a uniform phase shift of Φ; only relative phase information matters.
- **Shift invariance of T:** only differences and local progression of the internal time field matter; no term should depend on the absolute value of T itself.
- **No universal preferred frame:** directional order (χ^μ) is state-dependent and does not impose a global preferred frame; compatibility with relativistic kinematics is maintained once the emergent geometry is assembled.
- **Density sets scales:** the density facet ρ modulates stiffness and characteristic speeds; effective couplings may be environment-dependent.
- **Arrow of time requires activity:** temporal ordering is provided by T, but a physical arrow emerges only when dissipative/nonlinear channels are active (i.e., when $\eta > 0$).

Allowed vs. disallowed patterns (conceptual)

- *Allowed:* couplings that depend on changes/rates rather than absolute values of T; interactions that respect the global phase symmetry of Φ; state-dependent transport organized by χ^μ without implying a universal frame.
- *Disallowed:* any term that breaks the global phase symmetry of Φ at this stage; any direct dependence on the absolute value of T (violating shift invariance); constructions that assume a fixed, universal directional frame.

Topological and discrete aspects (brief) Because the phase of Φ is angular, ordered domains can support winding and defect structures; discrete symmetries (parity/time reversal) may be respected or effectively broken by the state of the medium. Precise statements are deferred to later chapters.

Bridge to Chapter 2 Chapter 2 introduces the mathematical toolkit that turns these constraints into equations: the relevant derivatives and connections, the effective action and variational principles, and the decomposition between spectral organization and emergent geometry. The symmetries and constraints listed above will be enforced explicitly there.

Chapter 2

Chapter 2:Mathematical Foundations of the Quantum Aether: Bundles, Spectral Derivatives, and the Emergent Geometry

2.1 Introduction

This chapter upgrades the conceptual picture of the quantum aether into a precise, self-contained mathematical framework. The guiding idea is that all "structural" data of the medium are encoded in a single *internal state*

$$\mathcal{S}(x) \in F, \qquad \dim F = 7,$$

viewed as a smooth section of an associated bundle above spacetime. The familiar "matter" field Φ is then modeled as a section of a (complex) associated line bundle *constructed from the same principal data*, so that matter is not external to the medium, but is *built on* its internal structure.

Why a bundle-theoretic formulation? We assume spacetime is a smooth four-dimensional manifold M (no a priori metric is fixed at this stage). The aether carries an internal configuration space F (the *spectral fiber*) of dimension seven; the choice $\dim F = 7$ is minimal yet sufficient to host the five functional roles identified in Chapter 1 (density, nonlinearity/viscosity, internal time, directionality, and phase support/anisotropy) with the right number of constraints. Technically, we posit a principal bundle $P(M, G)$ with structure group G, and the associated bundle

$$E \;=\; P \times_G F \xrightarrow{\pi} M,$$

so that the structural state is a section $s : M \to E$, written $\mathcal{S}(x) \equiv s(x) \in F$. This ensures that changes of internal frames act as genuine gauge transformations on F, and it gives us the correct notion of differentiation *along* spacetime while $\mathcal{S}(x)$ lives *in* the fiber.

Spectral differentiation and the need for a connection. Given any smooth function $f : F \to \mathbb{R}$, its pullback $f \circ s : M \to \mathbb{R}$ can be differentiated along M only after we specify an Ehresmann connection on E, which splits TE into horizontal and vertical subbundles. Writing $ds_x : T_xM \to T_{s(x)}E$, we define the vertical projection $\mathbf{V} \circ ds_x$ as the intrinsic change of the internal state along spacetime. This leads to the *spectral derivative*

$$D_\mu[f \circ s](x) \;:=\; (df)_{\mathcal{S}(x)}\big(\, \mathbf{V} \circ ds_x(\partial_\mu)\,\big),$$

which captures how fiber-data vary as we move in M. Non-integrability of the horizontal distribution yields the *spectral curvature*

$$F_{\text{spec}}(\partial_\mu, \partial_\nu) \;\sim\; [D_\mu, D_\nu],$$

a genuinely new geometric ingredient of the theory. Unlike standard curvature built from a spacetime connection, F_{spec} measures the failure of internal coordinates to organize globally in a path-independent

way; it is a structural property of the medium. One of the central theses of this book is that *spectral curvature can found new physics*: it can stabilize composite configurations at short range, modify long-range interaction tails, and render effective couplings environment-dependent, all without introducing fundamental extra forces.

From spectral to covariant: the full derivative and emergent geometry. To act on tensorial fields (including Φ) we extend D_μ by an emergent connection ω_μ valued in the Lie algebra of G, and define the full derivative

$$\nabla_\mu \;=\; D_\mu + \omega_\mu.$$

State-dependence of ω_μ (through $\mathcal{S}$ and its gradients) allows the geometry felt by excitations to be *emergent*. We encode kinematics via an effective Lorentzian metric g_{eff} built from normalized internal directions (a timelike flow u^μ aligned with the ordering set by the internal time field, and a spacelike director $\hat{\chi}^\mu$ encoding anisotropy). The causal cone, hyperbolicity, and positivity of energy are imposed as *consistency gates* on the admissible constitutive choices. A Palatini-type gravitational sector with an aether-dependent effective scale $M_{\text{eff}}(\rho)$ will later recover Einstein dynamics in the appropriate low-energy, homogeneous limit.

Guiding symmetries and standing assumptions. Two structural symmetries constrain all admissible terms:

- Global phase invariance of the matter field Φ ($U(1)$ at the level of the chapter), implemented by using currents/tensors (not Φ itself) in couplings.
- Shift invariance of the internal time field T (no absolute T appears, only its derivatives).

We work in natural units $c = \hbar = 1$, take $[\mathcal{L}] = +4$ (mass dimension), and ensure that all kinetic principal symbols are positive and subluminal with respect to g_{eff}.

What this chapter delivers. We will:

1. Formalize the structural state $\mathcal{S}(x) \in F$ as a section and set the bundle-theoretic tools (projections, sections, horizontal/vertical split).
2. Define the spectral derivative D, the full covariant derivative $\nabla = D + \omega$, and identify the split of curvature into "spectral" and "geometric" parts.
3. Construct the effective metric $g_{\text{eff}}(\mathcal{S})$ and state the causality/positivity gates.
4. Present a unified action for the structural sector and for Φ, derive the general stress–energy tensor $T^{\mu\nu}_{\text{eff}}$, and analyze homogeneous backgrounds and first-order fluctuations (mass matrices, propagation speeds, anisotropy).
5. Fix a consistent units/scaling scheme for all fields and couplings to be used in the rest of the book.

The result is a complete mathematical backbone on which Chapter 3 will erect the equations of motion and the solution theory. Throughout, we emphasize where *spectral curvature* leaves distinctive, potentially observable fingerprints, marking the conceptual departure point from conventional field theories on fixed spacetime.

2.2 Bundles, Projections, and Sections: The Structural State as a Section

This section lays the geometric backbone of the chapter. The quantum aether is modeled by a single *structural state* that lives in a seven–dimensional internal (spectral) fiber and varies smoothly over spacetime. Mathematically, the structural state is a smooth *section* of an associated bundle above the spacetime manifold M. This bundle–theoretic viewpoint clarifies: (i) what transforms under internal ("gauge") frame changes, (ii) how to differentiate fiber–valued data along directions in M, and (iii) where the notion of *spectral curvature* will later come from (non–integrability of the chosen connection on the associated bundle).

We keep the presentation self–contained and purely differential–geometric. No background metric on M is assumed at this stage; the *effective* metric will be constructed later from the structural state itself in §2.6. Throughout, smoothness is tacitly assumed (class C^∞, unless stated otherwise).

2.2.1 Principal vs. Associated Bundles (Concept and Notation)

Base manifold. Let M be a smooth, oriented, four–dimensional manifold (to be interpreted as spacetime at the kinematical level). No metric is fixed yet.

Principal bundle. Let G be a Lie group (the structural or internal frame group). A *principal G–bundle* $P(M,G)$ over M is a smooth manifold P with a smooth, free and transitive right action $R_g : P \to P$ by G and a smooth surjective submersion $\wp : P \to M$ such that $\wp(p \cdot g) = \wp(p)$ for all $p \in P$, $g \in G$. Fibers are G–orbits $\wp^{-1}(x) \cong G$.

Spectral fiber and associated bundle. Let F be a smooth 7–dimensional manifold (the *spectral fiber*) endowed with a smooth left action $\rho : G \times F \to F$, written $\rho(g,\xi) = g \cdot \xi$. The *associated bundle* with typical fiber F is

$$E \;\equiv\; P \times_G F \;:=\; (P \times F)\big/\!\sim, \qquad (p,\xi) \sim (p \cdot g,\, g^{-1} \cdot \xi),$$

together with the projection $\pi : E \to M$, $\pi([p,\xi]) = \wp(p)$. Points $e \in E$ are equivalence classes $[p,\xi]$, and the fiber $E_x := \pi^{-1}(x) \simeq F$.

Structural state as a section. A *structural state* of the medium is a smooth section $s : M \to E$, $x \mapsto s(x) = [p(x), \mathcal{S}(x)]$, with $\mathcal{S}(x) \in F$ defined modulo the G–action. We shall write simply $\mathcal{S}(x) \in F$ for the fiber coordinate of the state in a chosen local trivialization.

Physical reading. Internal frame changes (local G–transformations) act on the *fiber coordinate* of the state, not on the base point $x \in M$. This aligns with the ontology of a single medium with internal degrees of freedom, rather than a collection of unrelated fields.

2.2.2 The Spectral Fiber F with $\dim F = 7$ and Gauge Action

Local coordinates and minimal content. We assume that F admits local charts with coordinates

$$S^A \;=\; \big(\rho,\; \eta,\; T,\; \chi^1,\; \chi^2,\; \chi^3,\; \alpha\big), \qquad A = 1, \dots, 7,$$

where $\rho > 0$ is the density–like modulus, $\eta \geq 0$ controls nonlinearity/viscosity, T is the internal time coordinate (later entering only through derivatives), $\boldsymbol{\chi} = (\chi^1, \chi^2, \chi^3)$ encodes a spacelike director (orientation), and α is an anisotropy/intensity parameter. This chart realizes the five functional roles identified in Chapter 1 within $\dim F = 7$ by assigning three degrees of freedom to directionality and one to its intensity, plus the three scalars (ρ, η, T).

Gauge action on the fiber. The group G acts on F via $\rho(g,\cdot)$; in local coordinates S^A one has $S^A \mapsto (g \cdot S)^A$. If F is endowed with additional structures (e.g. a Riemannian metric $\mathcal{G}_{AB}$, or constraints $\varphi_i(S) = 0$ defining an admissible submanifold), we assume ρ acts by diffeomorphisms preserving the relevant structures (or mapping admissible points to admissible points).

Physical reading. Choosing a local trivialization is tantamount to choosing an internal frame. Under frame changes, $\mathcal{S}(x)$ re–expresses in the new chart via the G–action. The physics is invariant under such reparametrizations of the internal fiber.

2.2.3 Sections, Pullbacks, and Local Trivializations

Projection and section. The bundle projection $\pi : E \to M$ forgets the internal coordinate: $\pi([p,\xi]) = \wp(p)$. A section $s : M \to E$ assigns to each $x \in M$ a point $s(x) \in E_x$. Locally, on a trivializing neighborhood $U \subset M$ with diffeomorphism $\Psi : \pi^{-1}(U) \to U \times F$, one writes $\Psi \circ s(x) = (x, \mathcal{S}(x))$.

Pullbacks and fiber functions. Given a smooth function $f : F \to \mathbb{R}$, its pullback along s is the spacetime function $f \circ s : U \to \mathbb{R}$, $x \mapsto f(\mathcal{S}(x))$. Likewise, fiber tensors pulled back along s become spacetime tensors with only base indices.

Local trivializations and transitions. Let $\{U_i\}$ be a cover of M by trivializing neighborhoods with maps $\Psi_i : \pi^{-1}(U_i) \to U_i \times F$. On overlaps $U_i \cap U_j$, transition functions $g_{ij} : U_i \cap U_j \to G$ relate trivializations by

$$\Psi_i \circ \Psi_j^{-1}(x, \xi) \;=\; \big(x,\; g_{ij}(x) \cdot \xi\big),$$

and satisfy the cocycle condition $g_{ij} g_{jk} = g_{ik}$ on triple overlaps. Components of $\mathcal{S}$ transform accordingly: $\mathcal{S}_i(x) = g_{ij}(x) \cdot \mathcal{S}_j(x)$.

Physical reading. Transition functions encode how we must "twist" the internal frame to compare states in neighboring charts. This is the geometric seed of the later notion of a connection and its curvature.

2.2.4 Ehresmann Connection and the Horizontal/Vertical Split

Vertical subbundle. The *vertical* subbundle $\mathcal{V} \subset TE$ is defined fiberwise by $\mathcal{V}_e := \ker(d\pi_e)$. Vectors in $\mathcal{V}_e$ are tangent to the fiber $E_{\pi(e)}$.

Ehresmann connection. An *Ehresmann connection* on E is a smooth choice of a complementary, rank–dim M subbundle $\mathcal{H} \subset TE$ such that for each $e \in E$,

$$T_e E \;=\; \mathcal{H}_e \oplus \mathcal{V}_e.$$

Vectors in $\mathcal{H}_e$ are "horizontal" lifts of base vectors. The connection yields projectors $\mathbf{H}_e : T_e E \to \mathcal{H}_e$ and $\mathbf{V}_e : T_e E \to \mathcal{V}_e$.

Horizontal lift and differential of a section. Given $X_x \in T_x M$, its horizontal lift at $e \in E_x$ is the unique $X_e^H \in \mathcal{H}_e$ with $d\pi_e(X_e^H) = X_x$. For a section $s : M \to E$, the differential $ds_x : T_x M \to T_{s(x)} E$ decomposes as

$$ds_x \;=\; \mathbf{H}_{s(x)} \circ ds_x \;+\; \mathbf{V}_{s(x)} \circ ds_x.$$

The *vertical* part, $\mathbf{V} \circ ds_x : T_x M \to \mathcal{V}_{s(x)} \cong T_{\mathcal{S}(x)} F$, encodes the intrinsic change of the internal state as one moves along M.

Local expression. In a local trivialization $E|_U \cong U \times F$ with coordinates (x^μ, S^A), an Ehresmann connection can be represented by a set of *connection coefficients* $\Gamma^A{}_\mu(x, S)$ so that horizontal lifts satisfy

$$X^H \;=\; X^\mu \Big(\partial_\mu + \Gamma^A{}_\mu(x, S)\, \partial_{S^A}\Big),$$

and for a section $S^A = S^A(x)$,

$$\mathbf{V} \circ ds(\partial_\mu) \;=\; \Big(\partial_\mu S^A + \Gamma^A{}_\mu(x, S(x))\Big)\, \partial_{S^A}.$$

This is the raw, geometrically intrinsic object from which the spectral derivative in §2.4 will be defined.

Integrability remark. The horizontal distribution $\mathcal{H}$ is generally *non–integrable*; its curvature two–form measures the failure of horizontal lifts to close under commutators. This curvature will be the geometric origin of the *spectral curvature* entering commutators of fiber–differentiations.

Physical reading. The split $T_e E = \mathcal{H}_e \oplus \mathcal{V}_e$ cleanly separates (i) motion along the base manifold from (ii) intrinsic reconfigurations of the internal state. Non–integrability means that following two different spacetime paths can lead to distinct internal configurations even if they close at the same base point: this path–dependence is the hallmark of spectral curvature and the seed of novel physical effects in the model.

2.2.5 Constraints and the Admissible Constraint Subbundle $C \subset E$

Normalization and inequality constraints. The internal state is not arbitrary. We enforce:

$$u^\mu u_\mu = -1, \qquad \hat{\chi}^\mu \hat{\chi}_\mu = +1, \qquad u^\mu \hat{\chi}_\mu = 0, \qquad \rho > 0, \qquad \eta \geq 0,$$

where u^μ (timelike flow) and $\hat{\chi}^\mu$ (spacelike director) will be constructed from the spectral coordinates S^A in §2.6 (e.g. combining the ordering set by T with the raw directional data $\boldsymbol{\chi}$ and anisotropy α, then normalizing). The set of points in each fiber satisfying the equality constraints defines a smooth submanifold of F; inequalities define open domains.

Constraint subbundle. Let $C \subset E$ be the subbundle whose fiber $C_x \subset E_x$ consists of those internal states satisfying the equalities above.[1] The physical structural state is then a section $s : M \to C \subset E$. In the unified action (later in §2.7) these equalities are enforced by Lagrange multipliers, so that normalization and orthogonality hold *on–shell* without overconstraining variations.

No mixing of indices. We stress that internal indices (attached to objects living in F or to internal frames) must not be identified with spacetime indices. Contractions and normalizations above refer to the *effective* metric and its induced internal frames which will be constructed later; at this stage they serve to indicate the target geometric structure of admissible states.

Physical reading. The constraints encode two physical requirements: (i) there is a singled–out timelike direction (the medium's internal flow), (ii) there is an emergent spacelike director (anisotropy axis), orthogonal to the flow. Together with positivity of ρ and non–negativity of η, these conditions delimit the "phase space" of admissible aether configurations and prevent unphysical regimes.

Notation summary for §2.2.

M	Base manifold (spacetime), smooth, 4D, no prior metric.
G	Internal frame (structure) group.
$P(M, G)$	Principal G–bundle with projection $\wp : P \to M$.
F	Spectral fiber, $\dim F = 7$, with a smooth left action of G.
$E = P \times_G F$	Associated bundle with projection $\pi : E \to M$.
$s : M \to E$	Structural state (section), locally $s(x) = (x, \mathcal{S}(x))$.
$\mathcal{V} = \ker d\pi$	Vertical subbundle of TE.
$\mathcal{H} \subset TE$	Horizontal subbundle (Ehresmann connection), $TE = \mathcal{H} \oplus \mathcal{V}$.
$\mathbf{H}, \mathbf{V}$	Projectors onto $\mathcal{H}$ and $\mathcal{V}$.
$C \subset E$	Constraint subbundle (normalizations and orthogonality).

This completes the bundle–theoretic groundwork. In §2.4 we shall use the vertical component $\mathbf{V} \circ ds$ to define the *spectral derivative* D_μ acting on fiber functions pulled back along s, and in §2.5 we will extend it to the full covariant derivative $\nabla_\mu = D_\mu + \omega_\mu$ with a state–dependent emergent connection ω_μ.

2.3 The Structural State $\mathcal{S}(x) \in F$ and Field Decomposition

Having set the bundle–theoretic backbone, we now specify the *internal* structure of the spectral fiber F and the precise way the aether state $\mathcal{S}(x)$ is encoded in it. The goal is twofold: (i) to make explicit which coordinates on F we use and what redundancies/constraints they carry; (ii) to define gauge–invariant building blocks (invariants and an internal metric $\mathcal{G}_{AB}$) that will feed directly into kinetic/potential terms in §2.7 and into the spectral derivative in §2.4.

Throughout this section we keep the discussion at the level of the fiber F and the section $\mathcal{S} : M \to F$ (via a chosen trivialization of the associated bundle). No background spacetime metric is assumed; the *effective* metric g_{eff} will be constructed in §2.6.

[1] Inequality constraints (e.g. $\rho > 0$, $\eta \geq 0$) define admissible open domains in C.

2.3.1 Coordinates on F: Minimal Atlas and Physical Facets

Local coordinates. We posit that F admits a local atlas with coordinates

$$S^A = (\rho,\ \eta,\ T,\ \chi^1,\ \chi^2,\ \chi^3,\ \alpha), \qquad A = 1, \dots, 7, \tag{2.1}$$

where:

- $\rho > 0$ is a density–like modulus controlling inertia and coupling strengths.
- $\eta \geq 0$ measures the nonlinearity/viscosity level (dissipative propensity).
- $T \in \mathbb{R}$ is the internal time coordinate; *only its derivatives* will be allowed to appear in the action, implementing a global shift symmetry $T \mapsto T + c$.
- $\boldsymbol{\chi} = (\chi^1, \chi^2, \chi^3)$ encodes a *spatial director* (orientation) internal to the medium.
- $\alpha \geq 0$ is an anisotropy intensity (ordering strength) controlling how sharply the director is expressed.

The choice $\dim F = 7$ is minimal yet sufficient to host the five functional roles identified in Chapter 1: (ρ, η, T, directionality $\boldsymbol{\chi}$, and anisotropy α).

Gauge action on F. Let G act smoothly on F by diffeomorphisms $S \mapsto g \cdot S$. We assume: (i) ρ, η, T, α transform as scalars under G (possibly reparametrized by internal frame changes); (ii) $\boldsymbol{\chi}$ transforms in a representation compatible with its role as an internal director (e.g. internal rotations act by $R \in \mathrm{SO}(3)$: $\boldsymbol{\chi} \mapsto R\,\boldsymbol{\chi}$). All physical statements are required to be invariant under this action.

Physical remark. The seven coordinates are *not* independent fields in the Standard Model sense; rather, they are *coordinates* of a single internal state $\mathcal{S}(x) \in F$. Frame changes on F are gauge choices.

2.3.2 Redundancies and Constraints: From Raw Coordinates to the Admissible State Space

Redundancies. The following identifications capture internal symmetries that do not alter the physical state:

1. **Head–tail symmetry (nematic–like).** In phases where the sign of direction is immaterial, we identify $\boldsymbol{\chi} \sim -\boldsymbol{\chi}$. This reduces the orientation space from S^2 to $\mathbb{RP}^2$ for the direction factor.[2]
2. **Time shift.** $T \sim T + c$ for any constant c. The action and all invariants will depend on ∂T only.
3. **Gauge frames.** Internal frames related by $g \in G$ are identified; this is automatic at the associated-bundle level.
4. **Anisotropy rescalings (optional).** In some phases a monotone reparametrization $\alpha \mapsto h(\alpha)$ may be used; we fix a convenient normalization and regard α as dimensionless.

Inequality constraints (domains). We enforce $\rho > 0$ and $\eta \geq 0$ as domain conditions. The variable $\alpha \geq 0$ is taken nonnegative, with a possible threshold $\alpha_c \geq 0$ marking the onset of directional order (see below).

Constraint submanifold and reduced space. Equality constraints (e.g. unit–norm director in ordered phases) define a smooth submanifold $C \subset F$ on which the physical state lives; inequalities select an open domain $\mathfrak{D} \subset C$. The *reduced* internal state space is then

$$F_{\mathrm{red}} := (\mathfrak{D} \cap C)/\!\sim, \tag{2.2}$$

where $\sim$ collects the redundancies above (gauge frames and, if applicable, $\boldsymbol{\chi} \sim -\boldsymbol{\chi}$).

[2] Whether this $\mathbb{Z}_2$ identification is enforced depends on the phase; see §2.3.4.

Physical remark. The reduction ensures that predictions are independent of internal frame conventions and sign conventions for the director, when these have no physical meaning (nematic–like regime). Interfaces where the identification changes (e.g. across phases) will support topological defects.

2.3.3 A Structural Metric $\mathcal{G}_{AB}(\mathcal{S})$ and Gauge–Invariant Invariants

Internal metric on F. We introduce a smooth, positive–definite metric $\mathcal{G}_{AB}(\mathcal{S})$ on the fiber F, to be used in kinetic terms for the structural sector. Positivity of $\mathcal{G}_{AB}$ guarantees positive kinetic energy for vertical variations of the state. We require $\mathcal{G}_{AB}$ to be G–equivariant so that $\mathcal{G}$ descends to F_{red}.

Gauge–invariant scalars (examples). Out of the coordinates S^A and $\mathcal{G}_{AB}$ we can form invariant scalars:

$$I_1 = \rho, \quad I_2 = \eta, \quad I_3 = \alpha, \quad I_4 = \|\boldsymbol{\chi}\|_{\mathcal{G}}^2 := \sum_{i=1}^{3}\sum_{j=1}^{3} \mathcal{G}_{ij}(\mathcal{S})\,\chi^i\chi^j, \quad I_5 = \langle \chi, \chi \rangle_{\mathcal{G}} \cdot f(\alpha), \tag{2.3}$$

and so on, where f is a smooth function setting how the anisotropy α weights the director magnitude. In ordered phases we will enforce $\|\boldsymbol{\chi}\|_{\mathcal{G}} = 1$ by constraints (see below).

Internal potential (Landau–like template). At the level of F alone, a general internal energy density can be written as a G–invariant function

$$U(\mathcal{S}) \;=\; U_\rho(\rho) \;+\; U_\eta(\eta) \;+\; U_T(\partial T \text{ only}) \;+\; U_\alpha(\alpha) \;+\; U_{\chi\alpha}(\boldsymbol{\chi}, \alpha; \mathcal{G}), \tag{2.4}$$

with the following properties:

- U_ρ bounded below with $U_\rho'(\rho) > 0$ for $\rho \to \infty$ (stability).
- $U_\eta \geq 0$, minimal at $\eta = 0$ (no dissipation in the ideal limit).
- U_T depends on ∂T only at the Lagrangian level (enforcing shift symmetry).
- U_α has either a single minimum at $\alpha = 0$ (isotropic phase) or a minimum at $\alpha = \alpha_* > 0$ (ordered phase).[3]
- $U_{\chi\alpha}$ penalizes deviations from the desired director normalization in the ordered phase, e.g. $U_{\chi\alpha} = \frac{\kappa}{2}\left(\|\boldsymbol{\chi}\|_{\mathcal{G}}^2 - 1\right)^2 g(\alpha)$, with $g(0) = 0$, $g(\alpha > 0) > 0$.

In the unified action we will *replace* such penalization by exact constraints via Lagrange multipliers, so that normalization holds on–shell without stiff potentials; here (2.4) is a template.

Physical remark. $\mathcal{G}_{AB}$ sets the *cost* of moving in different internal directions; its state–dependence allows anisotropic responses even before spacetime geometry is constructed. The invariants (2.3) are the only allowed arguments of U, ensuring gauge invariance.

2.3.4 Phases, Charts, and Stratification of F

Isotropic phase. When U_α minimizes at $\alpha = 0$, the director $\boldsymbol{\chi}$ carries no physical meaning ($\mathbb{Z}_2$ head–tail identification is automatic, and the effective cost of rotations in $\boldsymbol{\chi}$ is flat). Local charts on F can be chosen without singling out a direction.

Ordered (nematic–like) phase. When U_α minimizes at $\alpha = \alpha_* > 0$, a *directional order* emerges. The physical space of directions becomes $\mathbb{RP}^2$ if head–tail symmetry is enforced ($\boldsymbol{\chi} \sim -\boldsymbol{\chi}$), or S^2 otherwise. Charts on F must then accommodate this nontrivial topology (no global, nonsingular director field on S^2 without defects).

[3] A minimal polynomial form is $U_\alpha = \frac{1}{2}m_\alpha^2\alpha^2 + \frac{\lambda_\alpha}{4}\alpha^4$, with $m_\alpha^2 > 0$ for isotropic, $m_\alpha^2 < 0$ for ordered; $\lambda_\alpha > 0$ ensures boundedness.

Stratification. Altogether, F is *stratified* into regions (strata) with different stabilizers under G (isotropic vs. oriented strata). Transitions between strata correspond to structural phase transitions in the medium; these will reflect in the effective metric g_{eff} (e.g. through anisotropic wave speeds) and in the spectral curvature once gradients are present.

Physical remark. Phase structure on F is the internal seed of macroscopic phenomena: emergence of an axis, anisotropic propagation, and defect formation—all before invoking any external spacetime geometry.

2.3.5 Admissible Domains, Interfaces, and Defects

Admissible domains. A domain $\mathcal{U} \subset M$ is *admissible* for a given chart on F if the image $\mathcal{S}(\mathcal{U}) \subset F$ stays within that chart's coordinate patch and within the reduced space F_{red}. This excludes singular charts near defect cores or phase boundaries.

Interfaces. Interfaces are codimension–one regions in M where $\mathcal{S}(x)$ crosses from one stratum to another (e.g. α crossing α_c). Across an interface, the admissible chart on F may change, and transition functions (the G–valued glue) become dynamically relevant.

Topological defects. In ordered phases with head–tail symmetry, the director space $\mathbb{RP}^2$ supports nontrivial homotopy. Maps from loops in M to $\mathbb{RP}^2$ can carry $\mathbb{Z}_2$ charges; defects appear where no global lift to S^2 exists. In three dimensions, this leads to line defects (disclinations) or point defects, depending on the embedding and the detailed coupling to gradients (treated later in the kinetic sector).

Physical remark. Defects and interfaces are the natural *loci* for localized structures (solitons/molecules of the aether). They will couple strongly to spectral curvature once gradients of $\mathcal{S}$ are turned on and a connection is chosen (next sections).

2.3.6 Summary and Notational Commitments

Summary. We modeled the structural state as $\mathcal{S}(x) \in F$ with local coordinates (2.1), identified redundancies (gauge frames, time shift, optional head–tail), and imposed domains/constraints to define the reduced space F_{red}. An internal, positive–definite metric $\mathcal{G}_{AB}(\mathcal{S})$ and G–invariant scalars (2.3) provide the building blocks for potentials and, later, for kinetic couplings. Phase structure (isotropic vs. oriented) stratifies F and anticipates defects and anisotropy.

Notation (for later use).

- S^A — fiber coordinates on F; $\mathcal{S}(x) = (S^A(x))$.
- F_{red} — reduced internal state space after quotienting by redundancies.
- $\mathcal{G}_{AB}(\mathcal{S}) \succ 0$ — internal metric on F.
- $U(\mathcal{S}) = U(I_1, \dots)$ — internal potential written in terms of invariants.
- Isotropic phase: $\alpha = 0$; ordered phase: $\alpha = \alpha_* > 0$ (nematic–like).

With the internal state space now structured and its invariants in hand, we are ready to define *differentiation along spacetime* for fiber–valued data. In §2.4 we introduce the *spectral derivative* D_μ using the Ehresmann connection of §2.2, and we analyze the resulting *spectral curvature* that will play a central role in the dynamics.

2.4 The Spectral Derivative D_μ and Spectral Curvature

Spectral coordinates σ^A and the section $S^A(x)$

We distinguish between *fiber coordinates* and *field components*:

- A local chart on the internal fiber F is denoted by

$$\sigma^A \equiv (\sigma^1, \dots, \sigma^7) \quad \text{with} \quad A = 1, \dots, 7,$$

  and we adopt the identification

$$(\sigma^1, \dots, \sigma^7) \equiv (\rho,\, \eta,\, T,\, \chi^1,\, \chi^2,\, \chi^3,\, \alpha).$$

  We write $\partial_A \equiv \partial/\partial\sigma^A$. All σ^A are taken *dimensionless*.

- A structural state (section) is a map $s : M \to E$; its components in the chart (x^μ, σ^A) are

$$S^A(x) := \sigma^A \circ s(x).$$

  Thus $S^A(x)$ are the *fields* on spacetime obtained by evaluating the spectral coordinates along the section.

Horizontal/vertical split. Given an Ehresmann connection ${\Gamma^A}_\mu(x, \sigma)$, the horizontal lift and its curvature read

$$D_\mu \;=\; \partial_\mu + {\Gamma^A}_\mu(x,\sigma)\,\partial_A, \qquad {F^A}_{\mu\nu} \;=\; \partial_{[\mu}{\Gamma^A}_{\nu]} \;+\; {\Gamma^B}_{[\mu}\,\partial_B{\Gamma^A}_{\nu]}\,. \tag{2.5}$$

The vertical derivative acts only along the fiber: $d_V = d\sigma^A\,\partial_A$.

Relation between $D_\mu S^A$ and ${\Gamma^A}_\mu$. Evaluating (2.5) on the section gives

$$D_\mu S^A(x) \;=\; \partial_\mu S^A(x) \;+\; {\Gamma^A}_\mu\big(x, S(x)\big). \tag{2.6}$$

Change of spectral coordinates. Under a fiber reparametrization $\sigma'^A = f^A(x, \sigma)$ (local change of trivialization), the horizontal lift is invariant iff the coefficients transform as

$${\Gamma'^A}_\mu(x,\sigma') \;=\; \partial_\mu f^A(x,\sigma) \;+\; \frac{\partial f^A}{\partial\sigma^B}(x,\sigma)\,{\Gamma^B}_\mu(x,\sigma), \qquad {F'^A}_{\mu\nu} \;=\; \frac{\partial f^A}{\partial\sigma^B}\,{F^B}_{\mu\nu}. \tag{2.7}$$

If f^A is x–independent, the inhomogeneous piece $\partial_\mu f^A$ vanishes and Γ transforms tensorially.

Spectral metrics and bilinears. Any fiber bilinear $\mathcal{G}_{AB}(\sigma)$ (or $Z_{AB}(\sigma)$, $Y_{AB}(\sigma)$) is expressed in the σ–chart; pulled back to the section it yields the spacetime tensors used in §2.6:

$$\mathcal{G}_{AB}\big(S(x)\big), \quad Z_{AB}\big(S(x)\big), \quad Y_{AB}\big(S(x)\big).$$

In particular, the material form $C_{\mu\nu} = \mathcal{G}_{AB}(S)\,D_\mu S^A D_\nu S^B$ and all invariants built from it are chart–independent.

We now define differentiation *along spacetime* for quantities that live *in the fiber* F. Given a section $s : M \to E$, the Ehresmann connection introduced in §2.2.4 provides a canonical vertical part of the change of the internal state as one moves in M. This vertical part will be the seed of the *spectral derivative* D_μ, and its non–integrability gives rise to a curvature two–form that we call the *spectral curvature*. These objects will play a central role in the kinematics and dynamics of the quantum aether.

All constructions below are intrinsic to the associated bundle $E = P \times_G F$ and require no background metric on M. Local coordinate expressions are given for clarity.

2.4.1 Definition and Basic Properties of D_μ

Convention for the connection coefficients. On a trivializing neighborhood $U \subset M$ with coordinates (x^μ, S^A) on $E|_U \cong U \times F$, we represent an Ehresmann connection by coefficients $\Gamma^A{}_\mu(x,S)$ so that the *horizontal lift* of ∂_μ is

$$\big(\partial_\mu\big)^H \;=\; \partial_\mu \;+\; \Gamma^A{}_\mu(x,S)\,\partial_{S^A}. \tag{2.8}$$

For a section $S^A(x)$, the differential decomposes as

$$ds(\partial_\mu) \;=\; \big(\partial_\mu\big)^H \;+\; \big(\partial_\mu S^A - \Gamma^A{}_\mu(x,S(x))\big)\,\partial_{S^A}, \tag{2.9}$$

so the vertical part is $\mathbf{V}\circ ds(\partial_\mu) = \big(\partial_\mu S^A - \Gamma^A{}_\mu\big)\partial_{S^A}$. This fixes the sign convention we shall use throughout.[4]

Spectral gradient of the internal state. We define the *spectral gradient* of the structural coordinates as the vertical velocity

$$D_\mu S^A(x) \;:=\; \partial_\mu S^A(x) \;-\; \Gamma^A{}_\mu\big(x,S(x)\big). \tag{2.10}$$

This is a fiber–tangent (vertical) vector at $S(x)\in F$, independent of the choice of local trivialization (see the gauge transformation law below).

Spectral derivative on fiber functions. For any smooth $f : F \to \mathbb{R}$ we define

$$D_\mu\big[f\circ s\big](x) \;:=\; \partial_A f\big(S(x)\big)\,D_\mu S^A(x) \;=\; \partial_A f\big(S(x)\big)\,\big(\partial_\mu S^A - \Gamma^A{}_\mu\big), \tag{2.11}$$

where $\partial_A f := \partial f/\partial S^A$. This is the directional derivative of f along the *vertical* change induced by moving in M.

Leibniz and chain rules. By construction,

$$D_\mu\big[(fg)\circ s\big] = (D_\mu f\circ s)\,(g\circ s)+(f\circ s)\,(D_\mu g\circ s), \qquad D_\mu\big[(h\circ\varphi)\circ s\big] = \partial_a h\big(\varphi(S)\big)\,\varphi^a{}_{,A}(S)\,D_\mu S^A, \tag{2.12}$$

for any smooth $f, g : F \to \mathbb{R}$, $h : F' \to \mathbb{R}$, and smooth map $\varphi : F \to F'$ with local Jacobian $\varphi^a{}_{,A}$.

Gauge covariance. Under a change of trivialization $(x,S)\mapsto(x,S')$ with $S' = g(x)\cdot S$ ($g : U \to G$), the coefficients transform so that the horizontal lift (2.8) is invariant:

$$\Gamma'^A{}_\mu(x,S') \;=\; \frac{\partial S'^A}{\partial S^B}(x,S)\,\Gamma^B{}_\mu(x,S) \;-\; \frac{\partial S'^A}{\partial x^\mu}(x,S). \tag{2.13}$$

Then the spectral gradient transforms *tensorially* on the fiber:

$$D_\mu S'^A \;=\; \frac{\partial S'^A}{\partial S^B}\,D_\mu S^B, \tag{2.14}$$

and thus $D_\mu[f\circ s]$ in (2.11) is gauge–invariant (a scalar on M).

Physical reading. The compensator $\Gamma^A{}_\mu$ subtracts the spurious change due to reparametrizations of the internal frame as we move in M, leaving only the *intrinsic* vertical reconfiguration of the aether.

[4] Some authors define the horizontal lift with the opposite sign in front of $\Gamma^A{}_\mu$. Our choice (2.8) implies the vertical variation (2.9) with a minus sign in $\partial_\mu S^A - \Gamma^A{}_\mu$. All identities below are consistent with this convention.

2.4.2 Commutator and the Spectral Curvature

Operator form. It is convenient to regard D_μ as a horizontal differential operator acting on fiber functions:

$$D_\mu \;=\; \partial_\mu \;-\; \Gamma^A{}_\mu(x,S)\,\partial_{S^A}. \tag{2.15}$$

With this, the commutator on any $f \in C^\infty(F)$ reads

$$[D_\mu, D_\nu]\, f \;=\; -\,F^A{}_{\mu\nu}(x,S)\,\partial_{S^A} f, \tag{2.16}$$

where the *spectral curvature coefficients* are

$$F^A{}_{\mu\nu} \;=\; \partial_\mu \Gamma^A{}_\nu \;-\; \partial_\nu \Gamma^A{}_\mu \;+\; \Gamma^B{}_\mu\,\partial_{S^B}\Gamma^A{}_\nu \;-\; \Gamma^B{}_\nu\,\partial_{S^B}\Gamma^A{}_\mu. \tag{2.17}$$

Thus $F^A{}_{\mu\nu}\,\partial_{S^A}$ is a vertical vector field measuring the failure of horizontal lifts to commute (non–integrability of $\mathcal{H}$). By (2.7), $F^A{}_{\mu\nu}$ transforms tensorially on the fiber:

$$F'^A{}_{\mu\nu}(x,S') \;=\; \frac{\partial S'^A}{\partial S^B}(x,S)\,F^B{}_{\mu\nu}(x,S). \tag{2.18}$$

Bianchi–type identity. Define the horizontal derivative acting on fiber–tangent fields $V^A(S)$ by

$$(D_\mu V)^A \;:=\; \partial_\mu V^A \;-\; \Gamma^B{}_\mu\,\partial_{S^B} V^A \;+\; \big(\partial_{S^B}\Gamma^A{}_\mu\big)\,V^B. \tag{2.19}$$

Then $F^A{}_{\mu\nu}$ satisfies the identity

$$D_{[\lambda} F^A{}_{\mu\nu]} \;=\; 0, \tag{2.20}$$

which is the coordinate expression of the Cartan structure equation $d\Gamma + \Gamma\wedge\Gamma = F$ and $DF = 0$ in the non–linear (fiber–dependent) setting.[5]

Physical reading. The spectral curvature is a *structural* field strength: it quantifies path–dependence of the internal configuration under spacetime displacements. Where $F_{\mu\nu} = 0$, the medium admits flat (path–independent) spectral transport; where $F_{\mu\nu} \neq 0$, closed loops in M induce nontrivial holonomies in F, which will later feed directly into the effective interaction energy and stress.

Intrinsic spectral coordinates and a constitutive connection. Let $\Sigma^\mu = \Sigma^\mu(\sigma)$ be four intrinsic spectral coordinates built from (T, χ^μ) only, and define the fiber frame $\mathcal{E}^A{}_\mu(\sigma) = \partial\sigma^A/\partial\Sigma^\mu$. We choose the Ehresmann connection

$$\Gamma^A{}_\mu(x,\sigma) = \sqrt{\rho(\sigma)}\,\mathcal{E}^A{}_\mu(\sigma), \tag{2.21}$$

so that for any fiber scalar $f(\sigma)$ one has

$$D_\mu f = \Gamma^A{}_\mu\,\partial_A f = \sqrt{\rho}\,\frac{\partial f}{\partial\Sigma^\mu}, \qquad F^A{}_{\mu\nu} = \rho\,[\mathcal{E}_\mu, \mathcal{E}_\nu]^A + \tfrac{1}{2}(\mathcal{E}_\mu\rho)\,\mathcal{E}^A{}_\nu - \tfrac{1}{2}(\mathcal{E}_\nu\rho)\,\mathcal{E}^A{}_\mu. \tag{2.22}$$

Evaluated on the section $S^A(x)$ this yields $D_\mu S^A = \partial_\mu S^A + \sqrt{\rho(S)}\,\mathcal{E}^A{}_\mu(S)$. This construction is purely internal (metric-free) and removes any logical circularity.

2.4.3 Action of D_μ on Fiber Tensors and Compatibility

Vertical tensors pulled back along s. Let $V^A(S)$ be a fiber vector field (vertical). Its pullback along s is $V^A\big(S(x)\big)$. We define the spectral derivative along M by:

$$D_\mu V^A\big(S(x)\big) \;:=\; \partial_{S^B} V^A\big(S(x)\big)\,D_\mu S^B(x), \tag{2.23}$$

which coincides with the *vertical* Lie derivative of V along the vertical velocity $D_\mu S$. For a fiber covector $W_A(S)$, set

$$D_\mu W_A \;:=\; \partial_{S^A} W_B\,D_\mu S^B, \qquad D_\mu\big(V^A W_A\big) \;=\; (D_\mu V^A)\,W_A + V^A\,D_\mu W_A, \tag{2.24}$$

so that contractions are preserved.

[5] A short derivation proceeds by writing $\Gamma = \Gamma^A{}_\mu\,dx^\mu \otimes \partial_{S^A}$ and computing $F = d\Gamma + \frac{1}{2}[\Gamma,\Gamma]$ with the vertical Lie bracket (which is trivial for the coordinate frame ∂_{S^A}); the fiber dependence produces the terms with $\partial_{S^B}\Gamma$.

Internal metric compatibility (fiber level). Given a positive–definite $\mathcal{G}_{AB}(\mathcal{S})$ on F (see §2.3.3), we define

$$D_\mu \mathcal{G}_{AB} \;:=\; \partial_{S^C}\mathcal{G}_{AB}\, D_\mu S^C. \tag{2.25}$$

We *impose* fiber–level compatibility $D_\mu \mathcal{G}_{AB} = 0$ as a constitutive relation if desired; this constrains the admissible connections ${\Gamma^A}_\mu$ (they must be $\mathcal{G}$–preserving along s). In full generality we keep $D_\mu \mathcal{G}_{AB}$ allowed, as its presence can encode state–dependent stiffness.

Physical reading. Equation (2.23) says: the only way a fiber tensor changes when we move in M is by how $\mathcal{S}(x)$ moves vertically in F. This is the purely internal kinematics induced by the medium.

2.4.4 Parallel Transport, Holonomy, and Path Dependence

Spectral parallelism. Given a curve $\gamma : [0,1] \to M$, a *spectrally parallel* evolution of the state along γ satisfies

$$\frac{dS^A}{d\lambda} \;=\; {\Gamma^A}_\mu\big(\gamma(\lambda), S(\lambda)\big)\, \dot\gamma^\mu(\lambda), \qquad \lambda \in [0,1], \tag{2.26}$$

i.e. $D_{\dot\gamma} S^A = 0$. Solutions define a transport map $\mathcal{P}_\gamma : F_{\gamma(0)} \to F_{\gamma(1)}$. If $F_{\mu\nu} = 0$ in a simply connected domain, $\mathcal{P}_\gamma$ depends only on the endpoints (path–independence). Otherwise, the *holonomy* around closed loops encodes the spectral curvature:

$$S^A(1) \;=\; S^A(0) \;+\; \frac{1}{2}\, {F^A}_{\mu\nu}\, \Sigma^{\mu\nu} \;+\; \mathcal{O}(\text{area}^2), \tag{2.27}$$

for a small loop with area bivector $\Sigma^{\mu\nu}$.

Physical reading. Spectral parallelism corresponds to *adiabatic following* of the internal frame as we move in spacetime. Holonomy measures the net, path–dependent reconfiguration of the medium; it will be a natural source of interaction energy and can generate short–range repulsion/long–range tails once dynamics are turned on.

2.4.5 Coordinate Changes, Domains of Validity, and Regularity

Charts and transitions. On overlaps $U_i \cap U_j$, the transformation law (2.7) ensures that $D_\mu S^A$ and ${F^A}_{\mu\nu}$ patch together consistently. All scalar quantities built from $D_\mu S^A$ and fiber invariants are globally defined.

Admissible domains and singularities. Near defect cores or across phase interfaces (§2.3.5), a chosen chart may break down; the spectral derivative remains well–defined by switching to an overlapping trivialization. Regularity of solutions will require boundedness of ${\Gamma^A}_\mu$ and $\partial_S {\Gamma^A}_\mu$ on the image of s.

Physical reading. Breakdown of a chart does not signal a physical pathology; it simply reflects the nontrivial topology of the internal state space. The spectral derivative is geometric and survives chart changes.

2.4.6 Summary and Outlook to ∇_μ

Summary. Given an Ehresmann connection on E, the spectral derivative D_μ is defined by subtracting the horizontal–lift compensator from the raw spacetime derivative, (2.10)–(2.11). Its commutator yields the spectral curvature (2.17) that transforms covariantly on the fiber and satisfies a Bianchi–type identity (2.20). The operation extends to fiber tensors by vertical differentiation along $D_\mu S$, and supports a well–posed notion of spectral parallel transport.

Bridge to the full covariant derivative. In §2.5 we will extend D_μ to a full covariant derivative $\nabla_\mu = D_\mu + \omega_\mu$ that acts on all fields (including the complex matter field Φ) and on internal frames themselves. The emergent connection ω_μ will encode additional state–dependent structure (e.g. metric–compatibility and contortion–like terms), and its curvature will combine with F_{spec} to determine transport, dynamics, and the effective geometry.

2.5 The Full Covariant Derivative ∇_μ and the Emergent Connection ω_μ

We now upgrade the spectral derivative D_μ of §2.4 into a covariant derivative that acts on *all* objects carrying internal indices. The key idea is to supplement the horizontal operator D_μ with an *emergent* internal connection ω_μ that encodes the effective geometry felt by excitations. This step is purely kinematic and does not assume a background spacetime metric; the effective metric g_{eff} will be constructed from the structural state in §2.6.

Throughout, internal Lorentz indices are $a, b, \dots \in \{0, 1, 2, 3\}$, with the internal Minkowski form $\eta_{ab} = \mathrm{diag}(-, +, +, +)$ used only to raise/lower internal indices. Fiber indices on F are $A, B, \dots \in \{1, \dots, 7\}$ as in §2.3. We continue to work in a local trivialization of the associated bundle E when convenient, but all constructions are intrinsic.

2.5.1 Motivation and Definition: From D_μ to ∇_μ

Definition on internal tensors. Let $X^a(x)$ transform in the fundamental internal representation (a fiber scalar with one internal index). We define

$$\nabla_\mu X^a \;:=\; D_\mu X^a \;+\; \omega_\mu{}^a{}_b(x, S(x))\, X^b, \tag{2.28}$$

where D_μ acts as in (2.15)–(2.11) on the fiber dependence of X^a, and $\omega_\mu{}^a{}_b \in \mathfrak{so}(1,3)$ is antisymmetric in a, b: $\omega_{\mu ab} = -\omega_{\mu ba}$. For a covector Y_a we set

$$\nabla_\mu Y_a \;:=\; D_\mu Y_a \;-\; \omega_{\mu a}{}^b\, Y_b,$$

and extend by Leibniz to arbitrary internal tensors $T^{a\dots}{}_{b\dots}$.

On fiber scalars. If $f(F)$ is a scalar function on the fiber (no internal indices), then $\nabla_\mu(f \circ s) = D_\mu(f \circ s)$. Thus ∇_μ reduces to the spectral derivative on fiber scalars.

Gauge covariance under internal frame changes. Let $\Lambda(x, S) \in \mathrm{SO}(1,3)$ be a local change of internal frame. Internal vectors transform as $X'^a = \Lambda^a{}_b X^b$. We demand

$$\omega'_\mu \;=\; \Lambda\, \omega_\mu\, \Lambda^{-1} \;-\; (D_\mu \Lambda)\, \Lambda^{-1}, \qquad (D_\mu \Lambda) := \partial_\mu \Lambda - \Gamma^A{}_\mu\, \partial_{S^A} \Lambda, \tag{2.29}$$

so that $\nabla_\mu X$ transforms tensorially: $(\nabla_\mu X)' = \Lambda\, \nabla_\mu X$. Because D_μ already compensates the fiber reparametrizations $(x, S) \mapsto (x, S')$ of §2.4.1, the law (2.29) ensures full gauge covariance.

Physical reading. The operator D_μ compensates changes of *fiber coordinates* along spacetime; ω_μ compensates changes of *internal frames*. Together they produce a derivative that singles out the true, physical variation of internal tensors along M.

2.5.2 The Emergent Connection ω_μ: Palatini Split and Constraints Preservation

Palatini split. We split

$$\omega_\mu{}^{ab} \;=\; \omega_\mu{}^{ab}[e] \;+\; K_\mu{}^{ab}(\mathcal{S}), \tag{2.30}$$

where $\omega[e]$ is the Levi–Civita spin connection determined by an emergent tetrad $e^a{}_\mu(\mathcal{S})$ (to be constructed in §2.6), and $K_\mu{}^{ab}(\mathcal{S})$ is a *state–dependent, antisymmetric* distortion-like term that encodes structural effects.

A constitutive ansatz for K. On the constraint subbundle (normalizations and orthogonality of u^a and $\hat\chi^a$; cf. §2.2.5), a convenient, gauge-covariant and dimensionless ansatz is

$$K_\mu{}^{ab} \;=\; \alpha_1(\mathcal{S})\, \hat\chi^{[a}\, D_\mu \hat\chi^{b]} \;+\; \alpha_2(\mathcal{S})\, u^{[a}\, D_\mu u^{b]}, \qquad u^a u_a = -1, \;\; \hat\chi^a \hat\chi_a = +1, \;\; u\cdot\hat\chi = 0, \tag{2.31}$$

with $\alpha_{1,2}(\mathcal{S})$ to be *determined variationally* from the unified action using Lagrange multipliers for the constraints. This construction preserves the constraint surface under transport.

Metric compatibility (internal). We impose antisymmetry $\omega_{\mu ab} = -\omega_{\mu ba}$, which is equivalent to $\nabla^{\omega}_{\mu}\eta_{ab} = 0$. Later, the tetrad postulate will relate $\omega[e]$ to an effective spacetime connection $\tilde{\Gamma}^{\sigma}{}_{\mu\nu}$ compatible with $g_{\rm eff} = e^{a}{}_{\mu}e^{b}{}_{\nu}\eta_{ab}$.

Torsion choice. The default choice in this chapter is torsionless *on-shell* for the spacetime connection determined by $e^{a}{}_{\mu}$ (Levi–Civita for $g_{\rm eff}$). Any torsion induced by K is algebraic (EFT-suppressed) and does not introduce additional propagating modes at low energies; it can be set to zero by field equations or regarded as a finite renormalization of $\omega[e]$ in the homogeneous limit.

Physical reading. The split (2.30) isolates a purely geometric piece $\omega[e]$ from a structural, state-driven part $K(\mathcal{S})$. The latter is where spectral physics enters transport laws.

2.5.3 Curvatures and Mixed Terms: $[\nabla_\mu, \nabla_\nu]$ on General Objects

Operator identity. For any internal tensor field Ψ (possibly fiber-dependent), we have the exact decomposition

$$[\nabla_\mu, \nabla_\nu]\Psi \;=\; -[D_\mu, D_\nu]\Psi \;+\; \Big(D_\mu\omega_\nu - D_\nu\omega_\mu + [\omega_\mu, \omega_\nu]\Big)\cdot \Psi, \tag{2.32}$$

where D_μ acts on the fiber dependence of ω_ν according to (2.15), and $[\ ,\ ]$ is the matrix commutator in the internal representation.

Spectral vs. geometric curvature. Define the *spectral curvature* as in §2.4: $[D_\mu, D_\nu] = F^{A}_{{\rm spec}\mu\nu}\partial_{S^A}$ (notice the minus sign in (2.32)). Define the *emergent (internal) curvature*

$$R_{\mu\nu}(\omega) \;:=\; D_\mu\omega_\nu \;-\; D_\nu\omega_\mu \;+\; [\omega_\mu, \omega_\nu], \tag{2.33}$$

which transforms as $R'_{\mu\nu} = \Lambda R_{\mu\nu}\Lambda^{-1}$ under (2.29). Then

$$[\nabla_\mu, \nabla_\nu]\Psi \;=\; -\,F^{A}_{{\rm spec}\mu\nu}\,\partial_{S^A}\Psi \;+\; R_{\mu\nu}(\omega)\cdot\Psi. \tag{2.34}$$

Equation (2.34) exhibits a clean split between spectral non–integrability in F and internal curvature associated with ω.

Bianchi-type identities. The spectral curvature satisfies the horizontal Bianchi identity $D_{[\lambda}F_{\mu\nu]} = 0$ (§2.4.2). The emergent curvature obeys

$$\mathbb{D}_{[\lambda}R_{\mu\nu]} \;=\; 0, \qquad \mathbb{D}_\lambda R_{\mu\nu} \;:=\; D_\lambda R_{\mu\nu} + [\omega_\lambda, R_{\mu\nu}], \tag{2.35}$$

the usual identity with D in place of ∂, ensuring gauge consistency in the presence of fiber dependence.

Physical reading. Dynamics can couple to either or both curvatures. Regions with large $F_{\rm spec}$ encode strong path–dependence of internal configurations; large $R(\omega)$ encodes strong internal frame rotation or distortion along transport. Both leave imprints on effective stresses and interaction potentials.

2.5.4 Action of ∇_μ on the Matter Field Φ and Gauge Considerations

Matter as an associated line-bundle section. The complex matter field Φ is modeled as a section of a complex line bundle associated to the same principal data. In this chapter we *keep only a global U(1)* phase invariance, and therefore no independent Abelian gauge connection is introduced at this stage.

Covariant derivative on Φ. Since Φ carries no internal Lorentz index, we set

$$\nabla_\mu\Phi \;\equiv\; D_\mu\Phi, \tag{2.36}$$

with D_μ acting on its fiber dependence (via $\mathcal{S}$) as in (2.11). All couplings in §2.7 will respect the global $U(1)$ by involving only the current $J^{\mu}_{\Phi} := {\rm Im}(\Phi^*\nabla^\mu\Phi)$ and $|\Phi|^2$.

Commutator on Φ. From (2.34) we obtain

$$[\nabla_\mu, \nabla_\nu]\Phi \;=\; -\,F^{A}_{\mathrm{spec}\,\mu\nu}\,\partial_{S^A}\Phi, \tag{2.37}$$

i.e. Φ "feels" spectral curvature only through its dependence on the internal state $\mathcal{S}$.

Physical reading. At this stage the matter sector does not carry its own gauge connection. Any effective gauge structure felt by Φ is emergent (through ω in the kinematics or through state-dependent couplings); a local $U(1)$ or larger gauge group can be activated later if/when demanded by phenomenology.

2.5.5 Compatibility, Metricity, and Energy Positivity

Internal metricity. Antisymmetry of ω ensures $\nabla^{\omega}_{\mu}\eta_{ab} = 0$. When the emergent tetrad $e^{a}{}_{\mu}$ is introduced (in §2.6), we will impose the tetrad postulate with a spacetime connection $\tilde{\Gamma}^{\sigma}{}_{\mu\nu}$:

$$\nabla^{(\omega,\tilde{\Gamma})}_{\mu} e^{a}{}_{\nu} \;:=\; D_\mu e^{a}{}_{\nu} + \omega_\mu{}^{a}{}_{b} e^{b}{}_{\nu} - \tilde{\Gamma}^{\sigma}{}_{\mu\nu} e^{a}{}_{\sigma} \;=\; 0, \tag{2.38}$$

which implies $\nabla^{\tilde{\Gamma}}_{\mu} g_{\mathrm{eff}\,\alpha\beta} = 0$ for $g_{\mathrm{eff}\,\alpha\beta} = e^{a}{}_{\alpha} e^{b}{}_{\beta}\eta_{ab}$. In the low-energy limit we shall take $\tilde{\Gamma}$ to be the Levi–Civita connection of g_{eff}.

Positivity and hyperbolicity gates. In §2.7 the kinetic terms will involve $\mathcal{G}_{AB}(\mathcal{S}) \succ 0$ (fiber metric) and positive coefficients $\kappa_\Phi > 0$. Combined with metric-compatibility, these ensure a positive principal symbol and subluminal propagation with respect to g_{eff} for small fluctuations, thus excluding ghosts and gradient instabilities.

Physical reading. Equation (2.38) ties the emergent internal connection ω to the spacetime geometry felt by excitations, while the positivity gates ensure well-posed propagation. The state-dependent part $K(\mathcal{S})$ is the controlled locus for new physics, not a source of pathologies.

2.5.6 Worked Examples

Example 1: $[\nabla_\mu, \nabla_\nu]\Phi$. Using (2.37) we have

$$[\nabla_\mu, \nabla_\nu]\Phi \;=\; -\,F^{A}_{\mathrm{spec}\,\mu\nu}\,\partial_{S^A}\Phi.$$

If $\Phi = \Phi(\rho, \eta, \alpha)$ only (no explicit dependence on $\boldsymbol{\chi}, T$), then

$$[\nabla_\mu, \nabla_\nu]\Phi \;=\; -\sum_{A\in\{\rho,\eta,\alpha\}} F^{A}{}_{\mu\nu}\,\partial_A\Phi,$$

so spectral curvature along ρ, η, α directly feeds a "force" on Φ.

Example 2: $[\nabla_\mu, \nabla_\nu]X^a$. For an internal vector X^a,

$$[\nabla_\mu, \nabla_\nu]X^a \;=\; -\,F^{A}_{\mathrm{spec}\,\mu\nu}\,\partial_{S^A}X^a \;+\; R_{\mu\nu}{}^{a}{}_{b}(\omega)\,X^b, \tag{2.39}$$

with $R_{\mu\nu}{}^{a}{}_{b}$ the components of (2.33). The first term measures the failure of internal components to return to themselves after a spacetime loop due to fiber holonomy; the second is the usual frame-rotation curvature.

Example 3: Constraint preservation along transport. On the constraint subbundle, $u^a u_a = -1$ and $\hat{\chi}^a\hat{\chi}_a = +1$. Along any curve in M,

$$\nabla_\mu(u^a u_a) \;=\; 2\,u_a\,\nabla_\mu u^a \;=\; 0, \qquad \nabla_\mu(\hat{\chi}^a\hat{\chi}_a) \;=\; 2\,\hat{\chi}_a\,\nabla_\mu\hat{\chi}^a \;=\; 0,$$

provided the Lagrange multipliers enforce the constraints and K is chosen as in (2.31). Thus normalization and orthogonality propagate under ∇.

2.5.7 Regularity, Domains, and Gauge Patching

Transformation laws across patches. On overlaps of trivializations, ${\Gamma^A}_\mu$ transforms as in (2.7) and ω_μ as in (2.29), with $\Lambda(x, S)$ the internal frame transition. With these laws, ∇_μ is globally well-defined.

Regularity assumptions. We assume $\omega_\mu(x, S)$ and its fiber derivatives $\partial_S \omega_\mu$ are bounded on the image of the section $s : M \to F$, and are at least C^1 in x and S. These conditions ensure that the commutator (2.32) and the curvatures are well-defined distributions and that evolution problems posed later are well-posed.

Physical reading. No physical pathology is tied to a change of trivialization. Singularities (if any) arise from genuine defects or phase boundaries in the internal state, not from gauge choices.

2.5.8 Summary and Checklist

Summary. We have defined the full covariant derivative $\nabla_\mu = D_\mu + \omega_\mu$ acting on internal tensors, with $\omega_\mu \in \mathfrak{so}(1, 3)$ transforming as in (2.29). Its commutator splits cleanly into a spectral part and an emergent curvature $R(\omega)$, cf. (2.34). A Palatini split $\omega = \omega[e] + K(\mathcal{S})$ separates universal geometric transport from state-dependent distortions that preserve constraints. Metric-compatibility and positivity gates prepare the ground for stable, causal dynamics.

Checklist (consistency gates).

- **Gauge covariance:** $\omega'_\mu = \Lambda \omega_\mu \Lambda^{-1} - (D_\mu \Lambda)\Lambda^{-1}$.
- **Antisymmetry:** $\omega_{\mu ab} = -\omega_{\mu ba} \Rightarrow$ internal metricity.
- **Curvatures:** F_{spec} from $[D, D]$; $R(\omega)$ from (2.33); Bianchi identities (2.20) and (2.35).
- **Constraint propagation:** normalization/orthogonality preserved under ∇ on-shell.
- **No ghosts:** positive fiber metric $\mathcal{G}_{AB}$ and $\kappa_\Phi > 0$; subluminal principal symbol relative to g_{eff}.

In §2.6 we build g_{eff} and the tetrad ${e^a}_\mu(\mathcal{S})$ explicitly, making the relation between ω and spacetime geometry concrete. In §2.7 we assemble the unified action and derive the associated stress–energy and linearized spectra.

2.6 The Emergent Tetrad ${e^a}_\mu(\mathcal{S})$ and Effective Metric g_{eff}

We now construct the *effective spacetime geometry* felt by excitations directly from the structural state $\mathcal{S}(x)$. The construction is purely intrinsic: starting from the spectral derivative D_μ and the internal metric $\mathcal{G}_{AB}(\mathcal{S})$ on the fiber F, we build a coframe (tetrad) ${e^a}_\mu(\mathcal{S})$ and the Lorentzian metric

$$g_{\text{eff}\,\mu\nu}(\mathcal{S}) \;=\; {e^a}_\mu(\mathcal{S})\, {e^b}_\nu(\mathcal{S})\, \eta_{ab}, \qquad \eta_{ab} = \text{diag}(-, +, +, +). \tag{2.40}$$

Causality (hyperbolicity) and positivity conditions will be formulated as algebraic constraints on the constitutive scalars entering this construction.

The guiding principle is to extract from $\mathcal{S}$ two preferred covectors: one associated with the *internal time* T (shift symmetry $\Rightarrow$ only derivatives enter), and one associated with the *director* $\boldsymbol{\chi}$ and its anisotropy intensity α. These will seed a coframe via a Gram–Schmidt process performed with respect to a canonical, state–dependent, symmetric form on covectors built from the deformation gradient of $\mathcal{S}$.

2.6.1 Pre-metric data from the deformation gradient

Deformation gradient and material form. Define the *deformation gradient* of the internal state as the fiber–tangent map

$$\mathcal{F}_\mu{}^A(x) \;:=\; D_\mu S^A(x), \tag{2.41}$$

and the symmetric, positive semidefinite bilinear form on covectors

$$C_{\mu\nu}(\mathcal{S}) \;:=\; \mathcal{G}_{AB}(\mathcal{S})\,\mathcal{F}_\mu{}^A\,\mathcal{F}_\nu{}^B. \tag{2.42}$$

Intuitively, $C_{\mu\nu}$ measures the "material cost" (in internal metric $\mathcal{G}$) of changing $\mathcal{S}$ along spacetime directions. It provides a canonical inner product on covectors that we will use to orthonormalize the premetric data produced below.[6]

Preferred covectors. We extract two covectors from the state:

$$\tau_\mu \;:=\; N_T(\mathcal{S})\,\partial_\mu T, \qquad \sigma_\mu \;:=\; N_\chi(\mathcal{S})\,\hat{\chi}_i(\mathcal{S})\,D_\mu\chi^i, \tag{2.43}$$

where $\hat{\chi}_i := \chi_i/\|\boldsymbol{\chi}\|_{\mathcal{G}}$ in the ordered phase ($\alpha > 0$), and $N_T, N_\chi > 0$ are dimensionless constitutive scalars ensuring appropriate normalizations below. The definition of σ_μ is invariant under internal rotations of $\boldsymbol{\chi}$ and under the nematic head–tail identification $\boldsymbol{\chi} \sim -\boldsymbol{\chi}$.

Physical reading. τ captures the ordering provided by the internal time field, and σ captures the spatial director encoded in $\boldsymbol{\chi}$ as it varies along M. Both are built from shift/gauge–invariant ingredients.

2.6.2 Orthonormal coframe via Gram–Schmidt (with respect to C)

Normalization in the C–inner product. Define the C–norm $\|v\|_C := \sqrt{C^{\mu\nu}v_\mu v_\nu}$ with $C^{\mu\nu}$ the inverse of $C_{\mu\nu}$ on its image.[7] Set

$$\theta^0{}_\mu \;:=\; \frac{\tau_\mu}{\|\tau\|_C}, \tag{2.44}$$

and orthogonalize σ against θ^0 in the C–inner product:

$$\tilde{\sigma}_\mu \;:=\; \sigma_\mu \;-\; \frac{C^{\alpha\beta}\,\sigma_\alpha\theta^0_\beta}{C^{\alpha\beta}\,\theta^0_\alpha\theta^0_\beta}\,\theta^0_\mu, \qquad \theta^3{}_\mu \;:=\; \frac{\tilde{\sigma}_\mu}{\|\tilde{\sigma}\|_C}. \tag{2.45}$$

Complete $\{\theta^0, \theta^3\}$ to a C–orthonormal coframe $\{\theta^0, \theta^1, \theta^2, \theta^3\}$ by choosing $\theta^{1,2}$ spanning the C–orthogonal complement, with $C^{\mu\nu}\theta^a{}_\mu\theta^b{}_\nu = \delta^{ab}$.

Physical reading. θ^0 is the normalized "clock" 1–form aligned with ∂T; θ^3 is the normalized spatial 1–form aligned with the internal director. The pair (θ^1, θ^2) spans the residual isotropic plane orthogonal to both.

2.6.3 State-built tetrad and effective metric

Anisotropic scalings. Introduce three positive, dimensionless constitutive scalars

$$a_0(\mathcal{S}) > 0, \qquad a_\perp(\mathcal{S}) > 0, \qquad a_\parallel(\mathcal{S}) > 0, \tag{2.46}$$

encoding the relative scaling of the temporal coframe and of the spatial directions orthogonal/parallel to the director. Define the emergent coframe

$$e^0{}_\mu \;:=\; a_0\,\theta^0{}_\mu, \qquad e^1{}_\mu \;:=\; a_\perp\,\theta^1{}_\mu, \qquad e^2{}_\mu \;:=\; a_\perp\,\theta^2{}_\mu, \qquad e^3{}_\mu \;:=\; a_\parallel\,\theta^3{}_\mu. \tag{2.47}$$

The effective metric then reads

$$g_{\text{eff}\,\mu\nu} \;=\; -\,e^0{}_\mu e^0{}_\nu \;+\; e^1{}_\mu e^1{}_\nu \;+\; e^2{}_\mu e^2{}_\nu \;+\; e^3{}_\mu e^3{}_\nu, \tag{2.48}$$

with signature $(-+++)$ by construction.

[6] When gradients are small (e.g. nearly homogeneous backgrounds), $C_{\mu\nu}$ can be regularized by adding a small, state–dependent isotropic seed $\epsilon(\mathcal{S})\,\delta_{\mu\nu}$ in local coordinates; this does not affect the low–gradient limit and keeps the Gram–Schmidt well–defined.

[7] If $C_{\mu\nu}$ is degenerate, restrict to the rank–4 domain where τ and σ live and use the Moore–Penrose pseudoinverse; the constitutive scalars N_T, N_χ are chosen to keep the resulting coframe regular in admissible domains.

Dual frame and kinematic vectors. Let $e_a{}^\mu$ be the dual frame: $e^a{}_\mu e_b{}^\mu = \delta^a_b$, $e^a{}_\mu e_a{}^\nu = \delta^\nu_\mu$. Define

$$u^\mu \;:=\; e_0{}^\mu, \qquad \hat{\chi}^\mu \;:=\; e_3{}^\mu, \tag{2.49}$$

so that $u^\mu u_\mu = -1$, $\hat{\chi}^\mu \hat{\chi}_\mu = +1$, and $u^\mu \hat{\chi}_\mu = 0$. The *spatial projector* orthogonal to u is

$$h_{\mu\nu} \;:=\; g_{\mathrm{eff}\,\mu\nu} + u_\mu u_\nu \;=\; e^1{}_\mu e^1{}_\nu + e^2{}_\mu e^2{}_\nu + e^3{}_\mu e^3{}_\nu. \tag{2.50}$$

Within the spatial slice, define the director projectors:

$$\Pi_{\parallel\,\mu\nu} \;:=\; \hat{\chi}_\mu \hat{\chi}_\nu, \qquad \Pi_{\perp\,\mu\nu} \;:=\; h_{\mu\nu} - \Pi_{\parallel\,\mu\nu}. \tag{2.51}$$

Physical reading. Equation (2.48) realizes a clean separation between temporal rate (a_0), spatial propagation orthogonal to the director ($a_\perp$), and propagation along the director ($a_\parallel$). The three scalars will later control characteristic speeds of small disturbances.

2.6.4 Characteristic cones and hyperbolicity gates

Principal symbol template. Kinetic terms for a field ψ with coupling $\kappa(\mathcal{S}) > 0$ will have a principal part of the form $\kappa\, g_{\mathrm{eff}}^{\mu\nu} \partial_\mu\psi\, \partial_\nu\psi$ (possibly with anisotropic modifications projected by $\Pi_\parallel, \Pi_\perp$). The *characteristic covectors* k_μ satisfy

$$\mathcal{P}(x,k) \;=\; \kappa(\mathcal{S})\, g^{\mu\nu}_{\mathrm{eff}}(\mathcal{S})\, k_\mu k_\nu \;=\; 0 \qquad \Rightarrow \qquad -(u\cdot k)^2 + \frac{1}{a_\perp^2}\, k_\perp^2 + \frac{1}{a_\parallel^2}\, k_\parallel^2 \;=\; 0, \tag{2.52}$$

with $k_\parallel := \hat{\chi}^\mu k_\mu$, $k_\perp^2 := k_\mu k_\nu \Pi_\perp^{\mu\nu}$.

Causality (subluminal propagation). In units $c = 1$, hyperbolicity and subluminal group speeds are ensured if

$$a_0 > 0, \qquad a_\perp \geq 1, \qquad a_\parallel \geq 1, \tag{2.53}$$

so that characteristic cones lie within or on the Minkowski lightcone defined by any local inertial frame of g_{eff}. More general anisotropic principal symbols can be written as $-(u\cdot k)^2 + c_\perp^2 k_\perp^2 + c_\parallel^2 k_\parallel^2 = 0$ with $c_\perp := 1/a_\perp \leq 1$, $c_\parallel := 1/a_\parallel \leq 1$.

Energy positivity. For quadratic kinetic terms of the structural sector weighted by $\mathcal{G}_{AB} \succ 0$ and matter sector with $\kappa_\Phi > 0$, the Hamiltonian density measured by u^μ is positive provided (2.53) holds and the potential energies are bounded below.

2.6.5 Spacetime connection and the tetrad postulate

Levi–Civita connection of g_{eff}. Let $\widetilde{\Gamma}^\sigma{}_{\mu\nu}$ be the Christoffel symbols of the Levi–Civita connection of g_{eff}. The tetrad postulate

$$\nabla_\mu^{(\omega,\widetilde{\Gamma})} e^a{}_\nu \;=\; D_\mu e^a{}_\nu \;+\; \omega_\mu{}^a{}_b\, e^b{}_\nu \;-\; \widetilde{\Gamma}^\sigma{}_{\mu\nu}\, e^a{}_\sigma \;=\; 0 \tag{2.54}$$

fixes $\omega[e]$ uniquely in terms of e (torsionless case) and constrains the state–dependent piece $K(\mathcal{S})$ introduced in §2.5.2.

Spin connection from the coframe. In terms of the coframe $e^a = e^a{}_\mu dx^\mu$ and the Levi–Civita exterior derivative d, $\omega[e]$ is the unique $\mathfrak{so}(1,3)$–valued 1–form solving the first Cartan structure equation (with zero torsion):

$$de^a + \omega^a{}_b[e] \wedge e^b \;=\; 0. \tag{2.55}$$

In components,

$$\omega_{\mu ab}[e] \;=\; \tfrac{1}{2}\big(C_{ab\mu} - C_{b\mu a} + C_{\mu ab}\big), \quad C_{ab\mu} := e_a{}^\nu\big(\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu}\big), \tag{2.56}$$

and $\widetilde{\Gamma}$ follows from (2.54).

Physical reading. Equations (2.54)–(2.56) make concrete how the state–built geometry encodes transport. The geometric piece $\omega[e]$ is completely fixed by e; any new physics sits in $K(\mathcal{S})$.

2.6.6 Constitutive choices and minimal ansatz (practical box)

Minimal practical ansatz. For implementations where a simpler parametric form is preferred, one may use the *state-built* deformation of a background frame $\bar{e}^a{}_\mu$:

$$e^a{}_\mu \;=\; \bar{e}^a{}_\mu \;+\; A(\mathcal{S})\, u^a u_\mu \;+\; B(\mathcal{S})\, \hat{\chi}^a \hat{\chi}_\mu, \tag{2.57}$$

with $u^a, u_\mu, \hat{\chi}^a, \hat{\chi}_\mu$ extracted from θ^0, θ^3 as in (2.44)–(2.45) and (2.47), and A, B dimensionless. This ansatz preserves the orthonormality relations on–shell and reproduces (2.48) to leading order around $\bar{e}$.

Choice of normalizations. A convenient normalization is $N_T = 1/\sqrt{C^{\mu\nu}\partial_\mu T \partial_\nu T}$ and $N_\chi = 1/\sqrt{C^{\mu\nu}(\hat{\chi}_i D_\mu \chi^i)(\hat{\chi}_j D_\nu \chi^j)}$, so that θ^0, θ^3 are unit C–covectors irrespective of overall gradients. The scalings $a_0, a_\perp, a_\parallel$ are then the only free positive functions controlling causal cones as in (2.52)–(2.53).

2.6.7 Examples and limiting regimes

Isotropic structural phase ($\alpha = 0$). When $\alpha = 0$ (and thus $\boldsymbol{\chi}$ is physically irrelevant), set $a_\parallel = a_\perp$ and drop θ^3–preferred scaling. The metric reduces to

$$g_{\mathrm{eff}\,\mu\nu} \;=\; -a_0^2\, \theta^0{}_\mu \theta^0{}_\nu \;+\; a_\perp^2 \big(\theta^1{}_\mu \theta^1{}_\nu + \theta^2{}_\mu \theta^2{}_\nu + \theta^3{}_\mu \theta^3{}_\nu\big),$$

which is isotropic in the spatial slice orthogonal to u.

Homogeneous backgrounds. If $D_\mu S^A = 0$ exactly, then $C_{\mu\nu} = 0$ and the Gram–Schmidt recipe becomes degenerate. In practice, physically relevant configurations always have small but finite gradients; alternatively one may select a reference coframe $\bar{e}^a{}_\mu$ and use (2.57) with small A, B to parametrize departures, keeping $\theta^0 \propto \partial T$ as the singled–out clock 1–form.

Anisotropic wave speeds. Linearized excitations propagating orthogonally/along the director experience effective speeds $c_\perp = 1/a_\perp,\ c_\parallel = 1/a_\parallel$ relative to u. Bounds $c_{\perp,\parallel} \le 1$ ensure hyperbolicity and causal cones contained in those of g_{eff}.

2.6.8 Summary of §2.6

Summary. Starting from the structural state $\mathcal{S}$, we built a canonical pair of covectors $\tau_\mu \propto \partial_\mu T$ and $\sigma_\mu \propto \hat{\chi}_i D_\mu \chi^i$, orthonormalized them with respect to the material form $C_{\mu\nu} = \mathcal{G}_{AB}\, D_\mu S^A D_\nu S^B$, and assembled an emergent coframe via anisotropic scalings $(a_0, a_\perp, a_\parallel)$. The effective metric $g_{\mathrm{eff}} = -e^0 \otimes e^0 + \sum_{i=1}^3 e^i \otimes e^i$ has signature $(-+++)$, defines a distinguished flow u^μ and director $\hat{\chi}^\mu$, and yields clear projectors and causal gates. The Levi–Civita connection $\widetilde{\Gamma}$ and the spin connection $\omega[e]$ follow from the tetrad postulate, while state–dependent distortions $K(\mathcal{S})$ encode novel spectral physics.

Outlook. With g_{eff} in hand, §2.7 will assemble the unified Lagrangian for the structural and matter sectors, derive the effective stress–energy, and set the stage for linearized spectra and stability.

2.7 Unified Action, Constraints, and Conserved Quantities

We assemble in this section a fully covariant Lagrangian density for the quantum-aether structural state $\mathcal{S}$, the emergent geometry $(e^a{}_\mu, \omega_\mu{}^{ab})$, and the complex matter field Φ. The action respects the bundle/gauge structure developed in §§2.3–2.6, and is organized so that dynamical (propagating) pieces are cleanly separated from topological (boundary) contributions. We also derive the field equations, Noether currents, and stress–energy, and formulate hyperbolicity and positivity gates needed for a healthy low-energy effective theory.

2.7.1 Measure, Variational Set, and Dimensions

Measure and sign conventions. All densities are integrated with the effective Riemannian measure

$$S \;=\; \int_M d^4x\,\sqrt{-g_{\text{eff}}}\,\mathcal{L} \;+\; S_{\partial M}, \qquad g_{\text{eff}\,\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}, \quad \eta_{ab} = \mathrm{diag}(-,+,+,+). \tag{2.58}$$

Spacetime indices $\mu,\nu,\ldots$ are raised/lowered with $g^{\mu\nu}_{\text{eff}}$ and $g_{\text{eff}\,\mu\nu}$.

Variational variables (first-order/Palatini-like). Unless stated otherwise, the fundamental fields varied in the action are

$$\big\{\, e^a{}_\mu,\ \omega_\mu{}^{ab} = -\omega_\mu{}^{ba},\ S^A = (\rho,\eta,T,\chi^1,\chi^2,\chi^3,\alpha),\ \Phi \,\big\}.$$

The Ehresmann data $\Gamma^A{}_\mu(x,S)$ that define D_μ and $F^A{}_{\mu\nu}$ (cf. §2.4) are taken as constitutive (fiber-covariant) structures; their explicit state dependence, when present, is accounted for in the Euler–Lagrange equations through $\partial_S\Gamma$ terms.

Natural units and engineering dimensions. We adopt $c = \hbar = 1$. A minimal reference table (mass dimensions) is:

$$[e^a{}_\mu] = 0, \quad [g^{\mu\nu}_{\text{eff}}] = 0, \quad [\omega] = 1, \quad [R] = 2, \quad [S^A] = 0, \quad [\Phi] = 1, \quad [D_\mu] = 1, \quad [F_{\mu\nu}] = 2.$$

Couplings are chosen to render each term of $\mathcal{L}$ of dimension $+4$.

2.7.2 Structural Sector $\mathcal{L}_{\text{struct}}$

Kinetic term (first order in derivatives). Let $Z_{AB}(\mathcal{S}) \succ 0$ be a positive-definite fiber metric (possibly different from $\mathcal{G}_{AB}$; cf. §2.3.3). Define

$$\mathcal{L}^{(S)}_{\text{kin}} \;=\; \frac{1}{2}\, Z_{AB}(\mathcal{S})\, g^{\mu\nu}_{\text{eff}}\, D_\mu S^A\, D_\nu S^B. \tag{2.59}$$

Positivity of Z ensures a bounded-from-below kinetic energy for vertical motions of $\mathcal{S}$.

Spectral curvature term (Maxwell-like). Let $Y_{AB}(\mathcal{S}) \succ 0$ be a positive-definite bilinear on the fiber. Set

$$\mathcal{L}_F \;=\; -\frac{1}{4}\, Y_{AB}(\mathcal{S})\, g^{\mu\alpha}_{\text{eff}}\, g^{\nu\beta}_{\text{eff}}\, F^A{}_{\mu\nu} F^B{}_{\alpha\beta}, \qquad F^A{}_{\mu\nu} \text{ as in (2.17)}. \tag{2.60}$$

This term penalizes non-integrability of the horizontal distribution (cf. §2.4.2).

Internal potential. The structural potential $U(\mathcal{S})$ is a G-invariant function of the invariants listed in §2.3.3, bounded below, with phase structure controlled by α (isotropic vs. ordered):

$$\mathcal{L}_U \;=\; -\,U(\mathcal{S}), \qquad U \;=\; U_\rho(\rho) + U_\eta(\eta) + U_\alpha(\alpha) + U_{\chi\alpha}(\boldsymbol{\chi}, \alpha; \mathcal{G})\,. \tag{2.61}$$

Structural sector summary.

$$\boxed{\mathcal{L}_{\text{struct}} \;=\; \frac{1}{2} Z_{AB}\, D_\mu S^A D^\mu S^B \;-\; U(\mathcal{S}) \;-\; \frac{1}{4} Y_{AB}\, F^A_{\mu\nu} F^{B\,\mu\nu}}\,. \tag{2.62}$$

Physical remark. Z and Y are the material (state-dependent) stiffnesses controlling vertical deformations and spectral nonintegrability; U selects the internal phase.

2.7.3 Matter Sector $\mathcal{L}_\Phi$ (global $U(1)$)

Anisotropic kinetic form. With $\nabla_\mu\Phi \equiv D_\mu\Phi$ (no local $U(1)$ yet; cf. §2.5.4), define

$$\mathcal{L}^{\text{kin}}_\Phi \;=\; \kappa_\Phi(\mathcal{S})\left(g^{\mu\nu}_{\text{eff}} + \delta c(\mathcal{S})\,\hat\chi^\mu\hat\chi^\nu\right)(\nabla_\mu\Phi)^*(\nabla_\nu\Phi), \quad \kappa_\Phi > 0, \quad \delta c \geq -1, \tag{2.63}$$

where $\hat\chi^\mu$ is the normalized director from §2.6.3.

Environmental mass and self-interaction.

$$V_\Phi(\mathcal{S}, |\Phi|^2) \;=\; \frac{1}{2}\, m_{\rm eff}^2(\mathcal{S})\, |\Phi|^2 \;+\; \frac{\lambda_{\rm eff}(\mathcal{S})}{4}\, |\Phi|^4, \qquad m_{\rm eff}^2 = m_0^2 + c_\eta \eta + c_\rho \rho \;\; \text{(example)}. \tag{2.64}$$

Matter sector summary.

$$\boxed{\mathcal{L}_\Phi \;=\; \kappa_\Phi\, G^{\mu\nu}_{\rm (ani)}\, (\nabla_\mu \Phi)^*(\nabla_\nu \Phi) \;-\; V_\Phi, \quad G^{\mu\nu}_{\rm (ani)} := g^{\mu\nu}_{\rm eff} + \delta c\, \hat{\chi}^\mu \hat{\chi}^\nu}\,. \tag{2.65}$$

Physical remark. The scalar Φ propagates with state-controlled anisotropic speeds $c_\perp = 1/a_\perp$, $c_\parallel = 1/a_\parallel$ (cf. §2.6.4), and acquires an effective mass from (η, ρ).

2.7.4 Symmetry-Allowed Mixing Terms

Derivative coupling to the $U(1)$ current. Let

$$J^\mu_\Phi \;:=\; \mathrm{Im}\big(\Phi^* \nabla^\mu \Phi\big), \tag{2.66}$$

then a symmetry-allowed derivative mixing is

$$\mathcal{L}^{(1)}_{\rm mix} \;=\; \xi_A(\mathcal{S})\, J^\mu_\Phi\, D_\mu S^A. \tag{2.67}$$

Anisotropic projectors on matter kinetics. Further (parity-even) mixings respecting symmetries:

$$\mathcal{L}^{(2)}_{\rm mix} \;=\; \zeta_\parallel(\mathcal{S})\, \Pi^{\mu\nu}_\parallel (\nabla_\mu \Phi)^*(\nabla_\nu \Phi) \;+\; \zeta_\perp(\mathcal{S})\, \Pi^{\mu\nu}_\perp (\nabla_\mu \Phi)^*(\nabla_\nu \Phi), \tag{2.68}$$

with $\Pi^{\mu\nu}_\parallel = \hat{\chi}^\mu \hat{\chi}^\nu$, $\Pi^{\mu\nu}_\perp = g^{\mu\nu}_{\rm eff} + u^\mu u^\nu - \Pi^{\mu\nu}_\parallel$.

Mixing summary.

$$\boxed{\mathcal{L}_{\rm mix} \;=\; \mathcal{L}^{(1)}_{\rm mix} + \mathcal{L}^{(2)}_{\rm mix}}\,. \tag{2.69}$$

Physical remark. These couplings allow state-dependent refraction and current–drag effects without breaking global $U(1)$ or the time-shift symmetry of T.

2.7.5 Gravitational Sector (Dynamical, Parity-Even)

Einstein–Hilbert–Palatini core. With $M^2_{\rm eff}(\rho) > 0$,

$$\mathcal{L}_{\rm grav} \;=\; \frac{M^2_{\rm eff}(\rho)}{2}\, R(\omega, e) \;+\; \mathcal{O}\!\left(\frac{R^2}{\Lambda^2_{\rm eff}}\right)_{\rm even}, \tag{2.70}$$

where $R(\omega, e)$ is the Ricci scalar built from the torsionless $\omega[e]$ on-shell (cf. §2.6.5). Higher-curvature parity-even corrections are EFT-suppressed and do not introduce ghosts at low energies.

Physical remark. Taking $M^2_{\rm eff} \propto \rho$ will later reproduce Newton's law with an effective $G_{\rm eff} \propto 1/\rho$.

2.7.6 Constraints (Normalization and Orthogonality)

Lagrange multipliers. To enforce normalization and orthogonality (cf. §2.6.3) we add

$$\mathcal{L}_{\rm constr} \;=\; \lambda_1\, (u^\mu u_\mu + 1) \;+\; \lambda_2\, (\hat{\chi}^\mu \hat{\chi}_\mu - 1) \;+\; \lambda_3\, (u^\mu \hat{\chi}_\mu). \tag{2.71}$$

Inequality domains $\rho > 0$, $\eta \geq 0$ are maintained either as domain restrictions or via soft barriers in U.

2.7.7 Master Lagrangian (Dynamical Pieces)

$$\boxed{\mathcal{L}^{(\rm dyn)}_{\rm tot} \;=\; \mathcal{L}_{\rm grav} \;+\; \mathcal{L}_{\rm struct} \;+\; \mathcal{L}_\Phi \;+\; \mathcal{L}_{\rm mix} \;+\; \mathcal{L}_{\rm constr}}\,. \tag{2.72}$$

2.7.8 Variations and Field Equations

We denote by $\widetilde{\nabla}_\mu$ the Levi–Civita covariant derivative of g_{eff} and by $\mathbb{D}_\mu$ the horizontal derivative acting on fiber-dependent internal objects (as in §2.5.3).

Variation with respect to Φ. From (2.63)–(2.64) and (2.67)–(2.68) one finds

$$\widetilde{\nabla}_\mu\Big(\kappa_\Phi\, G^{\mu\nu}_{(\text{ani})}\,\nabla_\nu\Phi\Big) + \frac{\partial}{\partial\Phi^*}\Big(\zeta_\parallel\,\Pi^{\mu\nu}_\parallel+\zeta_\perp\,\Pi^{\mu\nu}_\perp\Big)(\nabla_\mu\Phi)^*(\nabla_\nu\Phi) - \frac{\partial V_\Phi}{\partial\Phi^*} + i\,\xi_A\,D_\mu S^A\,\nabla^\mu\Phi \;=\; 0, \tag{2.73}$$

plus terms with derivatives of the state-dependent coefficients (all functions of $\mathcal{S}$).

Variation with respect to S^A. Grouping explicit and implicit dependences,

$$\begin{aligned} 0 \;=\; & \widetilde{\nabla}_\mu\big(Z_{AB}\,D^\mu S^B\big) \;-\; \partial_A U \;-\; \frac{1}{4}\,\partial_A Y_{BC}\,F^B_{\mu\nu}F^{C\,\mu\nu} \;+\; \mathcal{F}_A^{\,(S)} \\ & +\; \partial_A\kappa_\Phi\,G^{\mu\nu}_{(\text{ani})}(\nabla_\mu\Phi)^*(\nabla_\nu\Phi) \;+\; \kappa_\Phi\,\partial_A G^{\mu\nu}_{(\text{ani})}(\nabla_\mu\Phi)^*(\nabla_\nu\Phi) \;-\; \partial_A V_\Phi \;+\; \partial_A\big(\xi_B J^\mu_\Phi D_\mu S^B\big) \\ & +\; \partial_A\zeta_\parallel\,\Pi^{\mu\nu}_\parallel(\nabla_\mu\Phi)^*(\nabla_\nu\Phi) \;+\; \partial_A\zeta_\perp\,\Pi^{\mu\nu}_\perp(\nabla_\mu\Phi)^*(\nabla_\nu\Phi), \end{aligned} \tag{2.74}$$

where $\mathcal{F}_A^{\,(S)}$ collects the implicit dependence of $F^A{}_{\mu\nu}$ and D_μ on S via $\Gamma^A{}_\mu(x,S)$:

$$\mathcal{F}_A^{\,(S)} \;:=\; \frac{1}{2}\,Y_{BC}\,F^{B\,\mu\nu}\,\partial_{S^A}F^C{}_{\mu\nu} \;+\; Z_{BC}\,D_\mu S^B\,\partial_{S^A}\Gamma^C{}_\nu\,g^{\mu\nu}_{\text{eff}} \quad \Big(\text{in any local chart, fiber-covariantly}\Big). \tag{2.75}$$

The precise chart expression of $\partial_S\Gamma$ follows from (2.7)–(2.17).

Variation with respect to $e^a{}_\mu$ (or g_{eff}). Define the effective stress–energy by

$$T^{\mu\nu}_{\text{eff}} \;:=\; \frac{2}{\sqrt{-g_{\text{eff}}}}\,\frac{\delta S_{\text{matter+struct}}}{\delta g_{\text{eff}\,\mu\nu}} \;=\; T^{\mu\nu}_{(S)}+T^{\mu\nu}_{(F)}+T^{\mu\nu}_{(\Phi)}+T^{\mu\nu}_{(\text{mix})}. \tag{2.76}$$

The gravitational field equations take the Einstein-like form

$$M^2_{\text{eff}}(\rho)\,G^{\mu\nu}(g_{\text{eff}}) \;=\; T^{\mu\nu}_{\text{eff}} \;+\; \big(\text{derivatives of } M^2_{\text{eff}}(\rho)\big), \tag{2.77}$$

where $G^{\mu\nu}$ is the Einstein tensor of g_{eff}. The extra terms arise if M^2_{eff} varies in space.

Variation with respect to ω. Palatini variation yields the metric-compatibility/tetrad-postulate constraint (cf. (2.54)), fixing $\omega[e]$ as the torsionless spin connection of g_{eff} and constraining the allowed state-dependent distortion $K(\mathcal{S})$ (cf. §2.5.2).

Explicit stress–energy components.

$$T^{\mu\nu}_{(S)} \;=\; Z_{AB}\,D^{(\mu}S^A D^{\nu)}S^B \;-\; \frac{1}{2}\,g^{\mu\nu}_{\text{eff}}\,Z_{AB}\,D_\alpha S^A D^\alpha S^B \;+\; g^{\mu\nu}_{\text{eff}}\,U(\mathcal{S}), \tag{2.78}$$

$$T^{\mu\nu}_{(F)} \;=\; Y_{AB}\left(F^{A\,\mu\alpha}F^{B\,\nu}{}_\alpha - \frac{1}{4}\,g^{\mu\nu}_{\text{eff}}\,F^A_{\alpha\beta}F^{B\,\alpha\beta}\right), \tag{2.79}$$

$$T^{\mu\nu}_{(\Phi)} \;=\; \kappa_\Phi\left(G^{\mu\alpha}_{(\text{ani})}G^{\nu\beta}_{(\text{ani})} - \tfrac{1}{2}g^{\mu\nu}_{\text{eff}}G^{\alpha\beta}_{(\text{ani})}\right)(\nabla_\alpha\Phi)^*(\nabla_\beta\Phi) \;-\; g^{\mu\nu}_{\text{eff}}\,V_\Phi, \tag{2.80}$$

$$\begin{aligned} T^{\mu\nu}_{(\text{mix})} \;=\; & \xi_A\left(J^{(\mu}_\Phi D^{\nu)}S^A - \tfrac{1}{2}g^{\mu\nu}_{\text{eff}}J^\alpha_\Phi D_\alpha S^A\right) \;+\; \zeta_\parallel\left(\Pi^{\alpha(\mu}_\parallel\Pi^{\nu)\beta}_\parallel - \tfrac{1}{2}g^{\mu\nu}_{\text{eff}}\Pi^{\alpha\beta}_\parallel\right)(\nabla_\alpha\Phi)^*(\nabla_\beta\Phi) \\ & +\; \zeta_\perp\left(\Pi^{\alpha(\mu}_\perp\Pi^{\nu)\beta}_\perp - \tfrac{1}{2}g^{\mu\nu}_{\text{eff}}\Pi^{\alpha\beta}_\perp\right)(\nabla_\alpha\Phi)^*(\nabla_\beta\Phi). \end{aligned} \tag{2.81}$$

Physical remark. The structural and spectral-curvature parts are manifestly positive in the rest frame of u^μ if $Z, Y \succ 0$; the matter part is positive if $\kappa_\Phi > 0$ and V_Φ bounded below.

2.7.9 Noether Currents and Symmetries

Global $U(1)$ of Φ. Under $\Phi \to e^{i\theta}\Phi$ with constant θ, the Noether current is J^μ_Φ in (2.66) and

$$\widetilde{\nabla}_\mu J^\mu_\Phi \;=\; 0, \tag{2.82}$$

on-shell. If coefficients depend on $\mathcal{S}$, extra terms cancel by the Φ-equation of motion.

Time-shift symmetry of T. Because T enters only through derivatives (via $D_\mu S^A$ and $C_{\mu\nu}$ indirectly), there is a global shift symmetry $T \mapsto T + c$ whose Noether current schematically reads

$$J^\mu_T \;=\; Z_{AB}\, D^\mu S^A \,\frac{\partial S^B}{\partial(\partial_\nu T)}\, u_\nu \;+\; (\text{terms from mixing}/\,V_\Phi)\,, \qquad \widetilde{\nabla}_\mu J^\mu_T \;=\; 0 \text{ (on-shell)}. \tag{2.83}$$

A fully explicit formula depends on the chosen chart, but conservation is chart-independent.

Bianchi identities. The spectral curvature satisfies $D_{[\lambda}F_{\mu\nu]} = 0$ (cf. §2.4.2), and the emergent curvature obeys $\mathbb{D}_{[\lambda}R_{\mu\nu]} = 0$ (cf. §2.5.3).

2.7.10 Energy Conditions and Positivity Gates

Weak energy condition (WEC). For any timelike v^μ ($g_{\text{eff}}(v,v) = -1$), WEC requires $T^{\mu\nu}_{\text{eff}} v_\mu v_\nu \geq 0$. Sufficient (not necessary) gates:

$$Z_{AB} \succ 0, \quad Y_{AB} \succ 0, \quad \kappa_\Phi > 0, \quad V_\Phi \text{ bounded below}, \quad a_0 > 0,\ a_{\perp,\parallel} \geq 1, \quad |\delta c| \leq 1. \tag{2.84}$$

Dominant energy condition (DEC). In addition, the flux $T^\mu{}_\nu v^\nu$ is non-spacelike if anisotropy parameters obey $|\delta c| \leq 1$ and mixing coefficients $\xi_A, \zeta_{\parallel,\perp}$ are small enough that the principal symbol remains causal inside the cone of g_{eff}.

2.7.11 Linearization Around Homogeneous Backgrounds

Background and fluctuations. Consider a homogeneous/steady background $\mathcal{S}_0$ with $D_\mu S^A|_0 = 0$, $F^A{}_{\mu\nu}|_0 = 0$, and constant coefficients $Z_0, Y_0, \kappa_{\Phi,0}, \ldots$. Let $S^A = \mathcal{S}^A_0 + \delta S^A$, $\Phi = \Phi_0 + \delta\Phi$, $g_{\text{eff}} = g_0 + \delta g$. Keeping quadratic terms, the principal symbols are

$$\mathcal{P}_{(S)}(x,k) = \; Z^{(0)}_{AB}\, g_0^{\mu\nu}\, k_\mu k_\nu\, \delta S^A \delta S^B, \tag{2.85}$$

$$\mathcal{P}_{(\Phi)}(x,k) = \; \kappa_{\Phi,0}\left(g_0^{\mu\nu} + \delta c_0\, \hat{\chi}^\mu_0 \hat{\chi}^\nu_0\right) k_\mu k_\nu\, |\delta\Phi|^2. \tag{2.86}$$

Hence characteristic speeds relative to u^μ_0 are

$$c_\perp = \frac{1}{a_{\perp,0}} \leq 1, \qquad c_\parallel = \frac{1}{a_{\parallel,0}} \leq 1,$$

ensuring hyperbolicity if the gates (2.84) hold. Mass matrices follow from second derivatives of U and V_Φ at the background.

2.7.12 Topological Sector

We now add metric-independent or boundary-complete terms that encode global/topological information. With constant couplings they do not modify local equations of motion; if couplings depend on $\mathcal{S}$, they produce controlled parity-odd bulk responses.

Gravitational topological densities.

$$\mathcal{E}_{\text{GB}} = \epsilon_{abcd}\, R^{ab} \wedge R^{cd}, \qquad \mathcal{P}_{\text{grav}} = R^{ab} \wedge R_{ab}, \tag{2.87}$$

and

$$\mathcal{L}^{\text{grav}}_{\text{top}} = \lambda_{\text{GB}}\, \mathcal{E}_{\text{GB}} + \vartheta_{\text{grav}}\, \mathcal{P}_{\text{grav}}. \tag{2.88}$$

Nieh–Yan and Holst (torsion-sensitive). With torsion 2-form T^a,

$$\mathcal{N}_{\rm NY} = d(e^a \wedge T_a) - T^a \wedge T_a + R_{ab} \wedge e^a \wedge e^b, \qquad \mathcal{H}_{\rm Holst} = e^a \wedge e^b \wedge R_{ab}, \tag{2.89}$$

and

$$\mathcal{L}^{\rm tors}_{\rm top} = \frac{M^2_{\rm eff}}{\gamma_{\rm H}}\, \mathcal{H}_{\rm Holst} + \vartheta_{\rm NY}\, \mathcal{N}_{\rm NY}. \tag{2.90}$$

Spectral θ-term(s). Let $\Theta_{AB} = -\Theta_{BA}$ be a constant antisymmetric bilinear on the fiber. Then

$$\mathcal{L}^{\rm spec}_{\rm top} \;=\; \frac{\vartheta_{\rm spec}(\mathcal{S})}{8\pi^2}\, \Theta_{AB}\, \mathcal{F}^A \wedge \mathcal{F}^B, \qquad \mathcal{F}^A = \tfrac{1}{2} F^A{}_{\mu\nu}\, dx^\mu \wedge dx^\nu, \tag{2.91}$$

is topological if $\vartheta_{\rm spec}$ is constant; otherwise it yields axion-like responses and sources $\propto \partial_A \vartheta_{\rm spec}\, \mathcal{F} \wedge \mathcal{F}$ in the $\mathcal{S}$-equations.

Director Hopf term (ordered phase). In the ordered stratum, a local 1-form $\mathcal{A}$ satisfies

$$d\mathcal{A} = \tfrac{1}{2}\, \epsilon_{IJK}\, \hat{\chi}^I\, d\hat{\chi}^J \wedge d\hat{\chi}^K, \qquad (I = 1, 2, 3), \tag{2.92}$$

and the Hopf term

$$\mathcal{L}^{\rm Hopf}_{\rm top} \;=\; \frac{\vartheta_{\rm H}}{4\pi^2}\, \mathcal{A} \wedge d\mathcal{A}, \tag{2.93}$$

computes linking of $\hat{\chi}$-field lines.

Topological sector summary.

$$\boxed{\mathcal{L}_{\rm top} = \mathcal{L}^{\rm grav}_{\rm top} + \mathcal{L}^{\rm tors}_{\rm top} + \mathcal{L}^{\rm spec}_{\rm top} + \mathcal{L}^{\rm Hopf}_{\rm top}}. \tag{2.94}$$

2.7.13 Boundary Terms and Well-Posed Variational Principle

Gravitational boundary term. The Gibbons–Hawking–York-type term in the first-order formalism is

$$S^{(\rm EH)}_{\partial M} = \frac{M^2_{\rm eff}}{2} \int_{\partial M} \epsilon_{abcd}\, e^a \wedge e^b \wedge \omega^{cd}, \tag{2.95}$$

ensuring a well-posed Dirichlet problem for e.

Boundary Chern–Simons additions. Let $\mathrm{CS}_3(\omega)$ and $\mathrm{CS}_3(\Gamma)$ satisfy $d\,\mathrm{CS}_3(\omega) = \mathrm{Tr}(R \wedge R)$, $d\,\mathrm{CS}_3(\Gamma) = \Theta_{AB}\, \mathcal{F}^A \wedge \mathcal{F}^B$. Then

$$S^{\rm CS}_{\partial M} \;=\; \int_{\partial M} \Big[\vartheta_{\rm grav}\, \mathrm{CS}_3(\omega) + \frac{\vartheta_{\rm spec}}{8\pi^2}\, \mathrm{CS}_3(\Gamma) \Big]. \tag{2.96}$$

Total action.

$$S \;=\; \int_M d^4x\, \sqrt{-g_{\rm eff}}\, \Big(\mathcal{L}^{(\rm dyn)}_{\rm tot} + \mathcal{L}_{\rm top} \Big) \;+\; S^{(\rm EH)}_{\partial M} + S^{\rm CS}_{\partial M}. \tag{2.97}$$

2.7.14 Consistency Gates and Summary

Checklist (sufficient conditions).

- **Symmetries:** Global $U(1)$ of Φ respected; T-shift symmetry respected (no explicit T without derivatives).
- **Positivity/causality:** $Z_{AB} \succ 0,\ Y_{AB} \succ 0,\ \kappa_\Phi > 0,\ a_0 > 0,\ a_{\perp,\parallel} \geq 1,\ |\delta c| \leq 1$.
- **Geometric consistency:** $\omega[e]$ torsionless on-shell; tetrad postulate holds; $\mathbb{D}R = 0$, $DF = 0$.
- **Topological decoupling:** If $\lambda_{\rm GB}, \vartheta_{\rm grav}, \gamma_{\rm H}, \vartheta_{\rm NY}, \vartheta_{\rm H}$ are constant, they do not alter local Euler–Lagrange equations; $\vartheta_{\rm spec}(\mathcal{S})$ sources controlled axion-like terms only.
- **Well-posed variation:** Boundary terms (2.95)–(2.96) included.

Section summary. We presented a unified, symmetry-consistent action for the quantum-aether model where the spectral derivative D_μ, the emergent connection ω, and the state-built geometry g_{eff} cohere into a viable low-energy EFT. Field equations (2.73)–(2.77), Noether currents, and stress–energy follow in standard fashion; hyperbolicity and energy positivity reduce to transparent algebraic gates. The topological sector augments the theory with global invariants and boundary responses without jeopardizing the local dynamics. This sets the stage for Chapter 3, where we work out explicit equations of motion, solutions, and observational signatures.

2.8 Worked Mini-Examples and Consistency Checks

This section illustrates the kinematics and symmetries of the unified framework through solvable backgrounds and linearized modes. Each example is followed by explicit checks: characteristic cones, Bianchi identities, current conservation, constraint propagation, and energy positivity. These worked cases will serve as templates for the full dynamical analysis in Chapter 3.

2.8.1 Homogeneous Background and Characteristic Cones

Setup. Consider a homogeneous background state $\mathcal{S}_0$ such that

$$D_\mu S^A\big|_0 = 0, \qquad F^A{}_{\mu\nu}\big|_0 = 0, \tag{2.98}$$

with constant constitutive coefficients $Z^{(0)}_{AB}, Y^{(0)}_{AB}, \kappa_{\Phi,0}, \delta c_0$ and constant anisotropic scales $a_0, a_\perp, a_\parallel$. The emergent tetrad and metric $(e^a{}_\mu, g_{\text{eff}})$ are therefore constant on the patch, and we choose the coframe

$$e^0{}_\mu = a_0\,\delta^0_\mu, \qquad e^1{}_\mu = a_\perp\,\delta^1_\mu, \qquad e^2{}_\mu = a_\perp\,\delta^2_\mu, \qquad e^3{}_\mu = a_\parallel\,\delta^3_\mu, \tag{2.99}$$

so that

$$g^{\mu\nu}_{\text{eff}} = \operatorname{diag}\left(-\frac{1}{a_0^2}, \frac{1}{a_\perp^2}, \frac{1}{a_\perp^2}, \frac{1}{a_\parallel^2}\right), \qquad u^\mu = \frac{1}{a_0}\,\delta^\mu_0, \quad \hat\chi^\mu = \frac{1}{a_\parallel}\,\delta^\mu_3. \tag{2.100}$$

Characteristic cones. For any field with principal part $\propto g^{\mu\nu}_{\text{eff}}\partial_\mu\partial_\nu$, the characteristic equation reads

$$-(u\cdot k)^2 + c_\perp^2\,k_\perp^2 + c_\parallel^2\,k_\parallel^2 = 0, \qquad c_\perp := \frac{1}{a_\perp} \le 1, \quad c_\parallel := \frac{1}{a_\parallel} \le 1, \tag{2.101}$$

with $k_\parallel = \hat\chi^\mu k_\mu$ and $k_\perp^2 := k_\mu k_\nu \Pi_\perp^{\mu\nu}$. Thus cones are contained within the Minkowski cone defined by any local inertial frame of g_{eff} if $a_{\perp,\parallel} \ge 1$ (cf. §2.6.4).

2.8.2 Isotropic Limit $\alpha = 0$

Setup. In the isotropic stratum ($\alpha = 0$), the director is physically irrelevant and we take $a_\parallel = a_\perp =: a_s$. Equation (2.100) becomes

$$g^{\mu\nu}_{\text{eff}} = \operatorname{diag}\left(-\frac{1}{a_0^2}, \frac{1}{a_s^2}, \frac{1}{a_s^2}, \frac{1}{a_s^2}\right), \tag{2.102}$$

and the characteristic equation reduces to

$$-(u\cdot k)^2 + c_s^2\,k_i k^i = 0, \qquad c_s := \frac{1}{a_s} \le 1. \tag{2.103}$$

All directions are equivalent; anisotropic terms $\propto \hat\chi^\mu\hat\chi^\nu$ drop out of $\mathcal{L}_\Phi$ and $\mathcal{L}_{\text{mix}}$.

2.8.3 Linear Modes of the Structural Sector

Fluctuations. Let $S^A = \mathcal{S}_0^A + \delta S^A$. At quadratic order on the homogeneous background (2.98),

$$\delta^2\mathcal{L}_{\text{struct}} = \frac{1}{2}\,Z^{(0)}_{AB}\,g^{\mu\nu}_{\text{eff}}\,\partial_\mu\delta S^A\,\partial_\nu\delta S^B - \frac{1}{2}\,\delta S^A\,(\mathcal{M}^2)_{AB}\,\delta S^B - \frac{1}{4}\,Y^{(0)}_{AB}\,F^A_{\mu\nu}F^{B\,\mu\nu}, \tag{2.104}$$

where $(\mathcal{M}^2)_{AB} = \partial_A\partial_B U(\mathcal{S}_0)$ and $F^A_{\mu\nu} = \partial_\mu(\Gamma^A{}_\nu)_0 - \partial_\nu(\Gamma^A{}_\mu)_0$ for linearized Ehresmann data.

Plane waves and dispersion. For plane waves $\delta S^A \sim \mathsf{s}^A e^{i(k\cdot x)}$ with constant polarizations s^A, the principal symbol yields

$$Z^{(0)}_{AB}\Big(-\omega^2/a_0^2 + k_\perp^2/a_\perp^2 + k_\parallel^2/a_\parallel^2\Big)\,\mathsf{s}^B = (\mathcal{M}^2)_{AB}\,\mathsf{s}^B, \tag{2.105}$$

so each normal mode $\mathsf{s}^{(n)}$ with eigenvalue m_n^2 of $Z^{-1}\mathcal{M}^2$ obeys

$$\omega_n^2 = c_\perp^2\,k_\perp^2 + c_\parallel^2\,k_\parallel^2 + m_n^2, \qquad c_{\perp,\parallel} = \frac{1}{a_{\perp,\parallel}}. \tag{2.106}$$

Positivity $Z \succ 0$ and $m_n^2 \geq 0$ ensure no ghosts and no tachyons at the linear level.

Bianchi check (spectral). The linearized Bianchi identity $D_{[\lambda}F_{\mu\nu]} = 0$ holds identically for $F = d\Gamma$ on the patch, ensuring consistency of (2.104) with gauge-covariant linearization.

2.8.4 Linear Modes of the Matter Field

Fluctuations. Let $\Phi = \Phi_0 + \delta\Phi$ with constant Φ_0. Keeping quadratic terms,

$$\delta^2\mathcal{L}_\Phi = \kappa_{\Phi,0}\left(g^{\mu\nu}_{\text{eff}} + \delta c_0\,\hat\chi^\mu\hat\chi^\nu\right)\partial_\mu\delta\Phi^*\,\partial_\nu\delta\Phi - \frac{1}{2}\,m^2_{\text{eff},0}\,|\delta\Phi|^2, \tag{2.107}$$

where $m^2_{\text{eff},0} := m^2_{\text{eff}}(\mathcal{S}_0) + 3\lambda_{\text{eff}}(\mathcal{S}_0)|\Phi_0|^2$ if $\Phi_0 \neq 0$.

Dispersion. For $\delta\Phi \sim e^{i(k\cdot x)}$,

$$\kappa_{\Phi,0}\Big(-\omega^2/a_0^2 + k_\perp^2/a_\perp^2 + (1+\delta c_0)\,k_\parallel^2/a_\parallel^2\Big) = m^2_{\text{eff},0}. \tag{2.108}$$

Causality requires the directional speeds

$$c_\perp^{(\Phi)} = \frac{1}{a_\perp}, \qquad c_\parallel^{(\Phi)} = \frac{1}{a_\parallel}\sqrt{1+\delta c_0} \;\leq 1, \tag{2.109}$$

hence $\delta c_0 \geq -1$ and $a_{\perp,\parallel} \geq 1$ are sufficient gates (cf. §2.7.10).

Current conservation. The Noether current $J^\mu_\Phi = \mathrm{Im}(\Phi^*\nabla^\mu\Phi)$ satisfies $\widetilde{\nabla}_\mu J^\mu_\Phi = 0$ on-shell; for $\Phi_0 = \text{const.}$, linear modes preserve $\int_\Sigma d\Sigma_\mu J^\mu_\Phi$ (with Σ a Cauchy slice of g_{eff}).

2.8.5 Constraint Propagation and Tetrad Postulate

Constraints. On the constraint subbundle, $u^\mu u_\mu = -1$, $\hat\chi^\mu\hat\chi_\mu = +1$, $u^\mu\hat\chi_\mu = 0$. With $\omega = \omega[e] + K(\mathcal{S})$ as in §2.5.2 and the tetrad postulate $\nabla^{(\omega,\widetilde{\Gamma})}_\mu e^a{}_\nu = 0$ (cf. §2.6.5), we have

$$\nabla_\mu(u^\nu u_\nu) = 0, \qquad \nabla_\mu(\hat\chi^\nu\hat\chi_\nu) = 0, \qquad \nabla_\mu(u^\nu\hat\chi_\nu) = 0, \tag{2.110}$$

i.e. normalization and orthogonality propagate along any solution obeying the Euler–Lagrange equations with Lagrange multipliers $\lambda_{1,2,3}$ (cf. §2.7.6).

Spin connection consistency. In the torsionless sector (on-shell), $\omega[e]$ is uniquely fixed by the coframe via Cartan's first structure equation; any residual state-dependent distortion $K(\mathcal{S})$ must preserve (2.110) and metricity, ensuring no unphysical drift of the constraints.

2.8.6 Bianchi Identities and Mixed Commutators

Spectral and emergent curvatures. We recall

$$D_{[\lambda}F_{\mu\nu]} = 0, \qquad \mathbb{D}_{[\lambda}R_{\mu\nu]} = 0, \qquad [\nabla_\mu,\nabla_\nu]\Psi = -F^A_{\text{spec}\,\mu\nu}\,\partial_{S^A}\Psi + R_{\mu\nu}(\omega)\cdot\Psi. \tag{2.111}$$

These identities hold identically on the homogeneous background, and perturbatively order-by-order for small departures. They guarantee the mutual compatibility of fiber holonomy and internal frame holonomy.

Worked check on Φ. Since Φ is a fiber scalar with no internal Lorentz index, $[\nabla_\mu, \nabla_\nu]\Phi = -F^A_{\mu\nu}\partial_{S^A}\Phi$, which vanishes on backgrounds with $\partial_S\Phi_0 = 0$ or $F = 0$. Thus the only obstruction is spectral.

2.8.7 Degenerate $C_{\mu\nu}$ and Regularization

Issue. In exactly homogeneous states, $C_{\mu\nu} = \mathcal{G}_{AB}D_\mu S^A D_\nu S^B = 0$. The Gram–Schmidt procedure used in §2.6.2 to construct $\{\theta^a\}$ from $\tau_\mu \propto \partial_\mu T$ and $\sigma_\mu \propto \hat{\chi}_i D_\mu \chi^i$ becomes degenerate.

Regularization. We adopt one of the following equivalent cures:

1. Add a small, state-dependent isotropic seed to the material form: $C_{\mu\nu} \mapsto C_{\mu\nu} + \epsilon(\mathcal{S})\,\delta_{\mu\nu}$, with $\epsilon > 0$ arbitrarily small.
2. Select a reference coframe $\bar{e}^a{}_\mu$ and use the state-built ansatz (2.57) with small $A(\mathcal{S}), B(\mathcal{S})$ to parametrize deviations, keeping $\theta^0 \propto \partial T$ as the clock 1-form.

Both prescriptions leave the low-gradient physics unchanged and keep the coframe regular.

2.8.8 Energy Positivity and Causality Checks

Quadratic energy in the rest frame. With $v^\mu = u^\mu$, the structural energy density is

$$\mathcal{E}_{(S)} = \frac{1}{2}\,Z_{AB}\,(u\cdot DS^A)(u\cdot DS^B) + \frac{1}{4}\,Y_{AB}\left(F^A_{\alpha\beta}F^{B\,\alpha\beta}\right) + U(\mathcal{S}) \quad \geq 0 \tag{2.112}$$

if $Z, Y \succ 0$ and U bounded below. For matter,

$$\mathcal{E}_{(\Phi)} = \kappa_\Phi\left(|u\cdot\nabla\Phi|^2 + c_\perp^2\,|\nabla_\perp\Phi|^2 + c_\parallel^2\,|\nabla_\parallel\Phi|^2\right) + V_\Phi \quad \geq 0, \tag{2.113}$$

provided $\kappa_\Phi > 0$, $c_{\perp,\parallel} \leq 1$, and V_Φ bounded below.

Dominant energy and causal flux. For sufficiently small mixing coefficients $\xi_A, \zeta_{\perp,\parallel}$ (relative to the positive kinetic coefficients), the total flux $T^\mu{}_\nu u^\nu$ is future–causal, and the characteristic cones remain inside those of g_{eff}. This implements the gates listed in §2.7.10.

2.8.9 Summary Boxes and Checklists

Box A: Dispersion relations.

$$\begin{aligned}
&\text{Structural modes:} && \omega_n^2 = c_\perp^2\,k_\perp^2 + c_\parallel^2\,k_\parallel^2 + m_n^2,\\
&\text{Matter mode:} && \omega^2 = c_\perp^2\,k_\perp^2 + \left(c_\parallel^{(\Phi)}\right)^2 k_\parallel^2 + m_{\text{eff},0}^2,\\
& && c_\perp = \tfrac{1}{a_\perp} \leq 1, \quad c_\parallel = \tfrac{1}{a_\parallel} \leq 1, \quad c_\parallel^{(\Phi)} = \tfrac{1}{a_\parallel}\sqrt{1+\delta c_0} \leq 1.
\end{aligned}$$

Box B: Consistency gates (sufficient).

$$Z_{AB} \succ 0, \quad Y_{AB} \succ 0, \quad \kappa_\Phi > 0, \quad a_0 > 0, \quad a_{\perp,\parallel} \geq 1, \quad \delta c \geq -1.$$

Box C: Identities ensured.

$$D_{[\lambda}F_{\mu\nu]} = 0, \qquad \mathbb{D}_{[\lambda}R_{\mu\nu]} = 0, \qquad [\nabla_\mu, \nabla_\nu]\Psi = -F^A_{\mu\nu}\partial_{S^A}\Psi + R_{\mu\nu}\cdot\Psi.$$

Box D: Constraint propagation.

$$\nabla(u^2) = \nabla(\hat{\chi}^2) = \nabla(u\cdot\hat{\chi}) = 0 \;\text{ on-shell (with Lagrange multipliers).}$$

These examples confirm that the unified kinematics $\{D_\mu, \omega_\mu, g_{\text{eff}}\}$ produces a healthy low-energy EFT with controllable anisotropy, positive energy, and well-posed propagation. In Chapter 3 we lift the linear analysis and derive full equations of motion, exact/approximate solutions, and observational implications.

2.9 Dimensional Analysis and Units

In this section we fix engineering (mass) dimensions, establish EFT power counting, and record canonical normalizations and useful rescalings. We work in natural units $c = \hbar = 1$, so the action is dimensionless and the Lagrangian density has mass dimension $[\mathcal{L}] = +4$ in $3+1$ dimensions. Spacetime integrals carry $[d^4x] = -4$, and $\sqrt{-g_{\text{eff}}}$ is dimensionless.

2.9.1 Conventions and Core Assignments

Spacetime and geometry.

$$[x^\mu] = -1, \quad [\partial_\mu] = +1, \quad [e^a{}_\mu] = 0, \quad [g^{\mu\nu}_{\text{eff}}] = 0, \quad [\omega_\mu{}^{ab}] = +1, \quad [R] = +2.$$

The emergent tetrad and metric are kinematic (dimensionless), while the spin connection carries one derivative and the curvature two.

Fields and structural operators.

$$[S^A] = 0, \quad [D_\mu] = +1, \quad [F^A{}_{\mu\nu}] = +2, \quad [\Phi] = +1, \quad [\nabla_\mu \Phi] = +2.$$

Here $S^A = (\rho, \eta, T, \chi^I, \alpha)$ are taken as dimensionless order parameters; the Ehresmann covariant derivative D_μ has one derivative; its curvature F has two.

2.9.2 Dimensions of Couplings and Coefficients

Structural sector. From $\mathcal{L}^{(S)}_{\text{kin}} = \frac{1}{2}\, Z_{AB}\, g^{\mu\nu} D_\mu S^A D_\nu S^B$ and $\mathcal{L}_F = -\frac{1}{4}\, Y_{AB}\, F^A_{\mu\nu} F^{B\,\mu\nu}$ we read

$$[Z_{AB}] = +2, \qquad [Y_{AB}] = 0, \qquad [U] = +4.$$

Matter sector. From $\mathcal{L}^{\text{kin}}_\Phi = \kappa_\Phi\, G^{\mu\nu}_{(\text{ani})} (\nabla_\mu \Phi)^* (\nabla_\nu \Phi)$ and $V_\Phi = \frac{1}{2} m^2_{\text{eff}} |\Phi|^2 + \frac{\lambda_{\text{eff}}}{4} |\Phi|^4$ we obtain

$$[\kappa_\Phi] = 0, \quad [\delta c] = 0, \quad [m^2_{\text{eff}}] = +2, \quad [\lambda_{\text{eff}}] = 0.$$

Mixing, constraints, gravity.

$$[\xi_A] = 0, \quad [\zeta_{\parallel,\perp}] = 0, \quad [\lambda_{1,2,3}] = +4, \quad [M^2_{\text{eff}}] = +2, \quad [\Lambda_{\text{eff}}] = +1.$$

The multipliers λ_i multiply algebraic constraints and therefore carry $[+4]$.

Topological sector. Treating 4-form densities as mass-dimension $+4$,

$$[\lambda_{\text{GB}}] = 0, \quad [\vartheta_{\text{grav}}] = 0, \quad [\gamma_{\text{H}}] = 0, \quad [\vartheta_{\text{NY}}] = 0, \quad [\vartheta_{\text{spec}}] = 0, \quad [\vartheta_{\text{H}}] = 0.$$

The Holst piece appears as $(M^2_{\text{eff}}/\gamma_{\text{H}})\, \mathcal{H}_{\text{Holst}}$, with M^2_{eff} setting the correct mass dimension.

2.9.3 EFT Power Counting and Operator Hierarchy

Two-derivative core (relevant/marginal).

- Geometry: $M^2_{\text{eff}} R$.
- Structure: $Z_{AB}\, DS\, DS$, $-U(\mathcal{S})$.
- Spectral curvature: $-\frac{1}{4} Y_{AB} F^A F^B$.
- Matter: $\kappa_\Phi\, (\nabla\Phi)^2$, $m^2_{\text{eff}} |\Phi|^2$, $\lambda_{\text{eff}} |\Phi|^4$.

Higher-derivative corrections (suppressed). Allowed operators with ≥ 4 derivatives are included as

$$\Delta\mathcal{L} \;\sim\; \frac{1}{\Lambda_{\rm eff}^2}\Big\{c_1\,(D^2S)^2 + c_2\,(F^A_{\mu\nu}F^{A\,\mu\nu})^2 + c_3\,R_{\mu\nu}R^{\mu\nu} + c_4\,R^2 + \cdots\Big\},$$

with $\Lambda_{\rm eff}$ the EFT cutoff. Parity-even curvature squares may be kept in analytic studies, but we do *not* rely on them for propagation to avoid Ostrogradski ghosts; the dynamical graviton sector is carried by $M_{\rm eff}^2 R$.

Power-counting gates.

$$\frac{\partial}{\partial x} \ll \Lambda_{\rm eff}, \qquad \frac{F}{\Lambda_{\rm eff}^2} \ll 1, \qquad \frac{R}{\Lambda_{\rm eff}^2} \ll 1, \tag{2.114}$$

so that higher-derivative terms remain corrections.

2.9.4 Canonical Normalizations and Field-Space Diagonalization

Matter field. A position-dependent wavefunction renormalization can be absorbed locally by

$$\Phi \;\mapsto\; \Phi_c := \sqrt{\kappa_\Phi(\mathcal{S})}\,\Phi,$$

leading to a canonical kinetic term w.r.t. $g_{\rm eff}$ and moving $\mathcal{S}$-dependence to derivative couplings (via $\partial\kappa_\Phi$) and the potential.

Structural sector. Let $Z_{AB}(\mathcal{S})$ be positive definite. In a neighborhood of a background $\mathcal{S}_0$, choose a field-space vielbein $E^A{}_a(\mathcal{S}_0)$ so that $E^T Z\,E = \mathbb{1}$. Defining $\varphi^a := E^a{}_A\,\delta S^A$ renders the quadratic kinetic form canonical:

$$\delta^2\mathcal{L}^{(S)}_{\rm kin} = \tfrac{1}{2}\,\eta_{ab}\,g^{\mu\nu}\,\partial_\mu\varphi^a\,\partial_\nu\varphi^b, \qquad \eta_{ab} = \delta_{ab},$$

while the mass matrix becomes $\mathcal{M}^2_{ab} = E_a{}^A(\partial_A\partial_B U)\,E^B{}_b$.

Anisotropy bookkeeping. Directional speeds are encoded by $(a_\perp, a_\parallel)$ through $c_\perp = 1/a_\perp$, $c_\parallel = 1/a_\parallel$. For the matter sector, the longitudinal correction is $c_\parallel^{(\Phi)} = (1/a_\parallel)\sqrt{1+\delta c}$. Canonical normalization does not alter these dimensionless ratios; it only moves factors between kinetic and mixing terms.

2.9.5 Spurion Analysis for Symmetry-Breaking Patterns

Global $U(1)$ of Φ. Treat Φ as a spurion with charge $+1$. Coefficients multiplying Φ^n must be $U(1)$-neutral. Hence linear terms in Φ are forbidden, cubic terms are forbidden, while $|\Phi|^2$ and $|\Phi|^4$ are allowed.

Time-shift of T. Since T appears only via derivatives, any operator with undifferentiated T is forbidden. This excludes dangerous relevant deformations that would explicitly pick a time origin.

Internal rotations of $\boldsymbol{\chi}$ and nematicity. In the isotropic stratum ($\alpha = 0$) only rotational scalars built from χ^I via $\mathcal{G}_{IJ}$ are allowed. In the ordered stratum ($\alpha > 0$) spurions $\hat{\chi}^I$ and projectors $\Pi_\parallel, \Pi_\perp$ are admissible.

2.9.6 Mapping Back to SI Units (Quick Reference)

If desired, reinsert c and $\hbar$ using

$$[E] = [\text{mass}]\,c^2, \quad [p] = [\text{mass}]\,c, \quad [x] = \frac{\hbar}{[p]} = \frac{\hbar}{[\text{mass}]\,c}, \quad [t] = \frac{\hbar}{[E]} = \frac{\hbar}{[\text{mass}]\,c^2}.$$

Then the assignments above translate as

$$[\Phi]_{\rm SI} = \mathrm{J}^{1/2}\,\mathrm{m}^{-3/2}, \quad [m^2_{\rm eff}]_{\rm SI} = \mathrm{J}^2\,\hbar^{-2}c^{-4}, \quad [M^2_{\rm eff}]_{\rm SI} = \mathrm{J}^2\,\hbar^{-2}c^{-4}.$$

Dimensionless couplings remain dimensionless; anisotropy parameters $(a_0, a_\perp, a_\parallel, \delta c)$ are unchanged.

2.9.7 Ghost/Ostrogradski Avoidance and Causality Gates

Sufficient conditions (recap).

$$Z_{AB} \succ 0, \quad Y_{AB} \succ 0, \quad \kappa_\Phi > 0, \quad a_0 > 0,\ a_{\perp,\parallel} \geq 1,\ \delta c \geq -1, \quad \frac{R}{\Lambda_{\text{eff}}^2}, \frac{F}{\Lambda_{\text{eff}}^2} \ll 1. \tag{2.115}$$

Under these gates, the principal symbol is hyperbolic and energy densities measured by u^μ are nonnegative.

Topological safety. With constant coefficients, $\mathcal{L}_{\text{top}}$ does not change local equations and introduces no extra propagating degrees of freedom. If $\vartheta_{\text{spec}}(\mathcal{S})$ varies, it couples derivatively to $\mathcal{S}$ (axion-like) without raising the differential order of the EoMs.

2.9.8 Summary Boxes

Box A: Master dimension table (selected).

Quantity	Dim	Quantity	Dim
x^μ	-1	∂_μ	$+1$
$e^a{}_\mu,\ g_{\text{eff}}^{\mu\nu}$	0	$\omega_\mu{}^{ab}$	$+1$
$R,\ F$	$+2$	D_μ	$+1$
S^A	0	Φ	$+1$
Z_{AB}	$+2$	Y_{AB}	0
$\kappa_\Phi,\ \delta c$	0	m_{eff}^2	$+2$
λ_{eff}	0	M_{eff}^2	$+2$
Λ_{eff}	$+1$	λ_i	$+4$
Topological couplings $(\lambda_{\text{GB}}, \vartheta_\cdot, \gamma_{\text{H}}^{-1})$	0		

Box B: EFT and causality gates.

$$Z \succ 0,\ Y \succ 0,\ \kappa_\Phi > 0,\ a_{\perp,\parallel} \geq 1,\ \delta c \geq -1,\ \frac{R}{\Lambda_{\text{eff}}^2}, \frac{F}{\Lambda_{\text{eff}}^2} \ll 1.$$

Box C: Canonical normalizations.

$$\Phi \to \sqrt{\kappa_\Phi}\,\Phi, \qquad \delta S^A \to \varphi^a = E^a{}_A\,\delta S^A \text{ with } E^T Z E = \mathbb{1}.$$

With these assignments the unified Lagrangian of §2.7 is dimensionally consistent, the EFT hierarchy is explicit, and canonical normalizations are available around any background. We are now ready to conclude the chapter by summarizing the construction and previewing the dynamical program of Chapter 3.

2.10 Summary and Roadmap to Chapter 3

This chapter established the *geometric* and *dynamical* substrate of the Quantum Aether model. We built a fibered internal space F ($\dim = 7$) hosting the structural state $\mathcal{S} = (\rho, \eta, T, \chi^I, \alpha)$, introduced an Ehresmann connection $\Gamma^A{}_\mu$ with curvature $F^A{}_{\mu\nu}$ (the *spectral curvature*), constructed a state-built tetrad $e^a{}_\mu$ and an effective metric g_{eff}, and defined a covariant derivative $\nabla_\mu = D_\mu + \omega_\mu$ acting on matter and structural fields. The unified Lagrangian was organized into dynamical and topological sectors with clear symmetries and consistency gates.

2.10.1 Chapter Highlights (Technical Recap)

Spectral geometry and kinematics.

- Fiber bundle $E \to M$ with internal fiber F, horizontal lift via $\Gamma^A{}_\mu$, and curvature $F^A{}_{\mu\nu} = \partial_{[\mu}\Gamma^A{}_{\nu]} + \Gamma^B{}_{[\mu}\partial_{S^B}\Gamma^A{}_{\nu]}$ (cf. §2.4).
- Structural fields S^A carry a positive-definite fiber metric Z_{AB} for vertical kinetics and Y_{AB} for F-stiffness.

Emergent metric and connection.

- Coframe $\theta^0 \propto \partial_\mu T\, dx^\mu$, $\theta^3 \propto \Pi^\nu{}_\mu(\hat{\chi})\, dx^\mu$, completed to $\{\theta^a\}$ (regularized if $C_{\mu\nu} \to 0$); metric $g_{\text{eff}} = \eta_{ab}\, \theta^a \otimes \theta^b$ (cf. §2.6).
- Spin connection $\omega = \omega[e]$ (torsionless on-shell) with state-compatible distortion constrained by metricity and normalization/orthogonality of $(u^\mu, \hat{\chi}^\mu)$ (cf. §§2.5,2.6).

Unified action and symmetries.

- Dynamical sector (§2.7):

$$\mathcal{L}^{(\text{dyn})}_{\text{tot}} = \mathcal{L}_{\text{grav}} + \mathcal{L}_{\text{struct}} + \mathcal{L}_\Phi + \mathcal{L}_{\text{mix}} + \mathcal{L}_{\text{constr}},$$

  with $\mathcal{L}_{\text{grav}} = \frac{M^2_{\text{eff}}(\rho)}{2} R(\omega, e) + \mathcal{O}(R^2/\Lambda^2_{\text{eff}})$, $\mathcal{L}_{\text{struct}} = \frac{1}{2} Z_{AB} DS^A \cdot DS^B - U(\mathcal{S}) - \frac{1}{4} Y_{AB} F^A \cdot F^B$, and anisotropic matter kinetics for Φ controlled by $\hat{\chi}$ (cf. (2.59)–(2.72)).
- Topological sector (§2.7.12) separated explicitly: Gauss–Bonnet, Pontryagin, Holst/Nieh–Yan, spectral θ-term, and Hopf term for the director.
- Global $U(1)$ for Φ and time-shift symmetry $T \mapsto T + c$ preserved; Bianchi identities for F and R hold.

Field equations and positivity.

- Einstein-like equation with variable $M_{\text{eff}}(\rho)$ (cf. (2.77)); structural and matter EL-equations (cf. (2.74), (2.73)) include implicit $\partial_S \Gamma$ contributions.
- Energy and causality gates (§2.7.10): $Z, Y \succ 0$, $\kappa_\Phi > 0$, $a_{\perp,\parallel} \geq 1$, $\delta c \geq -1$; hyperbolic principal symbol; WEC/DEC satisfied in the gated regime.

Worked checks.

- Homogeneous backgrounds and linear modes yield dispersions $\omega^2 = c^2_\perp k^2_\perp + c^2_\parallel k^2_\parallel + m^2$ for structural modes, and $c^{(\Phi)}_\parallel = \frac{1}{a_\parallel}\sqrt{1+\delta c}$ for matter (cf. §2.8).
- Constraint propagation and metricity are consistent with the tetrad postulate; degeneracies are regularized harmlessly.

2.10.2 Key Equations at a Glance

- **Spectral curvature:** $F^A{}_{\mu\nu} = \partial_{[\mu}\Gamma^A{}_{\nu]} + \Gamma^B{}_{[\mu}\partial_{S^B}\Gamma^A{}_{\nu]}$.
- **Unified dynamical Lagrangian:** $\mathcal{L}^{(\text{dyn})}_{\text{tot}}$ as in (2.72).
- **Einstein-like equation:** $M^2_{\text{eff}}(\rho)\, G_{\mu\nu} = T^{\text{eff}}_{\mu\nu} + \partial_\mu\partial_\nu M^2_{\text{eff}} - \cdots$ (schematic of (2.77)).
- **Linear dispersions:** structural (2.106), matter (2.108).

2.10.3 Roadmap to Chapter 3: Equations of Motion and Solutions

Chapter 3 develops the *dynamics* implied by the unified action:

(I) Emergent connection and spectral forces.

- Compute $\omega[e]$ explicitly and quantify any admissible state-dependent distortion $K(\mathcal{S})$ subject to metricity and constraint preservation; evaluate $R(\omega)$ (cf. §§2.5,2.6).
- Derive the full EL-equations for $(S^A, \Phi; e, \omega)$ including the implicit $\partial_S\Gamma$ sources from the spectral curvature sector.

(II) Recovery of General Relativity at low gradients.

- Show that for slowly varying ρ with $\rho \approx \rho_0$, the equations reduce to Einstein's with $G_{\text{eff}} = (8\pi M_{\text{eff}}^2(\rho_0))^{-1}$ and small controlled corrections.
- Newtonian limit and Poisson equation in the weak-field, slow-motion regime; identify constraints on $\partial\rho$ from Solar-System tests.

(III) Prototypical solutions.

- **Spectral waves**: plane waves in δS^A and their backreaction on g_{eff}.
- **Localized solitons**: stationary finite-energy configurations of Φ stabilized by (Z, Y, U), including anisotropy via $\hat{\chi}$; compute masses, sizes, and stability bands.
- **Two-body sector**: effective potential $V_{\text{int}}(R)$ with spectral-curvature corrections (short-range repulsion / long-range tails), and conditions for bound states.

(IV) Dissipation, arrow of time, and decoherence.

- Include Rayleigh-type dissipation $\mathcal{R} \propto \eta\,(\nabla\Phi)^2$: derive entropy production $\partial_\mu(su^\mu) \geq 0$ and conditions for macroscopic irreversibility.
- Application: suppression of long-baseline coherence and phenomenology of measurement-like interactions in structured media.

(V) Observational/computational touchstones.

- Constraints on $c_\perp, c_\parallel, \delta c$ from wave propagation and birefringent-like effects.
- Bounds on spatial variations of $M_{\text{eff}}(\rho)$ from lensing and timing tests.
- Numerical solvers for coupled $(S^A, \Phi, g_{\text{eff}})$ in 1D/2D symmetry-reduced settings.

2.10.4 Reading Guide and Cross-References

- For spectral geometry foundations and the role of F_{spec}, see §§2.3–2.4.
- For state-built metric, projectors, and anisotropy, see §2.6.
- For the unified action, EL-equations, and energy/causality gates, see §2.7.
- For explicit linear modes, checks, and regularization strategies, see §2.8.
- For EFT hierarchy and canonical normalizations, see §2.9.

2.10.5 Open Technical Tasks Deferred to Chapter 3

1. Compute the explicit *implicit* variations $\partial_S\Gamma$ in the EL-equations and classify them by derivative order and symmetry.
2. Work out the precise conditions under which $M_{\text{eff}}^2(\rho)$ variations can be neglected and quantify the leading corrections to GR observables.
3. Construct and analyze canonical solitons and two-body bound states; map their dependence on (ρ, η, α).
4. Quantify dissipation/entropy production from $\eta > 0$ and connect to measurable decoherence rates.

With the spectral geometry, emergent metric, and unified action now in place, Chapter 3 turns to the *equations of motion* and *solutions*. The goal is twofold: (i) demonstrate the model's internal predictive power by exact/approximate configurations, and (ii) extract parameter windows and signatures that connect the theory to observations and experiments.

2.11 Appendix 2.A — Technical Derivations

This appendix supplies derivations that were quoted or summarized in §§2.4–2.7: (i) how the implicit state dependence $\Gamma^A{}_\mu(x, S(x))$ contributes to the Euler–Lagrange equations via $\partial_S\Gamma$; (ii) coordinate-free and chart formulas for δF under variations of S; (iii) why the topological sector $\mathcal{L}_{\text{top}}$ does not modify local equations of motion when its couplings are constant, and what changes when a coupling depends on $\mathcal{S}$; (iv) boundary terms. Throughout we keep the notation of Chapter 2.

2.11.1 Ehresmann Connection: Implicit Variations

Setup and notation. The spectral derivative uses an Ehresmann connection $\Gamma^A{}_\mu(x, S)$ on the bundle $E \to M$. The *horizontal* derivative acting on fiber scalars is

$$D_\mu := \partial_\mu + \Gamma^A{}_\mu\, \partial_{S^A}, \qquad F^A{}_{\mu\nu} = \partial_{[\mu}\Gamma^A{}_{\nu]} + \Gamma^B{}_{[\mu}\, \partial_{S^B}\Gamma^A{}_{\nu]}\,, \tag{2.116}$$

(cf. §2.4). In the unified Lagrangian (§2.7) we *do not* vary Γ independently (Palatini with respect to e, ω only). However, because $\Gamma = \Gamma(x, S(x))$, a variation δS^C induces $\delta\Gamma^A{}_\mu = (\partial_{S^C}\Gamma^A{}_\mu)\,\delta S^C$, and thus contributes to $\delta\mathcal{L}$ through the chain rule.

Variation of $D_\mu S^A$. Since $D_\mu S^A = \partial_\mu S^A + \Gamma^A{}_\mu$, we have

$$\delta(D_\mu S^A) = \partial_\mu \delta S^A + (\partial_{S^C}\Gamma^A{}_\mu)\,\delta S^C. \tag{2.117}$$

This produces the familiar kinetic contribution $\widetilde{\nabla}_\mu(Z_{AB}\, D^\mu S^B)$ in the EL-equations plus a source proportional to $Z_{AB}\,\partial_S\Gamma$.

Variation of $F^A{}_{\mu\nu}$ under δS. Write $\Gamma^A{}_\mu = \Gamma^A{}_\mu(x, S)$. A direct chart computation gives

$$\begin{aligned}\delta F^A{}_{\mu\nu} &= \partial_{[\mu}\delta\Gamma^A{}_{\nu]} + \delta\Gamma^B{}_{[\mu}\,\partial_{S^B}\Gamma^A{}_{\nu]} + \Gamma^B{}_{[\mu}\,\partial_{S^B}\delta\Gamma^A{}_{\nu]} \\ &= \partial_{[\mu}\big((\partial_{S^C}\Gamma^A{}_{\nu]})\,\delta S^C\big) + (\partial_{S^C}\Gamma^B{}_{[\mu})\,\partial_{S^B}\Gamma^A{}_{\nu]}\,\delta S^C + \Gamma^B{}_{[\mu}\,(\partial_{S^B}\partial_{S^C}\Gamma^A{}_{\nu]})\,\delta S^C,\end{aligned} \tag{2.118}$$

where we used $\delta\Gamma^A{}_\mu = (\partial_{S^C}\Gamma^A{}_\mu)\,\delta S^C$ and $\partial_{S^B}\delta\Gamma^A{}_\nu = (\partial_{S^B}\partial_{S^C}\Gamma^A{}_\nu)\,\delta S^C$. Expanding the first term,

$$\partial_{[\mu}\big((\partial_{S^C}\Gamma^A{}_{\nu]})\,\delta S^C\big) = (\partial_{S^C}\Gamma^A{}_{\nu]})\,\partial_{[\mu}\delta S^C + (\partial_\mu S^D)\,(\partial_{S^D}\partial_{S^C}\Gamma^A{}_{\nu]})\,\delta S^C + \partial_{[\mu}\partial_{S^C}\Gamma^A{}_{\nu]}\,\delta S^C. \tag{2.119}$$

Compact, covariantized form. It is convenient to reorganize (2.118)–(2.119) as

$$\delta F^A{}_{\mu\nu} = D_{[\mu}\big((\partial_{S^C}\Gamma^A{}_{\nu]})\,\delta S^C\big) + \mathcal{Q}^A{}_{\mu\nu C}(\partial S)\,\delta S^C, \qquad \mathcal{Q}^A{}_{\mu\nu C} := \Gamma^B{}_{[\mu}\,\partial_{S^B}\partial_{S^C}\Gamma^A{}_{\nu]} + (\partial_\rho S^D)\,\Xi^{A\rho}{}_{\mu\nu CD}, \tag{2.120}$$

for some smooth tensors Ξ built from $\partial_S\partial_S\Gamma$ (explicit but cumbersome). The first term is a horizontal divergence and will integrate by parts in the action.

Contribution to $\delta\mathcal{L}_F$. From $\mathcal{L}_F = -\frac{1}{4}\,Y_{AB}F^A\cdot F^B$ and (2.120),

$$\begin{aligned}\delta\mathcal{L}_F &= -\frac{1}{2}\,Y_{AB}\,F^{A\,\mu\nu}\,\delta F^B{}_{\mu\nu} - \frac{1}{4}\,(\partial_{S^C}Y_{AB})\,F^A_{\mu\nu}F^{B\,\mu\nu}\,\delta S^C \\ &= -\frac{1}{2}\,Y_{AB}\,F^{A\,\mu\nu}\,D_{[\mu}\big((\partial_{S^C}\Gamma^B{}_{\nu]})\,\delta S^C\big) - \frac{1}{2}\,Y_{AB}\,F^{A\,\mu\nu}\,\mathcal{Q}^B{}_{\mu\nu C}(\partial S)\,\delta S^C - \frac{1}{4}\,\partial_{S^C}Y_{AB}\,F^A\cdot F^B\,\delta S^C.\end{aligned} \tag{2.121}$$

Integrating the first term by parts (in the horizontal sense) and discarding boundary contributions, we obtain the bulk piece that appears in the EL-equations for S^C:

$$\Big[\delta\mathcal{L}_F\Big]_{\text{bulk}} = \frac{1}{2}\,D_{[\mu}\Big(Y_{AB}\,F^{A\,\mu\nu}\Big)\,(\partial_{S^C}\Gamma^B{}_{\nu]})\,\delta S^C - \frac{1}{2}\,Y_{AB}\,F^{A\,\mu\nu}\,\mathcal{Q}^B{}_{\mu\nu C}\,\delta S^C - \frac{1}{4}\,\partial_{S^C}Y_{AB}\,F^A\cdot F^B\,\delta S^C. \tag{2.122}$$

This is the explicit origin of the $\mathcal{F}^{(S)}_C$ term quoted in (2.75). On homogeneous backgrounds ($F = 0,\ \partial S = 0$) these contributions vanish, as used in §2.8.

Variation of the kinetic structural term. Similarly,

$$\begin{aligned}\delta\mathcal{L}^{(S)}_{\rm kin} &= \frac{1}{2}\,\partial_{S^C} Z_{AB}\, D_\mu S^A D^\mu S^B\,\delta S^C + Z_{AB}\, D_\mu S^A\,\delta(D^\mu S^B)\\ &= \frac{1}{2}\,\partial_{S^C} Z_{AB}\, DS^A\cdot DS^B\,\delta S^C + Z_{AB}\, D_\mu S^A\,\big(\partial^\mu\delta S^B + \partial_{S^C}\Gamma^{B\mu}\,\delta S^C\big).\end{aligned} \tag{2.123}$$

Integrating $\partial^\mu\delta S^B$ by parts produces $\widetilde{\nabla}_\mu(Z_{AB}D^\mu S^A)$ and a source term $-Z_{AB}(\partial_{S^C}\Gamma^B{}_\mu)D^\mu S^A\,\delta S^C$. Collecting with (2.122) and $\delta(-U) = -\partial_{S^C}U\,\delta S^C$ yields the full S-equations in (2.74).

2.11.2 Coordinate-Free Differential-Form View

Horizontal/vertical split. Let $d_H := dx^\mu(\partial_\mu + \Gamma^A{}_\mu\partial_{S^A})$ and $d_V := dS^A\,\partial_{S^A}$. Then the curvature 2-form $\mathcal{F}^A$ may be written as

$$\mathcal{F}^A = d_H\Gamma^A + \frac{1}{2}\,[\Gamma,\Gamma]^A_V, \qquad [\Gamma,\Gamma]^A_V := \Gamma^B\wedge(\partial_{S^B}\Gamma^A). \tag{2.124}$$

A vertical variation δ_S gives

$$\delta_S\mathcal{F}^A = d_H(\partial_{S^C}\Gamma^A)\,\delta S^C + \Gamma^B\wedge(\partial_{S^B}\partial_{S^C}\Gamma^A)\,\delta S^C, \tag{2.125}$$

which is the form counterpart of (2.120). Contracting with $Y_{AB}*\mathcal{F}^B$ and integrating by parts reproduces (2.122).

2.11.3 Topological Sector: Local Decoupling and Axion-Like Sources

Gauss–Bonnet and Pontryagin (gravitational). In 4D,

$$\mathcal{E}_{\rm GB} = \epsilon_{abcd}R^{ab}\wedge R^{cd} = d\,\mathrm{CS}^{\rm grav}_3(\omega), \qquad \mathcal{P}_{\rm grav} = R^{ab}\wedge R_{ab} = d\,\widetilde{\mathrm{CS}}^{\rm grav}_3(\omega), \tag{2.126}$$

so for *constant* $\lambda_{\rm GB},\vartheta_{\rm grav}$,

$$\delta\int_M\big(\lambda_{\rm GB}\mathcal{E}_{\rm GB} + \vartheta_{\rm grav}\mathcal{P}_{\rm grav}\big) = \int_{\partial M}\big(\lambda_{\rm GB}\delta\mathrm{CS}^{\rm grav}_3 + \vartheta_{\rm grav}\delta\widetilde{\mathrm{CS}}^{\rm grav}_3\big), \tag{2.127}$$

i.e. no bulk contribution to the Euler–Lagrange equations.

Holst/Nieh–Yan (torsion-sensitive). Recall $\mathcal{N}_{\rm NY} = d(e^a\wedge T_a) - T^a\wedge T_a + R_{ab}\wedge e^a\wedge e^b$, $\mathcal{H}_{\rm Holst} = e^a\wedge e^b\wedge R_{ab}$. In the torsionless sector ($T^a = 0$ on-shell), $\mathcal{H}_{\rm Holst}$ reduces to a boundary term modulo $\mathcal{N}_{\rm NY}$, and does not change bulk EoMs. With torsion allowed, Holst changes only the *constitutive relation* between spin connection and tetrad (Immirzi parameter), not the metric field equations in vacuum.

Spectral θ-term. For $\mathcal{L}^{\rm spec}_{\rm top} = \frac{\vartheta_{\rm spec}(\mathcal{S})}{8\pi^2}\,\Theta_{AB}\,\mathcal{F}^A\wedge\mathcal{F}^B$, if $\vartheta_{\rm spec}$ is constant, it is a total derivative (boundary Chern–Simons). If $\vartheta_{\rm spec} = \vartheta_{\rm spec}(\mathcal{S})$, then

$$\frac{\delta}{\delta S^C}\int_M\sqrt{-g}\,\mathcal{L}^{\rm spec}_{\rm top} = \int_M\sqrt{-g}\,\frac{1}{8\pi^2}\,\big(\partial_{S^C}\vartheta_{\rm spec}\big)\,\Theta_{AB}\,\frac{1}{2}\,\epsilon^{\mu\nu\rho\sigma}F^A{}_{\mu\nu}F^B{}_{\rho\sigma}, \tag{2.128}$$

which is the axion-like source quoted in §2.7.12. No higher time derivatives are introduced.

Director Hopf term. In the ordered stratum, the Hopf term $\mathcal{L}^{\rm Hopf}_{\rm top} = \frac{\vartheta_{\rm H}}{4\pi^2}\,\mathcal{A}\wedge d\mathcal{A}$ depends on a locally defined $\mathcal{A}$ with $d\mathcal{A} = \frac{1}{2}\epsilon_{IJK}\,\hat\chi^I d\hat\chi^J\wedge d\hat\chi^K$. Under patch changes $\mathcal{A}\to\mathcal{A}+d\lambda$, the action shifts by a boundary integral, so bulk equations are unchanged for constant $\vartheta_{\rm H}$.

2.11.4 Boundary Terms and Well-Posed Variation

Gravitational boundary term. The Gibbons–Hawking–York term in first-order variables $S^{(\rm EH)}_{\partial M} = \frac{M^2_{\rm eff}}{2}\int_{\partial M}\epsilon_{abcd}\,e^a\wedge e^b\wedge\omega^{cd}$ cancels boundary variations $\sim\delta\omega$ and ensures a Dirichlet problem for e.

Chern–Simons completions. Since $\mathcal{E}_{\rm GB} = d\,\mathrm{CS}_3^{\rm grav}$ and $\Theta_{AB}\,\mathcal{F}^A \wedge \mathcal{F}^B = d\,\mathrm{CS}_3(\Gamma)$, the boundary action $S_{\partial M}^{\rm CS} = \int_{\partial M} \left[\vartheta_{\rm grav}\,\mathrm{CS}_3(\omega) + \frac{\vartheta_{\rm spec}}{8\pi^2}\,\mathrm{CS}_3(\Gamma)\right]$ completes the variational principle and captures edge responses.

2.11.5 Minimal Chart Dictionary and Identities

Useful identities.

$$D_\mu D_\nu - D_\nu D_\mu = F^A{}_{\mu\nu}\,\partial_{S^A}, \qquad \mathbb{D}_{[\lambda} R_{\mu\nu]} = 0, \qquad D_{[\lambda} F_{\mu\nu]} = 0. \tag{2.129}$$

For a fiber scalar $\Psi(x, S)$,

$$[\nabla_\mu, \nabla_\nu]\Psi = -F^A{}_{\mu\nu}\,\partial_{S^A}\Psi. \tag{2.130}$$

Projectors and cones. $\Pi_\parallel^{\mu\nu} = \hat\chi^\mu\hat\chi^\nu$, $\Pi_\perp^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu - \Pi_\parallel^{\mu\nu}$, characteristic equation $-(u\cdot k)^2 + c_\perp^2 k_\perp^2 + c_\parallel^2 k_\parallel^2 = 0$ (cf. §2.8.1).

2.11.6 Summary Box

What this appendix provided. (i) A transparent derivation of the $\partial_S\Gamma$ terms entering the S-equations, Eqs. (2.117)–(2.122). (ii) A differential-form formula for $\delta_S\mathcal{F}$, Eq. (2.125). (iii) A proof that $\mathcal{L}_{\rm top}$ with constant couplings is locally inert (bulk EoMs unchanged), and that $\vartheta_{\rm spec}(\mathcal{S})$ yields axion-like sources, Eq. (2.128). (iv) Boundary completions guaranteeing a well-posed variational problem.

These results justify the compact EL-equations used in §2.7 and prepare the ground for the fully dynamical derivations in Chapter 3, where the implicit $\partial_S\Gamma$ contributions will be evaluated explicitly for each sector and background of interest.

2.12 Preview: Spectral Yukawa Couplings

This preview records the symmetry- and dimensionally-consistent form of *spectral Yukawa* couplings, to be developed fully when fermionic matter is introduced (Chs. 3–4). The guiding idea is that fermion masses and chiral responses arise from couplings to the structural state $\mathcal{S} = (\rho, \eta, T, \chi^I, \alpha)$ and to spectral geometry, rather than from an elementary Higgs field.

Fermionic Kinematics (to be elaborated later)

We will consider a Dirac spinor Ψ with kinetic term built from the state-built tetrad:

$$\mathcal{L}_\Psi^{\rm kin} = \bar\Psi\, i\,\gamma^a e_a{}^\mu \Big(\partial_\mu + \tfrac{1}{4}\,\omega_{\mu bc}\,\gamma^{bc}\Big)\Psi \;+\; \upsilon_u(\mathcal{S})\,\bar\Psi\,\gamma^\mu u_\mu \Psi \;+\; \upsilon_\chi(\mathcal{S})\,\bar\Psi\,\gamma^\mu\hat\chi_\mu\Psi, \tag{2.131}$$

where the last two terms are symmetry-allowed anisotropic but *derivative-free* couplings that renormalize the fermionic "speeds" along u^μ and $\hat\chi^\mu$. They preserve locality and hyperbolicity if $|\upsilon_{u,\chi}|$ are sufficiently small (EFT regime).

Spectral Yukawa Couplings: Minimal Set

Dimensional analysis in $3+1$ D gives $[\bar\Psi\Psi] = 3$. Hence any scalar spectral Yukawa of the form $y_S(\mathcal{S})\,\bar\Psi\Psi$ requires $[y_S] = +1$. With S^A dimensionless, the natural choice is to scale with the effective mass scale $M_{\rm eff}(\rho)$ (or the EFT cutoff $\Lambda_{\rm eff}$):

$$\boxed{\mathcal{L}_{\rm spec}^{\rm Yuk} = y_S(\mathcal{S})\,\bar\Psi\Psi + i\,y_5(\mathcal{S})\,\bar\Psi\gamma_5\Psi + y_\parallel(\mathcal{S})\,\bar\Psi\,\gamma^\mu\hat\chi_\mu\Psi + y_u(\mathcal{S})\,\bar\Psi\,\gamma^\mu u_\mu\Psi\ ,} \tag{2.132}$$

with

$$y_S(\mathcal{S}) = \tilde y_S\,M_{\rm eff}(\rho), \qquad y_5(\mathcal{S}) = \tilde y_5\,M_{\rm eff}(\rho), \qquad y_{\parallel,u}(\mathcal{S}) = \tilde y_{\parallel,u}\,, \tag{2.133}$$

where tilded couplings are dimensionless functions of $\mathcal{S}$ (slowly varying in the EFT regime). The term y_S yields an *environmental* mass $m_f(\mathcal{S}) = y_S(\mathcal{S})$; the pseudoscalar y_5 controls parity/CP response and will be constrained by phenomenology. Vectorial pieces $y_\parallel, y_u$ encode permissible anisotropies inherited from $\hat\chi^\mu$ and u^μ.

Symmetry gates. These operators are Lorentz covariant with respect to g_{eff} and preserve the global $U(1)_\Phi$ (they do not involve Φ). Time-shift symmetry for T is respected since T enters only through $\mathcal{S}$ (no undifferentiated T appears).

Hybrid Spectral–Scalar Couplings (Optional)

If one allows Yukawa-like mixing with the complex scalar Φ while preserving its global $U(1)$, the only allowed monomials are *neutral* in Φ:

$$\mathcal{L}^{\text{Yuk}}_{\Phi\text{-spec}} = y_{|\Phi|}(\mathcal{S})\,|\Phi|\,\bar{\Psi}\Psi \;+\; y_{|\Phi|^2}(\mathcal{S})\,|\Phi|^2\,\bar{\Psi}\Psi, \tag{2.134}$$

which effectively *modulate* m_f through the modulus $|\Phi|$. Linear terms in Φ or $\Phi^\dagger$ are *forbidden* by the global $U(1)_\Phi$.

Axion-like Spectral Couplings (Topology–Fermions)

If $\vartheta_{\text{spec}}(\mathcal{S})$ in the spectral θ-term varies, one may also include a pseudoscalar coupling

$$\mathcal{L}_{\text{ax}} = \frac{c_{\text{ax}}(\mathcal{S})}{M_{\text{eff}}(\rho)}\,\bar{\Psi}\gamma^\mu\gamma_5\Psi\,\partial_\mu\vartheta_{\text{spec}}(\mathcal{S}), \tag{2.135}$$

which is derivative-suppressed and consistent with EFT power counting; it ties chiral fermion response to spectral topology.

Dimensional and Causality Checks

In all cases $[\mathcal{L}] = +4$: $[y_S] = [y_5] = +1,\ [y_{\parallel,u}] = 0,\ [c_{\text{ax}}] = 0$. Hyperbolicity and WEC/DEC remain ensured provided $|v_{u,\chi}|, |y_{\parallel,u}| \ll 1$ (relative to the canonical kinetic terms), and $m_f(\mathcal{S}) = y_S(\mathcal{S}) \geq 0$.

Roadmap

A full treatment (Chs. 3–4) will: (i) derive the Dirac equation with $m_f(\mathcal{S})$ and anisotropic velocities; (ii) analyze spectra and stability in ordered vs. isotropic strata; (iii) obtain phenomenological bounds on $y_5, y_{\parallel,u}$ from parity and birefringence-like tests.

Remark (pre-geometric construction). Throughout this chapter, the spectral derivative D_μ is defined via a constitutive Ehresmann connection $\Gamma^A{}_\mu(x,\sigma)$ that does not depend on g_{eff}; the emergent metric is built afterwards from $(T, \chi^\mu, \ldots)$. Hence no logical circularity arises.

2.13 Tables and Illustrative Figures

Table 2.1: Core definitions and symbols used in Chapter 2. Compact summary of fields, geometric objects, and constraints.

Symbol	Type	Constraints	Meaning / Role
ρ	real scalar	$\rho > 0$	Spectral (aether) density; sets M_{eff} and local stiffness.
η	real scalar	$\eta \geq 0$	Spectral nonlinearity/viscosity; controls dissipation and entropy production.
T	real scalar	$\partial_\mu T$ timelike	Internal temporal phase; defines preferred time flow u^μ.
χ_I	internal comps.	$\hat{\chi}^2 = +1$	Polarization/direction field; induces anisotropy and prepares gauge emergence.
α	phase (real)	periodic	Spectral $U(1)$ phase; source of spectral gauge responses.
Φ	complex scalar	—	Matter carrier living on top of structural sector.
u^μ	vector	$u^\mu u_\mu = -1$	Timelike flow built from T (and χ-mixing if allowed).
$\hat{\chi}^\mu$	vector	$u \cdot \hat{\chi} = 0$	Unit spacelike direction selecting anisotropy axis.
$e^a{}_\mu$	coframe	$\det e \neq 0$	State-built tetrad; defines g_{eff}.
g_{eff}	metric	$(-,+,+,+)$	Emergent effective metric for propagation/causality.
$\Pi^{\mu\nu}$	projector	$u_\mu \Pi^{\mu\nu} = 0$	Projector onto the spatial subspace orthogonal to u^μ.
D_μ	derivation	Leibniz	Spectral covariant differential (horizontal lift on bundle).
$\omega_\mu{}^{ab}$	connection	antisym. in ab	Emergent spin-connection (Palatini-like split + contortion).
F_{spec}	curvature	two-form	Spectral curvature; drives geometric/gauge responses.
$\mathcal{L}_{\text{struct}}$	density	EFT	Structural Lagrangian (kinetic + potential + curvature-safe).
$\mathcal{L}_\Phi$	density	$U(1)$	Matter Lagrangian with spectral Yukawa couplings.
$M_{\text{eff}}(\rho)$	scale	> 0	Effective Planck scale (gravity sector) set by ρ.
$c_\perp,\ \delta c$	speeds	< 1	Anisotropic wave speeds in g_{eff}.

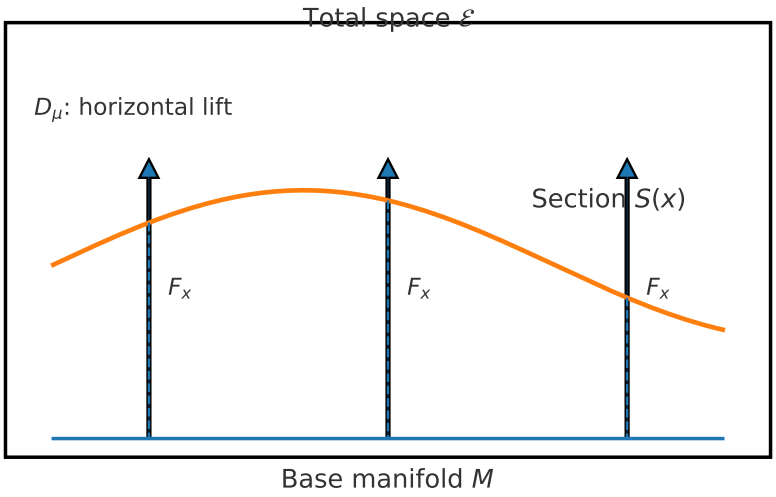

Figure 2.1: Bundle geometry and spectral section. The total space $\mathcal{E}$ fibers over the base M with fibers F_x. A smooth section $S(x)$ selects a point in each fiber; dashed lines indicate the projection. The spectral differential D_μ is the horizontal lift defining dynamics on $\mathcal{E}$ consistent with pre-geometric emergence.

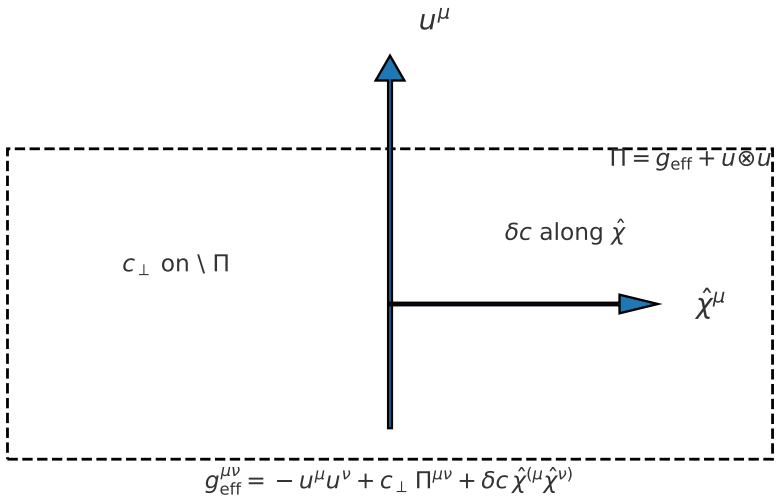
u^μ
$\Pi = g_{\text{eff}} + u \otimes u$
δc along $\hat{\chi}$
$c_\perp$ on $\setminus \Pi$
$\hat{\chi}^\mu$
$g^{\mu\nu}_{\text{eff}} = -u^\mu u^\nu + c_\perp \Pi^{\mu\nu} + \delta c\, \hat{\chi}^{(\mu} \hat{\chi}^{\nu)}$

Figure 2.2: Constructing the emergent metric. The unit timelike vector u^μ defines the temporal direction; $\Pi^{\mu\nu}$ projects orthogonally to u^μ. The unit spacelike $\hat{\chi}^\mu$ singles out an anisotropy axis within the spatial subspace. The effective metric takes the form $g^{\mu\nu}_{\text{eff}} = -u^\mu u^\nu + c_\perp \Pi^{\mu\nu} + \delta c\, \hat{\chi}^{(\mu} \hat{\chi}^{\nu)}$, ensuring controlled anisotropy and causal propagation.

Chapter 3

Chapter 3: Equations of Motion: From Action to Dynamics

3.0 Prelude

3.1 3.0 Introduction and Methodological Principles

This chapter turns the pre-geometric construction of Chapter 2 into a closed set of dynamical equations. Our goal is to derive, field by field, the Euler–Lagrange equations for the five structural (spectral) fields

$$\mathcal{S}(x) = \big(\rho(x),\, \eta(x),\, T(x),\, \chi^I(x),\, \alpha(x)\big), \qquad I = 1, 2, 3,$$

the complex matter field $\Phi(x)$, and the emergent metric g_{eff} built from the state. We adhere strictly to the pre-geometric ethos: *no external spacetime geometry is assumed a priori*; instead, geometry is reconstructed from the spectral state and its kinematics.

Guiding methodological choices. To avoid logical circularity and higher-derivative instabilities, we adopt:

- **Palatini split for geometry.** The tetrad $e^a{}_\mu$ and the spin-connection $\omega_\mu{}^{ab}$ are treated as independent variational variables; torsion is set to zero on-shell. The emergent metric is $g_{\text{eff}} = \eta_{ab}\, e^a \otimes e^b$.
- **Constitutive Ehresmann connection.** The spectral derivative is the horizontal lift

$$D_\mu \;=\; \partial_\mu + \Gamma^A{}_\mu(x,\sigma)\, \partial_A, \qquad F^A{}_{\mu\nu} \;=\; \partial_{[\mu}\Gamma^A{}_{\nu]} + \Gamma^B{}_{[\mu}\, \partial_B \Gamma^A{}_{\nu]}, \tag{3.1}$$

  where $\Gamma^A{}_\mu$ is *state-built/constitutive* (not varied independently). This keeps D_μ metric-free and avoids circularity.
- **State-built kinematics.** The timelike u^μ and the director $\hat{\chi}^\mu$ are constructed from (T, χ^I) and normalized by Lagrange multipliers to enforce $u^2 = -1$, $\hat{\chi}^2 = +1$, $u\cdot\hat{\chi} = 0$.
- **Topological sector isolated.** Gauss–Bonnet, Pontryagin, Holst/Nieh–Yan, and spectral θ-terms are included but, for constant couplings, do not alter bulk equations; they contribute only through boundary/quantization data.

What is varied and what is held constitutive. The dynamical variables are

$$\{\, S^A(x)\,\} \equiv \{\rho, \eta, T, \chi^I, \alpha\}, \quad \Phi(x), \quad e^a{}_\mu(x), \quad \omega_\mu{}^{ab}(x).$$

The constitutive (non-variational) ingredients are the Ehresmann coefficients $\Gamma^A{}_\mu(x,\sigma)$ and any state-dependent tensors (e.g. fiber metrics $Z_{AB}(\sigma)$, $Y_{AB}(\sigma)$) specified in Chapter **??**. Their implicit dependence on S^A enters the field equations via $\partial_S\Gamma$, $\partial_S Z$, $\partial_S Y$ after variation (cf. Appendix 2.A).

Causality and hyperbolicity gates. All principal symbols are tuned to lie within the null cone of $g_{\rm eff}$:

$$Z_{AB} \succ 0, \quad Y_{AB} \succ 0, \quad \kappa_\Phi > 0, \qquad \text{and} \quad c_\perp^2 > 0,\ c_\parallel^2 > 0,\ c_\perp^2 + \delta c > 0,$$

ensuring well-posedness and positive energy. These conditions will be stated explicitly after each equation.

Boundary terms and well-posed variational problem. We supplement the bulk action by the Gibbons–Hawking–York boundary term in first-order variables and by the Chern–Simons forms matching the chosen topological densities, so that Dirichlet data for $e^a{}_\mu$ (and appropriate boundary conditions for ω) render the variational problem well defined.

Roadmap of the chapter. Section 3.2 presents the full unified Lagrangian and identifies its sectors (gravitational, structural, matter, mixing, constraints, topological). Sections 3.4.1–3.4.5 derive, *field by field*, the Euler–Lagrange equations for $\rho, \eta, T, \chi^I, \alpha$, each followed by a dedicated *physical reading* paragraph and stability/causality gates. Section 3.5 treats the matter field Φ and its conserved current. Section 3.6 derives the effective Einstein equation in the Palatini formalism, displaying the Brans–Dicke–like terms induced by $M_{\rm eff}(\rho)$. Conservation laws and Bianchi identities are summarized in Section **??**. Linearized cones and characteristic speeds are discussed in Section 3.8. Short illustrative examples (homogeneous limit, plane spectral waves) appear in Section **??**, and entropy production/decoherence gates are previewed in Section 3.10. We conclude with observational signposts in Section **??** and a map to subsequent chapters in Section **??**.

With these principles and conventions fixed, we now present the complete unified Lagrangian.

3.2 Unified Lagrangian and Sector Decomposition

We collect here the full dynamical action and split it into clearly identified sectors. Spacetime indices are $\mu, \nu, \ldots$; internal spectral indices are $A, B = 1, \ldots, 7$ with $S^A \equiv (\rho, \eta, T, \chi^I, \alpha)$ and $I = 1, 2, 3$. The tetrad is $e^a{}_\mu$ $(a, b = 0, 1, 2, 3)$, $e \equiv \det(e^a{}_\mu)$, and $g_{\rm eff} = \eta_{ab}\, e^a \otimes e^b$. The spectral (Ehresmann) lift is

$$D_\mu \;=\; \partial_\mu + \Gamma^A{}_\mu(x, \sigma)\, \partial_A, \qquad F^A{}_{\mu\nu} \;=\; \partial_{[\mu}\Gamma^A{}_{\nu]} + \Gamma^B{}_{[\mu}\, \partial_B \Gamma^A{}_{\nu]}\,, \tag{3.2}$$

with $\Gamma^A{}_\mu$ *constitutive* (state-built, non-variational). Evaluated on the section, $D_\mu S^A(x) = \partial_\mu S^A(x) + \Gamma^A{}_\mu\big(x, S(x)\big)$. We use positive-definite fiber bilinears $Z_{AB}(S)$, $Y_{AB}(S)$ and a scalar potential $U(S)$.

Action and sector split. The total action reads

$$S_{\rm tot} \;=\; \int_M d^4x\, e \left[\mathcal{L}_{\rm grav} + \mathcal{L}_{\rm struct} + \mathcal{L}_\Phi + \mathcal{L}_{\rm mix} + \mathcal{L}_{\rm constr}\right] \;+\; S_{\rm top} \;+\; S_{\partial M}, \tag{3.3}$$

where:

(i) Gravitational sector (Palatini)

$$\boxed{\mathcal{L}_{\rm grav} \;=\; \frac{M_{\rm eff}^2(\rho)}{2}\, R(\omega, e)} \tag{3.4}$$

with $R(\omega, e)$ the Ricci scalar built from the spin-connection $\omega_\mu{}^{ab}$ and the tetrad $e^a{}_\mu$ in the first-order (Palatini) formalism.[1] The effective Planck mass $M_{\rm eff}(\rho)$ depends on the aether density, encoding environmental running of gravity. In units $\hbar = c = 1$, $[M_{\rm eff}] = +1$.

(ii) Structural (spectral) sector

$$\boxed{\mathcal{L}_{\rm struct} \;=\; \frac{1}{2}\, Z_{AB}(S)\, g_{\rm eff}^{\mu\nu}\, D_\mu S^A\, D_\nu S^B \;-\; U(S) \;-\; \frac{1}{4}\, Y_{AB}(S)\, F^A_{\mu\nu} F^{B\,\mu\nu}} \tag{3.5}$$

where $Z_{AB} \succ 0$ sets the kinetic geometry on the fiber, $U(S)$ governs the stratified phases (isotropic vs. ordered), and $Y_{AB} \succ 0$ weights the spectral curvature energy. All three may vary with S (constitutive response). No prior spacetime metric enters D_μ or $F_{\mu\nu}$.

[1] Torsion is set to zero on-shell by the ω-variation; cf. §3.6.

(iii) Matter (complex scalar) sector

We take an anisotropic-but-causal kinetic metric for Φ,

$$G^{\mu\nu}_{(\mathrm{ani})}(S) \;\equiv\; g^{\mu\nu}_{\mathrm{eff}} \;+\; \delta c(S)\,\hat{\chi}^{\mu}\hat{\chi}^{\nu}, \qquad \delta c > -1, \tag{3.6}$$

with $\hat{\chi}^{\mu}$ the normalized director (see below). Then

$$\boxed{\mathcal{L}_{\Phi} \;=\; \kappa_{\Phi}(S)\,G^{\mu\nu}_{(\mathrm{ani})}(S)\,(\nabla_{\mu}\Phi)^{*}(\nabla_{\nu}\Phi) \;-\; V_{\Phi}\big(|\Phi|;S\big),} \tag{3.7}$$

where $\kappa_{\Phi}(S) > 0$ and V_{Φ} may include an environmental mass $m^2_{\mathrm{eff}}(S)$ and stabilizing interactions. The covariant derivative ∇_{μ} is Levi–Civita with respect to g_{eff} on spacetime indices; $U(1)_{\Phi}$ is global at this stage.

(iv) Mixing sector (safe, symmetry-preserving)

We allow low-derivative, symmetry-consistent mixing between matter current and structural flows, keeping equations second order and preserving $U(1)_{\Phi}$ and T-shift:

$$\boxed{\mathcal{L}_{\mathrm{mix}} \;=\; \lambda_1(S)\,J^{\mu}_{\Phi}\,D_{\mu}\rho \;+\; \lambda_2(S)\,J^{\mu}_{\Phi}\,D_{\mu}\eta \;+\; \lambda_3(S)\,J^{\mu}_{\Phi}\,\hat{\chi}_{\mu} \;+\; \lambda_4(S)\,(\nabla_{\mu}\Phi)^{*}(\nabla_{\nu}\Phi)\,\hat{\chi}^{(\mu}u^{\nu)} \;+\; \cdots} \tag{3.8}$$

with $J^{\mu}_{\Phi} := \mathrm{Im}\big(\kappa_{\Phi}\,G^{\mu\nu}_{(\mathrm{ani})}\,\Phi^{*}\nabla_{\nu}\Phi\big)$. The ellipsis stands for additional, EFT-suppressed operators consistent with the same symmetries (e.g. contractions with the projector $\Pi^{\mu\nu} = g^{\mu\nu}_{\mathrm{eff}} + u^{\mu}u^{\nu} - \hat{\chi}^{\mu}\hat{\chi}^{\nu}$). All $\lambda_i(S)$ are dimensionless or inverse-mass–suppressed as needed to keep $[\mathcal{L}] = +4$.

(v) Constraint sector

The timelike flow u^{μ} and the spacelike director $\hat{\chi}^{\mu}$ are built from (T, χ^{I}) and normalized by Lagrange multipliers:

$$\boxed{\mathcal{L}_{\mathrm{constr}} \;=\; \lambda_u\,(u^{\mu}u_{\mu} + 1) \;+\; \lambda_{\chi}\,(\hat{\chi}^{\mu}\hat{\chi}_{\mu} - 1) \;+\; \lambda_{u\chi}\,(u^{\mu}\hat{\chi}_{\mu}) \;+\; \Lambda_{\mathrm{frame}}\cdot\mathcal{C}[u,\hat{\chi};S],} \tag{3.9}$$

where $\mathcal{C}[\cdot]$ collects mild frame-building constraints (e.g. orthogonality of $u^{a}, \hat{\chi}^{a}$ in the state-built tetrad). A convenient representative is $u^{\mu} \propto a(S)\,\partial^{\mu}T + b(S)\,\hat{\chi}^{\mu}$, followed by on-shell normalization via λ_u; see Chap. 2 for explicit constructions.

(vi) Topological sector and boundary terms

The topological action contains metric/topology densities with *constant* couplings (no bulk EL-contributions):

$$\boxed{S_{\mathrm{top}} \;=\; \int_{M} d^4x\,e\,\Big(\lambda_{\mathrm{GB}}\,\mathcal{G}_{\mathrm{GB}} \;+\; \frac{\theta_{\mathrm{P}}}{32\pi^2}\,R\widetilde{R} \;+\; \frac{\theta_{\mathrm{spec}}}{32\pi^2}\,\mathcal{F}\widetilde{\mathcal{F}} \;+\; \kappa_{\mathrm{Hopf}}\,\mathcal{H}[\hat{\chi}]\Big),} \tag{3.10}$$

with Gauss–Bonnet density $\mathcal{G}_{\mathrm{GB}} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ (topological in 4D), Pontryagin $R\widetilde{R}$, a spectral Pontryagin-type density $\mathcal{F}\widetilde{\mathcal{F}}$ built from $F^{A}{}_{\mu\nu}$ and a fiber metric, and a Hopf functional $\mathcal{H}[\hat{\chi}]$ for director textures. Boundary terms ensure a well-posed variational problem:

$$\boxed{S_{\partial M} \;=\; \frac{M^2_{\mathrm{eff}}}{2}\int_{\partial M}\epsilon_{abcd}\,e^{a}\wedge e^{b}\wedge\omega^{cd} \;+\; S_{\mathrm{CS}}[\,\omega\,] \;+\; S_{\mathrm{CS}}[\,\Gamma\,],} \tag{3.11}$$

with Chern–Simons forms matching the bulk topological densities.

Anisotropy gate and causality. The kinetic tensor $G^{\mu\nu}_{(\mathrm{ani})}$ in (3.6) satisfies $G^{\mu\nu}_{(\mathrm{ani})}v_{\mu}v_{\nu} \le 0$ for any causal v^{μ} of g_{eff}, provided $\delta c > -1$. Together with $Z_{AB} \succ 0$, $Y_{AB} \succ 0$, $\kappa_{\Phi} > 0$, these choices keep all principal characteristics inside (or on) the light cone of g_{eff}.

Dimensional bookkeeping (natural units).

$$[\mathcal{L}] = +4, \quad [e] = 0, \quad [x^\mu] = -1, \quad [e^a{}_\mu] = 0, \quad [\Phi] = +1, \quad [M_{\rm eff}] = +1, \quad [Z_{AB}] = [Y_{AB}] = 0.$$

State functions such as $\kappa_\Phi(S)$, $\delta c(S)$, and the $\lambda_i(S)$ are dimensionless unless otherwise indicated (EFT operators carry the appropriate inverse powers of a cutoff $\Lambda_{\rm eff} \sim M_{\rm eff}$).

What is varied. In the sequel we vary $\{S^A\}$, Φ, $e^a{}_\mu$, and $\omega_\mu{}^{ab}$, while keeping $\Gamma^A{}_\mu$ constitutive. Implicit S-dependences (e.g. $\partial_S\Gamma$, $\partial_S Z$, $\partial_S Y$, $\partial_S\kappa_\Phi$, $\partial_S G_{(\rm ani)}$) are included systematically after integration by parts. The resulting field equations appear, field-by-field, in §3.4.1–§3.4.5 and for Φ in §3.5; the geometric (Palatini) equations are derived in §3.6.

With the unified Lagrangian (3.3)–(3.11) specified, we now present the variational rules, identify implicit contributions from constitutive tensors, and then derive the Euler–Lagrange equations for each field with an immediate physical reading after each derivation.

3.3 Variational Rules: Explicit vs. Implicit Dependencies

Before deriving the field equations, we collect the variational identities needed to handle (i) explicit dependence of the Lagrangian on the structural fields $S^A = (\rho, \eta, T, \chi^I, \alpha)$ and (ii) *implicit* dependence through the constitutive tensors and the emergent kinematics ($\Gamma^A{}_\mu$, Z_{AB}, Y_{AB}, $g_{\rm eff}$, $G_{(\rm ani)}$, $M_{\rm eff}$). Throughout this section, the Ehresmann coefficients $\Gamma^A{}_\mu(x,\sigma)$ are *constitutive* (non-variational); their dependence on S contributes only via $\partial_S\Gamma$.

3.3.1 Notation and basic identities

Spacetime covariant derivatives compatible with $g_{\rm eff}$ are denoted by $\widetilde{\nabla}_\mu$, while the horizontal lift on the total space is

$$D_\mu \;=\; \partial_\mu + \Gamma^A{}_\mu(x,\sigma)\,\partial_A, \qquad F^A{}_{\mu\nu} \;=\; \partial_{[\mu}\Gamma^A{}_{\nu]} + \Gamma^B{}_{[\mu}\,\partial_B\Gamma^A{}_{\nu]}\,. \tag{3.12}$$

Evaluated on the section $S^A(x)$,

$$D_\mu S^A(x) = \partial_\mu S^A(x) + \Gamma^A{}_\mu\big(x, S(x)\big), \qquad \delta\big(D_\mu S^A\big) = \partial_\mu \delta S^A + \big(\partial_{S^C}\Gamma^A{}_\mu\big)\,\delta S^C. \tag{3.13}$$

The fiber metrics and weights are $Z_{AB}(S)$, $Y_{AB}(S)$; the scalar potential is $U(S)$. The matter-sector kinetic tensor is $G^{\mu\nu}_{(\rm ani)}(S) = g^{\mu\nu}_{\rm eff} + \delta c(S)\,\hat\chi^\mu\hat\chi^\nu$.

Metric variations. In Palatini variables, $e^a{}_\mu$ and $\omega_\mu{}^{ab}$ are independent. We keep $g_{\rm eff} = \eta_{ab}\,e^a\otimes e^b$ explicit and isolate:

$$\delta g^{\mu\nu}_{\rm eff} = \frac{\partial g^{\mu\nu}_{\rm eff}}{\partial e^a{}_\lambda}\,\delta e^a{}_\lambda + \frac{\partial g^{\mu\nu}_{\rm eff}}{\partial S^C}\,\delta S^C, \qquad \delta G^{\mu\nu}_{(\rm ani)} = \delta g^{\mu\nu}_{\rm eff} + (\partial_{S^C}\delta c)\,\hat\chi^\mu\hat\chi^\nu\,\delta S^C + \delta(\hat\chi^\mu\hat\chi^\nu). \tag{3.14}$$

The variation $\delta(\hat\chi^\mu\hat\chi^\nu)$ is handled via the constraint sector (orthonormalization enforced on-shell), so its contribution can be transferred to $\mathcal{L}_{\rm constr}$.

3.3.2 Structural kinetic term

The variation of the kinetic term $\mathcal{K}_{\rm struct} = \frac12\,Z_{AB}(S)\,g^{\mu\nu}_{\rm eff}D_\mu S^A D_\nu S^B$ splits as

$$\begin{aligned}\delta\mathcal{K}_{\rm struct} = {}& \tfrac12\,(\partial_{S^C}Z_{AB})\,\delta S^C\,g^{\mu\nu}_{\rm eff}D_\mu S^A D_\nu S^B + \tfrac12\,Z_{AB}\,\delta g^{\mu\nu}_{\rm eff}\,D_\mu S^A D_\nu S^B \\ & + Z_{CB}\,g^{\mu\nu}_{\rm eff}\,\delta(D_\mu S^C)\,D_\nu S^B.\end{aligned} \tag{3.15}$$

Using (3.13) and integrating by parts (w.r.t. $\widetilde{\nabla}_\mu$) we obtain the bulk contribution

$$\delta\mathcal{K}_{\rm struct}\Big|_{\rm bulk} = -\,\delta S^C\,\widetilde{\nabla}_\mu\big(Z_{CB}\,g^{\mu\nu}_{\rm eff}D_\nu S^B\big) + \tfrac12\,\delta S^C\,(\partial_{S^C}Z_{AB})\,g^{\mu\nu}_{\rm eff}D_\mu S^A D_\nu S^B + \delta S^C\,\mathfrak{S}^{(\rm kin)}_C, \tag{3.16}$$

where the "kinematic feedback" term gathers the implicit metric dependence:

$$\mathfrak{S}^{(\rm kin)}_C := \frac12\,\frac{\partial g^{\mu\nu}_{\rm eff}}{\partial S^C}\,Z_{AB}\,D_\mu S^A D_\nu S^B. \tag{3.17}$$

Boundary pieces are canceled by $S_{\partial M}$ (GHY) and need not be tracked further in the bulk EL-equations.

3.3.3 Spectral curvature term

Varying $\mathcal{C}_{\rm spec} = -\frac{1}{4}\, Y_{AB}(S)\, F^{A}_{\mu\nu} F^{B\,\mu\nu}$ and using that $\Gamma^{A}{}_{\mu}$ is constitutive yields

$$\delta\mathcal{C}_{\rm spec} = -\tfrac{1}{4}\,(\partial_{S^C} Y_{AB})\,\delta S^C\, F^{A}_{\mu\nu} F^{B\,\mu\nu} \;-\; \tfrac{1}{2}\, Y_{AB}\, F^{A\,\mu\nu}\,\delta F^{B}_{\mu\nu}. \tag{3.18}$$

A key identity (cf. Appendix 2.A) is

$$Y_{AB}\, F^{A\,\mu\nu}\,\delta F^{B}_{\mu\nu} = D_{[\mu}\big(Y_{AB} F^{A\,\mu\nu}\big)\,\delta\Gamma^{B}{}_{\nu} \;-\; \tfrac{1}{2}\, Y_{AB}\, F^{A\,\mu\nu}\,\mathcal{Q}^{B}{}_{\mu\nu C}\,\delta S^C, \qquad \delta\Gamma^{B}{}_{\nu} = (\partial_{S^C}\Gamma^{B}{}_{\nu})\,\delta S^C, \tag{3.19}$$

where $\mathcal{Q}^{B}{}_{\mu\nu C}$ collects algebraic vertical-derivative contributions (from $\partial_S\partial_\sigma\Gamma$). After integrating by parts, the total bulk variation becomes

$$\delta\mathcal{C}_{\rm spec}\Big|_{\rm bulk} = -\,\delta S^C\left[\frac{1}{4}\,(\partial_{S^C} Y_{AB})\, F^{A}\cdot F^{B} - \frac{1}{2}\, D_{[\mu}\big(Y_{AB} F^{A\,\mu\nu}\big)\,\partial_{S^C}\Gamma^{B}{}_{\nu} + \frac{1}{2}\, Y_{AB}\, F^{A\,\mu\nu}\,\mathcal{Q}^{B}{}_{\mu\nu C}\right]. \tag{3.20}$$

3.3.4 Potential, mixing, and constraint sectors

Variations of $U(S)$, $\mathcal{L}_{\rm mix}$, and $\mathcal{L}_{\rm constr}$ give

$$\delta\big(-U(S)\big) = -(\partial_{S^C}U)\,\delta S^C, \qquad \delta\mathcal{L}_{\rm mix} = \frac{\partial\mathcal{L}_{\rm mix}}{\partial S^C}\,\delta S^C + \frac{\partial\mathcal{L}_{\rm mix}}{\partial\Phi}\,\delta\Phi + \frac{\partial\mathcal{L}_{\rm mix}}{\partial\Phi^*}\,\delta\Phi^* + \cdots, \tag{3.21}$$

and similarly for $\mathcal{L}_{\rm constr}$ (the multipliers enforce orthonormality and orthogonality on-shell).

3.3.5 Matter-sector variation and implicit feedback

For $\mathcal{L}_\Phi = \kappa_\Phi(S)\, G^{\mu\nu}_{\rm (ani)}(S)(\nabla_\mu\Phi)^*(\nabla_\nu\Phi) - V_\Phi(|\Phi|;S)$ we obtain

$$\begin{aligned}\delta\mathcal{L}_\Phi = {}& \underbrace{\kappa_\Phi\,\delta G^{\mu\nu}_{\rm (ani)}\,(\nabla_\mu\Phi)^*(\nabla_\nu\Phi)}_{\text{metric/anisotropy feedback}} + \underbrace{(\partial_{S^C}\kappa_\Phi)\,\delta S^C\, G^{\mu\nu}_{\rm (ani)}\,(\nabla_\mu\Phi)^*(\nabla_\nu\Phi)}_{\text{state dependence}} \\ & + \kappa_\Phi\, G^{\mu\nu}_{\rm (ani)}\,\delta\big((\nabla_\mu\Phi)^*(\nabla_\nu\Phi)\big) - \frac{\partial V_\Phi}{\partial\Phi}\,\delta\Phi - \frac{\partial V_\Phi}{\partial\Phi^*}\,\delta\Phi^* - (\partial_{S^C}V_\Phi)\,\delta S^C.\end{aligned} \tag{3.22}$$

After integration by parts (w.r.t. $\widetilde{\nabla}_\mu$), the bulk contributions split into the EL-equation for Φ and the structural feedback term

$$\mathfrak{S}^{(\Phi)}_C := \frac{\partial G^{\mu\nu}_{\rm (ani)}}{\partial S^C}\,\kappa_\Phi\,(\nabla_\mu\Phi)^*(\nabla_\nu\Phi) + (\partial_{S^C}\kappa_\Phi)\, G^{\mu\nu}_{\rm (ani)}\,(\nabla_\mu\Phi)^*(\nabla_\nu\Phi) - \partial_{S^C}V_\Phi. \tag{3.23}$$

3.3.6 Gravitational sector (Palatini) and $M_{\rm eff}$-dependence

For $\mathcal{L}_{\rm grav} = \frac{1}{2}M^2_{\rm eff}(\rho)\,R(\omega,e)$, the variation with respect to S^C contributes

$$\delta\mathcal{L}_{\rm grav}\Big|_S = \frac{1}{2}\,R(\omega,e)\,\partial_{S^C}M^2_{\rm eff}\,\delta S^C, \tag{3.24}$$

while variation w.r.t. ω and e (with torsion set to zero on-shell) yields the effective Einstein equation displaying the Brans–Dicke–like derivatives of $M^2_{\rm eff}(\rho)$ (derived in §3.6).

3.3.7 Collecting the structural feedback

For compactness in the field-by-field EL-equations we package all implicit dependencies into

$$\boxed{\mathfrak{S}_C \;:=\; \underbrace{\tfrac{1}{2}\,\frac{\partial g^{\mu\nu}_{\rm eff}}{\partial S^C}\, Z_{AB}\, D_\mu S^A D_\nu S^B}_{\text{from structural kinetic}} \;+\; \underbrace{\Big[\text{Eq. (3.23)}\Big]}_{\text{from matter sector}} \;+\; \underbrace{\Big[\text{Eq. (3.20)}\Big]}_{\text{from spectral curvature}} \;+\; \underbrace{\tfrac{1}{2}\,R(\omega,e)\,\partial_{S^C}M^2_{\rm eff}}_{\text{from gravity}} \;+\; \cdots} \tag{3.25}$$

where the ellipsis accounts for admissible mixing terms and constraint-induced metric pieces. With these rules, the Euler–Lagrange equation for a generic structural component S^C takes the canonical form

$$-\widetilde{\nabla}_\mu\big(Z_{CB}\, g_{\rm eff}^{\mu\nu} D_\nu S^B\big) \;+\; \tfrac{1}{2}\,(\partial_{S^C} Z_{AB})\, g_{\rm eff}^{\mu\nu} D_\mu S^A D_\nu S^B \;-\; \partial_{S^C} U \;+\; \mathfrak{S}_C \;=\; 0, \tag{3.26}$$

to which we will add, when appropriate, the explicit $Y_{AB} F^A\cdot F^B$-pieces and the $D_{[\mu}(YF)\,\partial_S\Gamma$ term from (3.20).

Boundary terms. All integrations by parts generate surface terms absorbed by $S_{\partial M}$: the Gibbons–Hawking–York term for the gravitational sector and the appropriate Chern–Simons counterparts for the topological densities. We thus focus exclusively on the bulk equations in the sequel.

Equipped with these identities, we now proceed to derive the Euler–Lagrange equations for each of the five structural fields, one by one, followed in each case by a physical interpretation and stability/causality gates.

3.4 Field Equations for the Structural Fields — Part I

We now derive the Euler–Lagrange equations for the structural (spectral) fields one by one. For each field we present: (i) the full variational equation obtained from the master rule (3.26) including all implicit (feedback) contributions packaged in $\mathfrak{S}_C$ as defined in (3.25); (ii) a *physical reading* clarifying the role of each term; and (iii) minimal *stability/causality gates*.

3.4.1 Density field ρ

Derivation. Setting $C=\rho$ in the master structural EL-equation (3.26) and adding explicitly the spectral-curvature pieces from (3.20) yields

$$\begin{aligned}
0 = &-\widetilde{\nabla}_\mu\big(Z_{\rho B}\, g_{\rm eff}^{\mu\nu} D_\nu S^B\big) \;+\; \frac{1}{2}\,(\partial_\rho Z_{AB})\, g_{\rm eff}^{\mu\nu} D_\mu S^A D_\nu S^B \;-\; \partial_\rho U \\
&-\; \frac{1}{4}\,(\partial_\rho Y_{AB})\, F^A_{\mu\nu} F^{B\,\mu\nu} \;+\; \frac{1}{2}\, D_{[\mu}\Big(Y_{AB}\, F^{A\,\mu\nu}\Big)\,\partial_\rho \Gamma^B{}_\nu \;-\; \frac{1}{2}\, Y_{AB}\, F^{A\,\mu\nu}\, \mathcal{Q}^B{}_{\mu\nu\rho} \\
&+\; \underbrace{\frac{\partial g_{\rm eff}^{\mu\nu}}{\partial\rho}\,\frac{1}{2}\, Z_{AB}\, D_\mu S^A D_\nu S^B}_{\text{metric feedback from structural kinetics}} \;+\; \underbrace{\frac{\partial G^{\mu\nu}_{\rm (ani)}}{\partial\rho}\,\kappa_\Phi\,(\nabla_\mu\Phi)^*(\nabla_\nu\Phi)}_{\text{anisotropy feedback on matter}} \;+\; \underbrace{(\partial_\rho\kappa_\Phi)\, G^{\mu\nu}_{\rm (ani)}\,(\nabla_\mu\Phi)^*(\nabla_\nu\Phi) - \partial_\rho V_\Phi}_{\text{state dependence in } \mathcal{L}_\Phi} \\
&+\; \underbrace{\frac{1}{2}\, R(\omega,e)\,\partial_\rho M^2_{\rm eff}(\rho)}_{\text{gravitational coupling}} \;+\; \Big(\partial_\rho \mathcal{L}_{\rm mix}\Big) \;+\; \Big(\partial_\rho \mathcal{L}_{\rm constr}\Big).
\end{aligned} \tag{3.27}$$

Here $\mathcal{Q}^B{}_{\mu\nu\rho}$ encodes the vertical-derivative algebra defined in Appendix 2.A, and all covariant derivatives $\widetilde{\nabla}$ are compatible with $g_{\rm eff}$. Equation (3.27) is second order and fully covariant.

Physical reading.

- **Kinetic flow of density.** The first line shows ρ participates in a multi-component flow: $Z_{\rho B}$ couples its transport to all structural gradients $D_\mu S^B$. When $Z_{\rho B}$ is diagonal-dominant, the dominant propagation channel is set by $Z_{\rho\rho}$ on the emergent geometry $g_{\rm eff}$.
- **Spectral-curvature backreaction.** The second line displays how the spectral curvature $F^A{}_{\mu\nu}$ pushes on ρ via two gates: (i) direct stiffness change $(\partial_\rho Y_{AB})F\cdot F$, and (ii) the *constitutive* feed-through $D_{[\mu}(YF)\,\partial_\rho\Gamma$ which activates whenever the horizontal lift $\Gamma^A{}_\mu$ reacts to density. The algebraic term with $\mathcal{Q}$ accounts for vertical derivatives on the fiber and is crucial for gauge-like consistency of the spectral bundle.
- **Geometric feedback.** The third line contains the metric and anisotropy feedbacks: $\partial_\rho g_{\rm eff}^{\mu\nu}$ arises because the tetrad (and thus $g_{\rm eff}$) is state-built; $\partial_\rho G^{\mu\nu}_{\rm (ani)}$ transmits density changes to the matter-lightcone (modulated by $\delta c(\rho,\dots)$). Together with $(\partial_\rho\kappa_\Phi)$ and $-\partial_\rho V_\Phi$, these terms implement the *environmental* control of matter inertia and dispersion.

- **Gravitational gate.** The last line shows that ρ modulates the effective Planck mass: $M_{\rm eff}^2(\rho)$. In slowly varying regimes, $\partial_\rho M_{\rm eff}^2 \neq 0$ adds a Brans–Dicke–like source $\propto R(\omega, e)$ to the ρ-equation. In the homogeneous limit $\rho = \rho_0$ (const.), this reduces to a constant $M_{\rm eff}$ and General Relativity is recovered at leading order.

Stability and causality gates (sufficient conditions).

$$Z_{\rho\rho} > 0, \quad Z_{AB} \succ 0, \quad Y_{AB} \succ 0, \quad \kappa_\Phi > 0, \qquad \big|\,\partial_\rho g_{\rm eff}^{\mu\nu}\,\big|,\ \big|\,\partial_\rho G_{\rm (ani)}^{\mu\nu}\,\big| \text{ small on background}, \tag{3.28}$$

together with a convex local potential $\partial_\rho^2 U(\mathcal{S}_0) > 0$ to avoid tachyonic instabilities of $\delta\rho$ around a background $\mathcal{S}_0$. Hyperbolicity follows from the principal symbol inherited from $g_{\rm eff}$ (see §3.8) provided $\delta c > -1$ and the anisotropic corrections are bounded.

Special regimes.

1. **Homogeneous background.** For $S^A = S_0^A$ constant, (3.27) linearizes to

$$-Z_{\rho\rho}\,\Box_{g_{\rm eff}}\,\delta\rho + m_\rho^2\,\delta\rho \;=\; J_\rho^{(\rm spec)} + J_\rho^{(\rm mix)}, \qquad m_\rho^2 \equiv \partial_\rho^2 U\big|_{\mathcal{S}_0} \;+\; \frac{1}{2}\,R\,\partial_\rho^2 M_{\rm eff}^2\big|_{\rho_0} \;+\; \cdots,$$

   where $J_\rho^{(\rm spec)}$ collects linear sources from $F{\cdot}F$ and $\partial_\rho\Gamma$, and $J_\rho^{(\rm mix)}$ from $\mathcal{L}_{\rm mix}$. Thus $\delta\rho$ propagates as a scalar on $g_{\rm eff}$ with an *environmental mass* m_ρ.

2. **Newtonian/weak-field limit.** If $\rho = \rho_a + \delta\rho$ with ρ_a constant spatially and $|\nabla\delta\rho| \ll 1$, then $M_{\rm eff}(\rho) \approx M_{\rm eff}(\rho_a)$ is constant and GR is recovered to leading order. The gravitational potential satisfies $\nabla^2\phi = 4\pi G_{\rm eff}(\rho_a)\,\rho_m$, with $G_{\rm eff} = (8\pi M_{\rm eff}^2)^{-1}$, while $\delta\rho$ evolves according to a Klein–Gordon–type equation sourced by spectral curvature and mixing.

3. **Adiabatic density gradients.** When $\partial_\mu\rho$ is small but nonzero, the Brans–Dicke–like pieces in the emergent Einstein equation (see §3.6) produce suppressed corrections $\mathcal{O}(\nabla\nabla M_{\rm eff}^2)$ to gravitational dynamics, while (3.27) dictates a slow relaxation of ρ toward minimizing $U(\mathcal{S})$ under spectral-curvature forcing.

Equation (3.27) will be used repeatedly in Chapters 4–5 to quantify how density inhomogeneities of the quantum aether renormalize the effective lightcone (via $g_{\rm eff}$), modulate inertial masses (via κ_Φ and V_Φ), and feed back into the spectral curvature sector.

3.4.2 Viscosity/Nonlinearity field η

Derivation. Setting $C = \eta$ in the master structural EL-equation (3.26) and adding the spectral-curvature contributions from (3.20), we obtain

$$\begin{aligned}
0 = &-\widetilde{\nabla}_\mu\big(Z_{\eta B}\,g_{\rm eff}^{\mu\nu}D_\nu S^B\big) \;+\; \frac{1}{2}\,(\partial_\eta Z_{AB})\,g_{\rm eff}^{\mu\nu}D_\mu S^A D_\nu S^B \;-\; \partial_\eta U \\
&-\; \frac{1}{4}\,(\partial_\eta Y_{AB})\,F^A_{\mu\nu}F^{B\,\mu\nu} \;+\; \frac{1}{2}\,D_{[\mu}\Big(Y_{AB}\,F^{A\,\mu\nu}\Big)\,\partial_\eta\Gamma^B{}_\nu \;-\; \frac{1}{2}\,Y_{AB}\,F^{A\,\mu\nu}\,\mathcal{Q}^B{}_{\mu\nu\eta} \\
&+\; \underbrace{\frac{\partial g_{\rm eff}^{\mu\nu}}{\partial\eta}\,\frac{1}{2}\,Z_{AB}\,D_\mu S^A D_\nu S^B}_{\text{metric feedback from structural kinetics}} \;+\; \underbrace{\frac{\partial G_{\rm (ani)}^{\mu\nu}}{\partial\eta}\,\kappa_\Phi\,(\nabla_\mu\Phi)^*(\nabla_\nu\Phi)}_{\text{anisotropy feedback on matter}} \;+\; \underbrace{(\partial_\eta\kappa_\Phi)\,G_{\rm (ani)}^{\mu\nu}\,(\nabla_\mu\Phi)^*(\nabla_\nu\Phi) - \partial_\eta V_\Phi}_{\text{state dependence in }\mathcal{L}_\Phi} \\
&+\; \underbrace{\frac{1}{2}\,R(\omega,e)\,\partial_\eta M_{\rm eff}^2}_{\text{gravitational running (vanishes if } M_{\rm eff}=M_{\rm eff}(\rho) \text{ only)}} \;+\; \Big(\partial_\eta\mathcal{L}_{\rm mix}\Big) \;+\; \Big(\partial_\eta\mathcal{L}_{\rm constr}\Big).
\end{aligned} \tag{3.29}$$

As before, $\widetilde{\nabla}$ is compatible with $g_{\rm eff}$, and $\mathcal{Q}^B{}_{\mu\nu\eta}$ encodes vertical-derivative algebra on the fiber (Appendix 2.A). Equation (3.29) is strictly second order.

Physical reading.

- **Nonlinearity gate.** The field η controls the nonlinear response of the aether: it modulates the kinetic weights ($\partial_\eta Z_{AB}$), the spectral-curvature stiffness ($\partial_\eta Y_{AB}$), and the matter kinetic prefactor $\kappa_\Phi(\mathcal{S})$ as well as the anisotropy $G^{\mu\nu}_{(\mathrm{ani})}(\mathcal{S})$. Hence, spatial or temporal variations in η *renormalize* both structural propagation and the effective light-cone seen by Φ.
- **Matter coupling and effective masses.** Through $-\partial_\eta V_\Phi$ and $(\partial_\eta \kappa_\Phi)$ the field η acts as an environmental variable that can raise or lower the effective mass $m^2_{\mathrm{eff}}(\mathcal{S})$ and the inertial coefficient of Φ. This realizes, at the Lagrangian level, an "environmental mass" mechanism without introducing an independent Higgs field.
- **Geometric feedback.** The term $\frac{1}{2}(\partial_\eta g^{\mu\nu}_{\mathrm{eff}})\, Z_{AB} D_\mu S^A D_\nu S^B$ communicates changes of η to the emergent metric (via the state-built tetrad), and therefore to all principal symbols. When small on the background, it yields gentle, causal deformations of characteristic cones (cf. §3.8).
- **Gravitational running (optional).** If M_{eff} depends on η (a non-minimal choice), the $R\,\partial_\eta M^2_{\mathrm{eff}}$ term introduces a Brans–Dicke–like coupling. In the *minimal* realization adopted in Chap. 2, $M_{\mathrm{eff}} = M_{\mathrm{eff}}(\rho)$ only and this term vanishes.

Stability/causality gates (sufficient).

$$Z_{\eta\eta} > 0, \quad Z_{AB} \succ 0, \quad Y_{AB} \succ 0, \quad \kappa_\Phi > 0, \qquad \delta c > -1, \qquad \partial^2_\eta U(\mathcal{S}_0) > 0, \tag{3.30}$$

together with bounded anisotropic sensitivity $\left|\partial_\eta G^{\mu\nu}_{(\mathrm{ani})}\right| \ll 1$ on the background $\mathcal{S}_0$. These ensure positive energy for $\delta\eta$ fluctuations and that all characteristics remain within the cone of g_{eff}.

Special regimes and illustrative limits.

1. **Homogeneous medium ($\eta = \eta_0$).** Linearization yields a Klein–Gordon–type propagation for $\delta\eta$,
$$-Z_{\eta\eta}\,\Box_{g_{\mathrm{eff}}}\,\delta\eta + m^2_\eta\,\delta\eta \;=\; J^{(\mathrm{spec})}_\eta + J^{(\mathrm{mix})}_\eta, \qquad m^2_\eta \equiv \partial^2_\eta U\big|_{\mathcal{S}_0} \;+\; \cdots,$$
where sources collect linear pieces from $(\partial_\eta Y)F\cdot F$, $\partial_\eta \Gamma$, and mixing. The dots can include weak geometric feedback proportional to $\partial_\eta g^{\mu\nu}_{\mathrm{eff}}$.
2. **Slowly varying η and decoherence threshold (preview).** Although §3.10 will introduce a Rayleigh functional $\mathcal{R} \propto \eta\, g^{\mu\nu}_{\mathrm{eff}} (\nabla_\mu \Phi)^* (\nabla_\nu \Phi)$ for irreversible effects, the *conservative* equation (3.29) already shows how gradients in η modulate the coherent dynamics of Φ. In weakly inhomogeneous media, a local criterion for the onset of strong damping reads schematically $\eta\,\ell^2_{\mathrm{grad}} \gtrsim 1/\omega$ (heuristic; to be derived precisely in §3.10).
3. **Ordered vs. isotropic strata.** If $U(\mathcal{S})$ admits distinct minima in η correlated with α (order parameter), domain walls in η can co-localize with nematic textures in $\hat{\chi}$; the coupling $(\partial_\eta Y)F\cdot F$ provides a natural channel for pinning spectral curvature to such textures (relevant to soliton stability in Chap. 4).

Equation (3.29) will be pivotal when analyzing (i) damping of matter waves through their coupling to the medium, and (ii) spectral stability conditions for solitons where η acts as an internal "damping" knob that controls phase lines and directionality.

3.4.3 Internal time field T

Derivation (Noether current from time-shift symmetry). In the minimal realization adopted here, T enters the Lagrangian only through its derivatives (via the state-built flow $u^\mu = u^\mu(\partial T, \chi)$ and the induced tensors $g^{\mu\nu}_{\mathrm{eff}}(S)$ and $G^{\mu\nu}_{(\mathrm{ani})}(S)$).[2] Therefore, the Euler–Lagrange equation for T reduces to the conservation of the Noether current

$$\boxed{\widetilde{\nabla}_\mu J^\mu_T \;=\; 0, \qquad J^\mu_T \;:=\; \frac{\partial \mathcal{L}^{(\mathrm{dyn})}_{\mathrm{tot}}}{\partial(\partial_\mu T)}\,,} \tag{3.31}$$

[2] All constitutive functions respect a global shift symmetry $T \mapsto T + c$, i.e., $\partial_T Z_{AB} = \partial_T Y_{AB} = \partial_T U = \partial_T \kappa_\Phi = \partial_T \delta c = \partial_T M^2_{\mathrm{eff}} = 0$. If one relaxes this, explicit source terms appear below.

where $\mathcal{L}_{\rm tot}^{(\rm dyn)}$ denotes the total bulk Lagrangian density excluding pure boundary terms. Using the sector split (3.4)–(3.9), we can write

$$J_T^\mu = \underbrace{Z_{TB}\, g_{\rm eff}^{\mu\nu}\, D_\nu S^B}_{\text{explicit structural } T\text{-kinematics}} + \underbrace{\frac{\partial g_{\rm eff}^{\alpha\beta}}{\partial(\partial_\mu T)}\left[\tfrac{1}{2}\, Z_{AB}\, D_\alpha S^A D_\beta S^B \;+\; \kappa_\Phi\, (\nabla_\alpha \Phi)^*(\nabla_\beta \Phi)\right]}_{\text{metric feedback}} \tag{3.32}$$
$$+ \underbrace{\frac{\partial G_{(\rm ani)}^{\alpha\beta}}{\partial(\partial_\mu T)}\, \kappa_\Phi\, (\nabla_\alpha \Phi)^*(\nabla_\beta \Phi)}_{\text{anisotropy feedback}} + \underbrace{\frac{\partial \mathcal{L}_{\rm mix}}{\partial(\partial_\mu T)} + \frac{\partial \mathcal{L}_{\rm constr}}{\partial(\partial_\mu T)}}_{\text{flow/director building terms}} .$$

Here, $\partial(\cdot)/\partial(\partial_\mu T)$ acts through the $u^\mu(\partial T, \chi)$ dependence present in $g_{\rm eff}$, $G_{(\rm ani)}$, and in the mixing/constraint sectors; the Ehresmann connection $\Gamma^A{}_\mu$ does *not* depend on derivatives of T by construction.

Physical reading.

- **Conserved aether time-flow.** Equation (3.31) states that the internal time T generates a conserved current J_T^μ. The first term in (3.32) is the direct transport of T within the structural multiplet, while the remaining terms express how the emergent geometry (and matter kinetics) re-route the time-flow through the state-built light-cone.

- **Foliation and clocking.** On backgrounds where $\partial_\mu T$ is timelike with respect to $g_{\rm eff}$, hypersurfaces $T = \text{const}$ define a preferred foliation that clocks the aether. The normalization of u^μ (enforced by $\mathcal{L}_{\rm constr}$) aligns the material time-flow with the geometric cone, avoiding superluminal characteristics.

- **When sources appear.** If one allowed explicit T-dependence in constitutive functions (breaking $T \to T + c$), (3.31) generalizes to $\widetilde{\nabla}_\mu J_T^\mu = \partial_T \mathcal{L}_{\rm tot}^{(\rm dyn)} \neq 0$, turning the continuity equation into a balance law. In this chapter we keep the exact symmetry.

Stability/causality gates (sufficient).

$$Z_{TT} > 0, \qquad u^\mu u_\mu = -1, \quad \hat\chi^\mu \hat\chi_\mu = +1, \quad u^\mu \hat\chi_\mu = 0 \;\; \text{(on-shell)}, \qquad \frac{\partial g_{\rm eff}^{\alpha\beta}}{\partial(\partial_\mu T)} \text{ bounded}, \tag{3.33}$$

together with $\kappa_\Phi > 0$ and $\delta c > -1$. These ensure a well-posed principal symbol for the T-sector and compatibility of the time-flow with the emergent light-cone.

Special regimes.

1. **Homogeneous backgrounds.** If $S^A = S_0^A$ and $\partial_\mu T = \tau_0\, \delta_\mu^0$ in adapted coordinates, then $J_T^\mu = \tau_0\, Z_{TT}\, g_{\rm eff}^{\mu 0} + (\text{constant feedback})$ and $\widetilde{\nabla}_\mu J_T^\mu = 0$ holds identically.

2. **Quasi-static media.** For slowly varying $u^\mu(\partial T, \chi)$, the feedback pieces in (3.32) are small, and the time-flow is approximately transported along the integral curves of u^μ.

3. **Alignment with matter flow.** The mixing term $\lambda_4 (\nabla\Phi)^*(\nabla\Phi)\, \hat\chi^{(\mu} u^{\nu)}$ in (3.8) contributes to $\partial\mathcal{L}_{\rm mix}/\partial(\partial_\mu T)$ through $u^\nu(\partial T, \chi)$. This encodes a controllable alignment between the structural time-flow and the dominant matter energy-momentum transport.

The conservation law (3.31) will be used in Chapter 4 to define energetic functionals for stationary solitons and, in Chapter 5, to relate the global aether clocking to cosmological effective histories.

3.4.4 Direction/Polarization field χ^I

Derivation. We now vary the action with respect to the internal director components χ^I $(I = 1, 2, 3)$. Because the normalized spacetime director $\hat{\chi}^\mu$ is *built* from χ^I and normalized by Lagrange multipliers (cf. $\mathcal{L}_{\rm constr}$), the raw Euler–Lagrange equation contains constraint forces. It is therefore convenient to write the unprojected EL-equation and then project onto the subspace transverse to $\hat{\chi}^I$.

Starting from the master rule (3.26) and adding the spectral-curvature pieces (3.20) we obtain

$$\begin{aligned}\mathcal{E}_{\chi^I} :=& -\widetilde{\nabla}_\mu\big(Z_{\chi^I B}\, g_{\rm eff}^{\mu\nu} D_\nu S^B\big) \;+\; \frac{1}{2}\,(\partial_{\chi^I} Z_{AB})\, g_{\rm eff}^{\mu\nu} D_\mu S^A D_\nu S^B \;-\; \partial_{\chi^I} U \\ &-\; \frac{1}{4}\,(\partial_{\chi^I} Y_{AB})\, F^A_{\mu\nu} F^{B\,\mu\nu} \;+\; \frac{1}{2}\, D_{[\mu}\Big(Y_{AB}\, F^{A\,\mu\nu}\Big)\, \partial_{\chi^I}\Gamma^B{}_\nu \;-\; \frac{1}{2}\, Y_{AB}\, F^{A\,\mu\nu}\, \mathcal{Q}^B{}_{\mu\nu\chi^I} \\ &+\; \underbrace{\frac{\partial g_{\rm eff}^{\mu\nu}}{\partial\chi^I}\,\frac{1}{2}\, Z_{AB}\, D_\mu S^A D_\nu S^B}_{\text{metric feedback}} \;+\; \underbrace{\frac{\partial G^{\mu\nu}_{\rm (ani)}}{\partial\chi^I}\,\kappa_\Phi\,(\nabla_\mu\Phi)^*(\nabla_\nu\Phi)}_{\text{anisotropy feedback on matter}} \;+\; \underbrace{(\partial_{\chi^I}\kappa_\Phi)\, G^{\mu\nu}_{\rm (ani)}\,(\nabla_\mu\Phi)^*(\nabla_\nu\Phi) - \partial_{\chi^I} V_\Phi}_{\text{state dependence in } \mathcal{L}_\Phi} \\ &+\; \partial_{\chi^I}\mathcal{L}_{\rm mix} \;+\; \partial_{\chi^I}\mathcal{L}_{\rm constr}\,.\end{aligned} \tag{3.34}$$

The last term supplies the constraint forces that enforce $\hat{\chi}^\mu\hat{\chi}_\mu = +1$ and $u^\mu\hat{\chi}_\mu = 0$.

To remove the unphysical longitudinal pieces, we project with the internal transverse projector

$$\Pi_\perp{}^{IJ} \;:=\; \delta^{IJ} - \hat{\chi}^I\hat{\chi}^J\,, \qquad \Pi_\perp{}^{IJ}\,\hat{\chi}_J = 0, \tag{3.35}$$

obtaining the *physical* director equation

$$\boxed{\;\Pi_\perp{}^{IJ}\,\mathcal{E}_{\chi^J} \;=\; 0,\;} \tag{3.36}$$

which is manifestly second order and free of constraint multipliers.[3]

Physical reading.

- **Directional elasticity and transport.** The first line in (3.34) shows that gradients of χ are transported by the emergent geometry with elastic weights $Z_{\chi^I B}$. Diagonal-dominance of $Z_{\chi\chi}$ gives a principal part analogous to Oseen–Frank "bend/splay" energies for a nematic director, now on the aether background $g_{\rm eff}$.
- **Spectral-curvature torque.** The second line encodes how the spectral curvature $F^A{}_{\mu\nu}$ *exerts a torque* on the director: $(\partial_\chi Y)\,F\cdot F$ changes the stiffness with direction, while the constitutive feed-through $D_{[\mu}(YF)\,\partial_\chi\Gamma$ acts as a gauge-like force sourced by horizontal curvature. The algebraic $\mathcal{Q}$-term is required by vertical-derivative consistency of the fiber bundle.
- **Matter anisotropy backreaction.** Through $\partial_\chi G^{\mu\nu}_{\rm (ani)}$ and $\partial_\chi\kappa_\Phi$, the director couples to the *directional* kinetic energy of Φ. This drives alignment of $\hat{\chi}^\mu$ with dominant matter transport; the mixing term $\lambda_3\, J^\mu_\Phi\hat{\chi}_\mu$ in (3.8) supplies a direct, symmetry-allowed alignment channel.
- **Ordered vs. isotropic strata.** If $U(S)$ has a valley at $\alpha > 0$ (ordered phase), rotations of $\hat{\chi}$ about the preferred axis can be light modes (quasi-Goldstones) provided the transverse curvature of U in the χ-directions is small. In isotropic phases ($\alpha = 0$) the director is marginal and aligns only where spectral curvature or matter flow pick out a direction.

Stability/causality gates (sufficient).

$$Z^\perp_{\chi\chi} \succ 0 \quad \text{on the transverse block}, \qquad Y_{AB} \succ 0, \quad \kappa_\Phi > 0, \quad \delta c > -1, \qquad \partial^2_\chi U\big|_\perp \geq 0, \tag{3.37}$$

together with bounded sensitivities $\big|\partial_\chi g_{\rm eff}^{\mu\nu}\big|$, $\big|\partial_\chi G^{\mu\nu}_{\rm (ani)}\big|$ on the background. These prevent ghosts in the transverse sector and keep characteristics within the $g_{\rm eff}$ cone.

[3] An equivalent route is to solve the constraints explicitly (e.g. parametrize $\hat{\chi}^I$ by two angles) and vary in those; the projector method keeps covariance transparent.

Special regimes and illustrative limits.

1. **Homogeneous ordered background.** Linearizing (3.36) around $\hat{\chi}^\mu = \hat{\chi}_0^\mu$ and $\alpha = \alpha_0 > 0$ yields anisotropic dispersion for transverse director waves:

$$\omega^2 \;=\; c_\perp^2\, k_\perp^2 \;+\; c_\parallel^2\, k_\parallel^2 \;+\; m_\chi^2, \quad m_\chi^2 \equiv \partial_\chi^2 U\big|_\perp \;+\; (\text{small spectral/matter corrections}).$$

   Here $k_\parallel := k_\mu \hat{\chi}_0^\mu$ and $k_\perp^2 := k_\mu k^\mu - k_\parallel^2$.

2. **Isotropic background ($\alpha = 0$).** If U is flat in χ at $\alpha = 0$, director modes are marginal; alignment is then driven by spectral curvature F and by the matter current J_Φ^μ via λ_3. Small Y- and Z-gradients select a preferred axis dynamically (spontaneous directional ordering).

3. **Topological textures (preview).** With constant couplings in S_{top} there is no bulk EL-contribution, yet the Hopf functional $\mathcal{H}[\hat{\chi}]$ classifies director textures. When mixing terms pin F to $\hat{\chi}$, skyrmion-like configurations acquire energetic protection (relevant in Chap. 4).

Equation (3.36) provides the working dynamical law for the internal director, cleanly separated from constraint forces and ready to be coupled to soliton ansätze and wave solutions in subsequent chapters.

3.4.5 Order/Anisotropy parameter α

Derivation. Setting $C = \alpha$ in the master structural EL-equation (3.26) and adding the spectral-curvature contributions from (3.20), we obtain

$$\begin{aligned}
0 = &-\widetilde{\nabla}_\mu\big(Z_{\alpha B}\, g_{\text{eff}}^{\mu\nu} D_\nu S^B\big) \;+\; \frac{1}{2}\,(\partial_\alpha Z_{AB})\, g_{\text{eff}}^{\mu\nu} D_\mu S^A D_\nu S^B \;-\; \partial_\alpha U \\
&-\; \frac{1}{4}\,(\partial_\alpha Y_{AB})\, F^A_{\mu\nu} F^{B\,\mu\nu} \;+\; \frac{1}{2}\, D_{[\mu}\Big(Y_{AB}\, F^{A\,\mu\nu}\Big)\, \partial_\alpha \Gamma^B{}_\nu \;-\; \frac{1}{2}\, Y_{AB}\, F^{A\,\mu\nu}\, \mathcal{Q}^B{}_{\mu\nu\alpha} \\
&+\; \underbrace{\frac{\partial g_{\text{eff}}^{\mu\nu}}{\partial\alpha}\,\frac{1}{2}\, Z_{AB}\, D_\mu S^A D_\nu S^B}_{\text{metric feedback}} \;+\; \underbrace{\frac{\partial G^{\mu\nu}_{(\text{ani})}}{\partial\alpha}\,\kappa_\Phi\,(\nabla_\mu\Phi)^*(\nabla_\nu\Phi)}_{\text{anisotropy feedback on matter}} \;+\; \underbrace{(\partial_\alpha\kappa_\Phi)\, G^{\mu\nu}_{(\text{ani})}\,(\nabla_\mu\Phi)^*(\nabla_\nu\Phi) - \partial_\alpha V_\Phi}_{\text{state dependence in } \mathcal{L}_\Phi} \\
&+\; \partial_\alpha \mathcal{L}_{\text{mix}} \;+\; \partial_\alpha \mathcal{L}_{\text{constr}} \;+\; \underbrace{\frac{1}{2}\, R(\omega, e)\, \partial_\alpha M_{\text{eff}}^2}_{\text{optional; vanishes in minimal } M_{\text{eff}}(\rho)} \;.
\end{aligned} \tag{3.38}$$

All covariant derivatives are compatible with g_{eff}. The algebraic fiber tensor $\mathcal{Q}^B{}_{\mu\nu\alpha}$ (Appendix 2.A) encodes vertical-derivative consistency of the Ehresmann connection. Equation (3.38) is second order and fully covariant.

Physical reading.

- **Order parameter for directional symmetry.** The field α controls the passage between isotropic strata ($\alpha = 0$) and ordered/nematic strata ($\alpha > 0$). Through $\partial_\alpha Z_{AB}$ and $\partial_\alpha Y_{AB}$ it reshapes the elastic and spectral-curvature response of the medium, thereby biasing or suppressing director textures and spectral fluxes.
- **Anisotropic light-cone for matter.** The sensitivity $\partial_\alpha G^{\mu\nu}_{(\text{ani})}$ is the direct gate by which ordering modifies matter propagation: typically $\delta c(\alpha)$ grows with α, separating transverse and longitudinal phase velocities with respect to the director. The $(\partial_\alpha\kappa_\Phi)$ and $-\partial_\alpha V_\Phi$ terms implement an environmental control of inertia and mass of Φ across phases.
- **Geometric feedback.** Because the emergent tetrad $e^a{}_\mu$ is state-built, changing α modifies g_{eff} (hence gravitational characteristics) via $\partial_\alpha g_{\text{eff}}^{\mu\nu}$. When adiabatic on the background, this deforms cones gently and preserves hyperbolicity.
- **Spectral-curvature pinning.** The terms involving $(\partial_\alpha Y)$ and $\partial_\alpha \Gamma$ show that spectral curvature $F^A{}_{\mu\nu}$ can be *pinned* or *repelled* by ordering, depending on the sign structure. This mechanism underlies stabilization channels for solitons where α co-localizes with $\hat{\chi}$-textures (preview of Chap. 4).

Stability/causality gates (sufficient).

$$Z_{\alpha\alpha} > 0, \quad Z_{AB} \succ 0, \quad Y_{AB} \succ 0, \quad \kappa_\Phi > 0, \quad \delta c(\alpha) > -1, \qquad \partial_\alpha^2 U(\mathcal{S}_0) \geq 0, \tag{3.39}$$

together with bounded sensitivities $\left|\partial_\alpha g_{\rm eff}^{\mu\nu}\right|$ and $\left|\partial_\alpha G_{\rm (ani)}^{\mu\nu}\right|$ on the background $\mathcal{S}_0$. These avoid ghosts/tachyons for $\delta\alpha$ and keep all characteristics inside the $g_{\rm eff}$ cone.

Special regimes and illustrative limits.

1. **Near an ordering transition.** Suppose $U(\mathcal{S})$ admits a Landau-like form $U \sim a(T_\star)\,\alpha^2 + b\,\alpha^4 + \cdots$. For $a > 0$ (isotropic) we have a gapped $\delta\alpha$ with $m_\alpha^2 \simeq 2a$; for $a < 0$ (ordered) the vacuum sits at $\alpha_0 = \sqrt{-a/(2b)}$, and small oscillations acquire $m_\alpha^2 \simeq -4a > 0$. Couplings to $F\cdot F$ can shift the critical point, yielding a curvature-assisted ordering transition.

2. **Ordered homogeneous background.** Linearizing (3.38) around $(\alpha_0 > 0, \hat\chi_0^\mu)$ gives

$$-Z_{\alpha\alpha}\,\Box_{g_{\rm eff}}\,\delta\alpha + m_\alpha^2\,\delta\alpha \;=\; J_\alpha^{(\rm spec)} + J_\alpha^{(\rm mix)} + (\text{small metric/anisotropy feedback}),$$

   where $J_\alpha^{(\rm spec)}$ collects the linearized $(\partial_\alpha Y)F\cdot F$ and $\partial_\alpha\Gamma$ contributions. The sign and magnitude of m_α^2 control domain-wall thickness and the binding of $\delta\alpha$ to director textures.

3. **Isotropic background ($\alpha = 0$).** If U is locally convex at $\alpha = 0$, fluctuations $\delta\alpha$ are massive and decouple at long distances; if U is flat to quadratic order (near-critical), then $\delta\alpha$ becomes a soft mode and anisotropy in $G_{\rm (ani)}$ turns on to leading order through $\partial_\alpha \delta c\big|_{\alpha=0}$.

Equation (3.38) completes the structural sector. We now turn to the matter field equation, where the cumulative influence of $(\rho, \eta, T, \chi^I, \alpha)$ on inertia, dispersion, and anisotropy becomes most transparent.

3.5 Matter field Φ: Euler–Lagrange equation and conserved current

Derivation. We vary the total Lagrangian with respect to Φ^*, keeping the structural fields S^A fixed in this step (their EL-equations were derived in §3.4). From (3.7) we obtain

$$\delta_{\Phi^*}\mathcal{L}_\Phi = \kappa_\Phi(S)\,G_{\rm (ani)}^{\mu\nu}(S)\,(\nabla_\mu\delta\Phi^*)(\nabla_\nu\Phi) \;-\; \frac{\partial V_\Phi}{\partial\Phi^*}\,\delta\Phi^*. \tag{3.40}$$

Integrating by parts (with $\widetilde\nabla$ compatible with $g_{\rm eff}$) yields the bulk contribution

$$-\,\delta\Phi^*\,\widetilde\nabla_\mu\big(\kappa_\Phi\,G_{\rm (ani)}^{\mu\nu}\,\nabla_\nu\Phi\big) \;-\; \delta\Phi^*\,\frac{\partial V_\Phi}{\partial\Phi^*}.$$

Adding the mixing sector contribution in Euler–Lagrange form,

$$\mathcal{M}_\Phi \;:=\; \frac{\partial\mathcal{L}_{\rm mix}}{\partial\Phi^*} \;-\; \widetilde\nabla_\mu\left(\frac{\partial\mathcal{L}_{\rm mix}}{\partial(\nabla_\mu\Phi^*)}\right), \tag{3.41}$$

we arrive at the matter-field equation

$$\boxed{\widetilde\nabla_\mu\Big(\kappa_\Phi(S)\,G_{\rm (ani)}^{\mu\nu}(S)\,\nabla_\nu\Phi\Big) \;-\; \frac{\partial V_\Phi(|\Phi|;S)}{\partial\Phi^*} \;+\; \mathcal{M}_\Phi \;=\; 0\,.} \tag{3.42}$$

Here $G_{\rm (ani)}^{\mu\nu} = g_{\rm eff}^{\mu\nu} + \delta c(S)\,\hat\chi^\mu\hat\chi^\nu$, so the principal symbol is anisotropic but remains inside the $g_{\rm eff}$ cone provided $\kappa_\Phi > 0$ and $\delta c > -1$.

Conserved $U(1)$ current. If the total Lagrangian is invariant under the global phase $\Phi \to e^{i\theta}\Phi$ (i.e. $V_\Phi = V_\Phi(|\Phi|)$ and $\mathcal{L}_{\rm mix}$ is $U(1)$-symmetric), Noether's theorem gives the current

$$\boxed{J^\mu_\Phi \;=\; \mathrm{Im}\Big(\kappa_\Phi\, G^{\mu\nu}_{(\rm ani)}\, \Phi^*\nabla_\nu\Phi\Big) \;+\; J^\mu_{\rm mix},} \tag{3.43}$$

where $J^\mu_{\rm mix}$ collects derivative-only $U(1)$-invariant contributions from the mixing sector (e.g. symmetry-preserving deformations proportional to $\hat\chi^{(\mu}u^{\nu)}(\nabla_\nu|\Phi|^2)$). Using (3.42) and its complex conjugate, one finds the continuity equation

$$\boxed{\widetilde\nabla_\mu J^\mu_\Phi \;=\; 0 \quad (\text{for } U(1)_\Phi\text{-symmetric } V_\Phi \text{ and } \mathcal{L}_{\rm mix}).} \tag{3.44}$$

If $U(1)_\Phi$ is explicitly broken (e.g. by a phase-dependent potential), the right-hand side equals the corresponding source.

Physical reading.

- **Anisotropic propagation.** The principal operator in (3.42) is $\kappa_\Phi\, G^{\mu\nu}_{(\rm ani)}\,\nabla_\mu\nabla_\nu$ plus lower-derivative terms from $\widetilde\nabla_\mu(\kappa_\Phi G^{\mu\nu}_{(\rm ani)})$. Decomposing k^μ into components parallel/perpendicular to the director $\hat\chi^\mu$ gives distinct phase velocities:

$$\omega^2 \;\simeq\; c_\perp^2\, k_\perp^2 \;+\; (c_\perp^2 + \delta c)\, k_\parallel^2 \;+\; m^2_{\rm eff}(S), \quad m^2_{\rm eff}(S) := \left.\frac{\partial V_\Phi}{\partial|\Phi|^2}\right|_S .$$

  Thus ordering and directionality in the aether $(\alpha, \hat\chi)$ directly refract matter waves.

- **Environmental inertia and mass.** Spatial/temporal variations of $\kappa_\Phi(S)$ and $m^2_{\rm eff}(S)$ implement an *environmental* control of inertia and rest mass for Φ. Gentle gradients act as adiabatic lenses; sharp gradients can trap or scatter matter amplitudes (relevant to soliton pinning in Chap. 4).

- **Mixing-induced alignment and drag.** The operator $\mathcal{M}_\Phi$ encodes symmetry-preserving couplings to structural flows. For instance, a term $\lambda_4(\nabla_\mu\Phi)^*(\nabla_\nu\Phi)\,\hat\chi^{(\mu}u^{\nu)}$ contributes an anisotropic drift along the $(u, \hat\chi)$ plane; a current–gradient coupling $\lambda_1 J^\mu_\Phi D_\mu\rho$ implements drag against density gradients. These are second-order and causal under the EFT bounds stated in §3.2.

- **Conserved charge.** When $U(1)_\Phi$ is exact, $Q = \int_\Sigma d\Sigma_\mu\, J^\mu_\Phi$ is conserved. In stationary backgrounds, Q classifies nontrivial bound configurations (Q-balls/solitons) to be constructed in Chap. 4.

Stability/causality gates (sufficient).

$$\kappa_\Phi(S) > 0, \qquad \delta c(S) > -1, \qquad m^2_{\rm eff}(S) \geq 0 \,(\text{around the chosen vacuum}), \qquad \big|\widetilde\nabla(\kappa_\Phi G_{(\rm ani)})\big| \text{ adiabatic on the backgrou} \tag{3.45}$$

Together with the structural gates from §3.4, these ensure a hyperbolic principal symbol inside the $g_{\rm eff}$ cone and positive energy density for small fluctuations.

Special regimes and illustrative limits.

1. **Homogeneous ordered background.** For $S^A = S^A_0$ constant, (3.42) reduces to $\kappa_{\Phi,0}\, G^{\mu\nu}_{(\rm ani),0}\,\nabla_\mu\nabla_\nu\Phi - \partial V_\Phi/\partial\Phi^* = 0$, yielding the dispersion quoted above with constant $m^2_{\rm eff}$.
2. **Weak inhomogeneity (WKB).** When S^A vary slowly, a WKB ansatz $\Phi = A\, e^{i\Theta}$ gives the eikonal equation $G^{\mu\nu}_{(\rm ani)}\, k_\mu k_\nu + m^2_{\rm eff}/\kappa_\Phi = 0$ with transport law $\widetilde\nabla_\mu(\kappa_\Phi G^{\mu\nu}_{(\rm ani)} k_\nu A^2) = 0$. Thus κ_Φ acts as an "index of refraction" for flux.
3. **Nonrelativistic limit.** In stationary backgrounds and for small gradients, writing $\Phi = \frac{1}{\sqrt{2m_*}}\,\psi\, e^{-im_* T}$ with $m_*^2 \approx m^2_{\rm eff}$ leads to an anisotropic Schrödinger equation with effective masses $m_\parallel \neq m_\perp$ controlled by δc and κ_Φ (details in Chap. 4).

Equations (3.42)–(3.44) complete the matter sector at the conservative level. Dissipative extensions (Rayleigh functional proportional to η) and soliton-bound states will be developed in the next chapter.

3.6 Effective Einstein Equation (Palatini route)

We now vary the gravitational sector $\mathcal{L}_{\rm grav} = \frac{1}{2} M_{\rm eff}^2(\rho)\, R(\omega, e)$ together with the structural, matter, mixing and constraint sectors with respect to the spin-connection ${\omega_\mu}^{ab}$ and the tetrad ${e^a}_\mu$. Throughout, the Ehresmann coefficients ${\Gamma^A}_\mu$ are constitutive (non-variational). We keep torsion set to zero *on shell* by the ω-equation.

3.6.1 Connection equation and conformal compatibility

Variation w.r.t. ω. In first-order variables one finds

$$\delta_\omega S_{\rm grav} = \frac{1}{2}\int d^4x\, e\, M_{\rm eff}^2(\rho)\, \Big(e_a{}^{[\mu} e_b{}^{\nu]}\Big)\, \delta{\omega_\mu}^{ab}{}_{;\nu}\,, \tag{3.46}$$

which, after integration by parts and neglecting boundary terms (absorbed in $S_{\partial M}$), gives the Cartan equation

$$\nabla^\omega_\mu\Big(e\, M_{\rm eff}^2(\rho)\, e_a{}^{[\mu} e_b{}^{\nu]}\Big) \;=\; 0. \tag{3.47}$$

Equation (3.47) is solved by a torsion-free connection ω that is metric-compatible with the *conformally rescaled* metric

$$h_{\mu\nu} \;:=\; \Xi(\rho)\, g_{{\rm eff}\,\mu\nu}, \qquad \Xi(\rho) \equiv \frac{M_{\rm eff}^2(\rho)}{M_\star^2}, \tag{3.48}$$

for an arbitrary constant mass scale $M_\star$ (conveniently chosen as $M_\star = M_{\rm eff}(\rho_0)$ on a reference background). Thus

$$\omega \;=\; \omega\big[\,h\,\big] \;\;\text{(Levi–Civita of } h_{\mu\nu}), \quad \text{torsion } {T^\lambda}_{\mu\nu}(\omega) = 0. \tag{3.49}$$

Curvature objects. All curvature tensors entering $\mathcal{L}_{\rm grav}$ are therefore the Riemann/Ricci scalars of h:

$$R(\omega, e) \equiv R\big(h\big), \qquad G_{\mu\nu}(\omega, e) \equiv G_{\mu\nu}\big(h\big). \tag{3.50}$$

This is the Palatini hallmark: the affine structure is the Levi–Civita connection of the conformal metric $h_{\mu\nu} = \Xi g_{{\rm eff}\,\mu\nu}$.

3.6.2 Tetrad variation and Einstein equation

Variation w.r.t. ${e^a}_\mu$. Holding ω fixed (Palatini split) gives

$$\delta_e S_{\rm grav} = \frac{1}{2}\int d^4x\, e\Big[M_{\rm eff}^2(\rho)\, G_{\mu\nu}(\omega, e)\, \delta g_{\rm eff}^{\mu\nu} \;+\; R(\omega, e)\, \partial_\rho M_{\rm eff}^2\, \delta\rho\Big]. \tag{3.51}$$

Adding the matter/structural/mixing/constraint variations $\delta_e S_{\rm rest} = -\frac{1}{2}\int d^4x\, e\, T^{(\rm tot)}_{\mu\nu}\, \delta g_{\rm eff}^{\mu\nu}$ with $T^{(\rm tot)}_{\mu\nu} := -\frac{2}{e}\frac{\delta(e\mathcal{L}_{\rm rest})}{\delta g_{\rm eff}^{\mu\nu}}$, we obtain the *effective Einstein equation in Palatini form*

$$\boxed{\,M_{\rm eff}^2(\rho)\, G_{\mu\nu}(\omega, e) \;=\; T^{(\rm tot)}_{\mu\nu}\,.\,} \tag{3.52}$$

Upon inserting (3.49)–(3.50) this reads

$$\boxed{\,M_{\rm eff}^2(\rho)\, G_{\mu\nu}\big(h\big) \;=\; T^{(\rm tot)}_{\mu\nu}\,, \qquad h_{\mu\nu} = \Xi(\rho) g_{{\rm eff}\,\mu\nu}.\,} \tag{3.53}$$

Two equivalent frames.

- **Palatini–Einstein frame (h).** Geometry is governed by $h_{\mu\nu}$ with *constant* Planck scale $M_\star$: $M_\star^2\, G_{\mu\nu}(h) = \tilde{T}^{(\rm tot)}_{\mu\nu}$, where the *rescaled* stress tensor is $\tilde{T}^{(\rm tot)}_{\mu\nu} := \Xi^{-1} T^{(\rm tot)}_{\mu\nu}$ and indices are moved with h. This form displays pure Einstein gravity for h. Matter/structural sectors are non-minimally coupled to h through Ξ.

- **Jordan ($g_{\rm eff}$) frame.** Eliminating h in favor of $g_{\rm eff}$ and $\Xi(\rho)$ yields an Einstein equation with Brans–Dicke–like source terms (second derivatives of $\ln\Xi$). Using standard conformal identities, one can cast (3.53) into

$$\boxed{M_{\rm eff}^2\, G_{\mu\nu}(g_{\rm eff}) \;=\; T_{\mu\nu}^{(\rm tot)} \;+\; \mathcal{D}_{\mu\nu}\big[M_{\rm eff}^2\big],} \tag{3.54}$$

  where $\mathcal{D}_{\mu\nu}[M_{\rm eff}^2]$ collects the (frame-choice) differential terms generated by the conformal relation $h_{\mu\nu} = \Xi g_{{\rm eff}\,\mu\nu}$.[4]

3.6.3 Stress–energy content

Definition. The total stress tensor entering (3.52) splits as

$$T_{\mu\nu}^{(\rm tot)} \;=\; T_{\mu\nu}^{(\rm struct)} \;+\; T_{\mu\nu}^{(\rm spec)} \;+\; T_{\mu\nu}^{(\Phi)} \;+\; T_{\mu\nu}^{(\rm mix)} \;+\; T_{\mu\nu}^{(\rm constr)}. \tag{3.55}$$

Each piece is defined by $T_{\mu\nu} = -\frac{2}{e}\,\delta(e\mathcal{L})/\delta g_{\rm eff}^{\mu\nu}$.

Structural kinetic and potential.

$$\begin{aligned} T_{\mu\nu}^{(\rm struct)} &= Z_{AB}\, D_{(\mu}S^A\, D_{\nu)}S^B - \tfrac{1}{2}\, g_{{\rm eff}\,\mu\nu}\, Z_{AB}\, g_{\rm eff}^{\alpha\beta} D_\alpha S^A D_\beta S^B \;+\; g_{{\rm eff}\,\mu\nu}\, U(S) \\ &\quad + \underbrace{\frac{\partial g_{\rm eff}^{\alpha\beta}}{\partial g_{\rm eff}^{\mu\nu}}\,\frac{1}{2}\, Z_{AB}\, D_\alpha S^A D_\beta S^B}_{\text{small metric-building correction}}, \end{aligned} \tag{3.56}$$

where the last term encodes the state-built tetrad dependence (it vanishes if $e^a{}_\mu$ is held fixed in this variation).

Spectral curvature.

$$T_{\mu\nu}^{(\rm spec)} \;=\; Y_{AB}\big(F^A{}_{\mu\lambda}F^{B}{}_{\nu}{}^{\lambda} - \tfrac{1}{4}\, g_{{\rm eff}\,\mu\nu}\, F^A_{\alpha\beta}F^{B\,\alpha\beta}\big)\,. \tag{3.57}$$

Matter sector.

$$\begin{aligned} T_{\mu\nu}^{(\Phi)} &= \kappa_\Phi\left[(\nabla_\mu\Phi)^*(\nabla_\nu\Phi) + (\nabla_\nu\Phi)^*(\nabla_\mu\Phi)\right]\frac{1}{2} - g_{{\rm eff}\,\mu\nu}\,\mathcal{L}_\Phi \\ &\quad + \kappa_\Phi\,\delta c\left[\hat\chi_\mu\hat\chi_\nu\,(\nabla_\alpha\Phi)^*(\nabla_\beta\Phi)\,\Pi^{\alpha\beta} - \tfrac{1}{2}\, g_{{\rm eff}\,\mu\nu}\,\hat\chi^\alpha\hat\chi^\beta\,(\nabla_\alpha\Phi)^*(\nabla_\beta\Phi)\right] \;+\; \Xi_{\mu\nu}^{(\hat\chi)}, \end{aligned} \tag{3.58}$$

where $\Pi^{\alpha\beta} = g_{\rm eff}^{\alpha\beta}$ at this stage and $\Xi_{\mu\nu}^{(\hat\chi)}$ collects the (lengthy but second-order) variations due to the $g_{\rm eff}$-dependence of $\hat\chi^\mu$ through the normalization constraints. A complete expanded form is provided in Appendix 3.A.

Mixing and constraints. The mixing sector contributes algebraically via its explicit $g_{\rm eff}$-dependence, $T_{\mu\nu}^{(\rm mix)} = -\frac{2}{e}\,\delta(e\mathcal{L}_{\rm mix})/\delta g_{\rm eff}^{\mu\nu}$, while $T_{\mu\nu}^{(\rm constr)}$ enforces the on-shell normalization/orthogonality of $(u^\mu, \hat\chi^\mu)$. Topological terms with constant couplings do not contribute to bulk EL-equations.

3.6.4 Bianchi identities and covariant balance

Palatini–Einstein frame. Because $\nabla_\mu^{(h)} G^\mu{}_\nu(h) = 0$, equation (3.53) implies

$$\nabla_\mu^{(h)}\big(\Xi^{-1}T^\mu{}_\nu\big) \;=\; 0, \qquad \Xi = \frac{M_{\rm eff}^2}{M_\star^2}. \tag{3.59}$$

Thus the *rescaled* stress tensor is covariantly conserved with respect to h.

[4] Explicit formulas are standard; in four dimensions one may write, for $\Xi = M_{\rm eff}^2/M_\star^2$, $\mathcal{D}_{\mu\nu} = \nabla_\mu\nabla_\nu M_{\rm eff}^2 - g_{{\rm eff}\,\mu\nu}\Box M_{\rm eff}^2 - \frac{3}{2}\frac{1}{M_{\rm eff}^2}\Big(\nabla_\mu M_{\rm eff}^2\nabla_\nu M_{\rm eff}^2 - \frac{1}{2}g_{{\rm eff}\,\mu\nu}(\nabla M_{\rm eff}^2)^2\Big)$. We relegate the derivation to Appendix 3.A to keep the flow uncluttered.

Jordan frame. In terms of g_{eff} one obtains an exchange law

$$\widetilde{\nabla}_\mu T^\mu{}_\nu \;=\; -\big(\partial_\nu \ln \Xi\big)\, T \;+\; \text{terms covariant under } \mathcal{D}_{\mu\nu}\big[M_{\text{eff}}^2\big], \tag{3.60}$$

where $T := T^\mu{}_\mu$ and $\widetilde{\nabla}$ is the Levi–Civita derivative of g_{eff}. Equation (3.60) expresses energy–momentum exchange between the geometric sector (M_{eff}, i.e. ρ) and the matter/structural sectors, exactly consistent with the ρ-equation (3.27).

3.6.5 Weak-field and Newtonian limits

Homogeneous background. Let $\rho = \rho_a$ be constant and S^A homogeneous. Then $M_{\text{eff}}(\rho_a)$ is constant and $h_{\mu\nu} = \Xi_a g_{\text{eff}\,\mu\nu}$ differs by a constant conformal factor. Equation (3.53) reduces to standard GR for h, $M_\star^2 G_{\mu\nu}(h) = \tilde{T}^{(\text{tot})}_{\mu\nu}$, and hence for g_{eff} with a rescaled Newton constant $G_{\text{eff}}(\rho_a) = (8\pi M_{\text{eff}}^2(\rho_a))^{-1}$.

Newtonian limit. For small velocities and weak potentials, $h_{00} = -1 - 2\tilde{\phi}$, $h_{ij} = (1 - 2\tilde{\phi})\delta_{ij}$, one finds

$$\nabla^2 \tilde{\phi} \;=\; 4\pi\, G_\star\, \tilde{\rho}_m, \qquad G_\star = \frac{1}{8\pi M_\star^2}, \qquad \tilde{\rho}_m := \Xi^{-1}\rho_m, \tag{3.61}$$

which in the g_{eff}-frame amounts to $\nabla^2\phi = 4\pi\, G_{\text{eff}}(\rho_a)\, \rho_m$ at leading order, in agreement with Chap. 2.

3.6.6 Physical reading and gates

Physical reading.

- **Emergent gravity with density-controlled coupling.** Gravity is pure Einstein for the conformal metric $h_{\mu\nu} = \Xi(\rho) g_{\text{eff}\,\mu\nu}$, with an effective Newton coupling $G_{\text{eff}} \propto 1/M_{\text{eff}}^2(\rho)$. Spatial/temporal gradients of ρ therefore generate mild, Brans–Dicke–like deviations in the g_{eff} description.
- **Clean separation of geometry and medium.** The Palatini solution (3.49) avoids higher-derivative ghosts: the affine structure is the Levi–Civita connection of h; no extra propagating spin-2 or spin-0 modes are introduced beyond the structural fields already present.
- **Backreaction channels.** The total stress tensor (3.55) shows three distinct channels for backreaction: (i) structural flows $D_\mu S^A$ and potential U; (ii) spectral curvature $F^A{}_{\mu\nu}$ (Maxwell-like); (iii) matter anisotropy and mixing. Their relative weights are environmental (functions of S).

Stability/causality gates (sufficient).

$$M_{\text{eff}}^2(\rho) > 0, \qquad Z_{AB} \succ 0, \quad Y_{AB} \succ 0, \quad \kappa_\Phi > 0, \quad \delta c > -1, \tag{3.62}$$

with Ξ slowly varying on the background (adiabaticity). These ensure hyperbolicity of all principal symbols inside the light-cone of g_{eff} and positivity of the gravitational coupling.

Equations (3.52)–(3.54) complete the geometric sector. In the next sections we assemble the conservation laws, linearize the full system, and analyze characteristic cones and example solutions.

3.7 Covariant Balance Laws and Noether Structure

This section assembles the conservation/balance relations implied by the field equations derived in §3.4–§**??**. We distinguish: (i) internal/global symmetries (Noether currents), (ii) energy–momentum balance in the two natural frames (Palatini–Einstein frame $h_{\mu\nu}$ and Jordan/emergent frame g_{eff}), and (iii) spectral/ topological identities associated with the Ehresmann curvature and director textures.

3.7.1 Noether currents from internal symmetries

Global phase of Φ ($U(1)_\Phi$). If $\mathcal{L}$ is invariant under $\Phi \mapsto e^{i\theta}\Phi$ (with θ constant), then the current in (3.43) is conserved on the Φ-equation of motion:

$$\boxed{J^\mu_\Phi \;=\; \mathrm{Im}\Big(\kappa_\Phi\, G^{\mu\nu}_{(\mathrm{ani})}\, \Phi^*\nabla_\nu\Phi\Big) \;+\; J^\mu_{\mathrm{mix}}, \qquad \widetilde{\nabla}_\mu J^\mu_\Phi = 0.} \tag{3.63}$$

The associated charge $Q_\Phi = \int_\Sigma d\Sigma_\mu\, J^\mu_\Phi$ is frame-independent.

Global time-shift of T (clock symmetry). If the constitutive functions have no explicit T-dependence, $\mathcal{L}$ is invariant under $T \to T + c$ and the current in (3.31) is conserved:

$$\boxed{J^\mu_T \;=\; \frac{\partial \mathcal{L}^{(\mathrm{dyn})}_{\mathrm{tot}}}{\partial(\partial_\mu T)}, \qquad \widetilde{\nabla}_\mu J^\mu_T = 0.} \tag{3.64}$$

This current measures transport of the internal aether time through the medium and mediates alignment with matter flow via the mixing sector.

Spatial isometries (Killing vectors). Let ξ^μ be a Killing vector of g_{eff}: $\widetilde{\nabla}_{(\mu}\xi_{\nu)} = 0$. Then the Hilbert stress tensor of any sector that is minimally coupled to g_{eff} produces a conserved Noether current

$$\boxed{\mathcal{J}^\mu[\xi] \;:=\; T^\mu{}_\nu\, \xi^\nu, \qquad \widetilde{\nabla}_\mu \mathcal{J}^\mu[\xi] \;=\; 0 \quad \text{(sectorwise on-shell).}} \tag{3.65}$$

In the full theory the exchange terms detailed below must be included when $\Xi(\rho)$ varies.

3.7.2 Energy–momentum balance: h- vs. g_{eff}-frames

Palatini–Einstein frame. With $h_{\mu\nu} = \Xi(\rho)\, g_{\mathrm{eff}\,\mu\nu}$ and $M_\star^2 G_{\mu\nu}(h) = \tilde{T}^{(\mathrm{tot})}_{\mu\nu}$ (where $\tilde{T}^{(\mathrm{tot})}_{\mu\nu} := \Xi^{-1} T^{(\mathrm{tot})}_{\mu\nu}$), Bianchi identities give

$$\boxed{\nabla^{(h)}_\mu \big(\tilde{T}^\mu{}_\nu\big) \;=\; 0.} \tag{3.66}$$

Thus, in the h-frame, the *rescaled* stress tensor is covariantly conserved.

Jordan/emergent frame. In terms of g_{eff} one finds the exchange law

$$\boxed{\widetilde{\nabla}_\mu T^\mu{}_\nu \;=\; -\,(\partial_\nu \ln \Xi)\, T \;+\; \mathcal{X}_\nu\big[\Xi; g_{\mathrm{eff}}\big],} \tag{3.67}$$

where $T := T^\mu{}_\mu$ and $\mathcal{X}_\nu[\Xi; g_{\mathrm{eff}}]$ denotes the standard conformal-covariant terms generated by $h_{\mu\nu} = \Xi\, g_{\mathrm{eff}\,\mu\nu}$ (explicit formulae are collected in Appendix 3.A). Equation (3.67) exactly matches the density equation (3.27): energy–momentum is exchanged with the geometric coupling $M^2_{\mathrm{eff}}(\rho)$ when ρ varies.

Sectorwise balances (schematic). Decomposing $T^\mu{}_\nu = T^\mu{}_\nu(\mathrm{struct}) + \cdots + T^\mu{}_\nu(\Phi)$ and using the field equations, one obtains the identities

$$\widetilde{\nabla}_\mu T^\mu{}_\nu(\Phi) = -\,(\partial_\nu S^C)\left[\mathfrak{S}^{(\Phi)}_C\right] \;-\; \tfrac{1}{2}\,(\partial_\nu g^{\alpha\beta}_{\mathrm{eff}})\,\kappa_\Phi\,(\nabla_\alpha\Phi)^*(\nabla_\beta\Phi), \tag{3.68}$$

$$\widetilde{\nabla}_\mu T^\mu{}_\nu(\mathrm{struct}) = +\,(\partial_\nu S^C)\left[-\,\mathfrak{S}^{(\mathrm{kin})}_C + \partial_{S^C} U + \cdots\right] \;+\; \tfrac{1}{2}\,(\partial_\nu g^{\alpha\beta}_{\mathrm{eff}})\, Z_{AB}\, D_\alpha S^A D_\beta S^B, \tag{3.69}$$

with $\mathfrak{S}^{(\Phi)}_C$ and $\mathfrak{S}^{(\mathrm{kin})}_C$ as in (3.23)–(3.17). Summing all sectors reproduces (3.67). These balances display explicitly how gradients of the structural state $(\rho, \eta, T, \chi, \alpha)$ act as *force densities* that exchange energy–momentum with matter.

3.7.3 Spectral Bianchi identities and topological charges

Horizontal Bianchi identity for $F^A{}_{\mu\nu}$. By construction of the Ehresmann curvature on the total space,

$$D_{[\lambda}F^A{}_{\mu\nu]} \;=\; 0 \qquad\Longrightarrow\qquad \epsilon^{\lambda\mu\nu\rho}D_\lambda F^A{}_{\mu\nu} \;=\; 0. \tag{3.70}$$

With $Y_{AB}(S)$ positive definite, the Maxwell-like identity $D_\mu(Y_{AB}F^{A\,\mu\nu}) = \mathscr{J}_B^\nu$ defines a *spectral flux* $\mathscr{J}_B^\nu$ (the right-hand side is nonzero only through constitutive S-dependence and mixing couplings; there is no independent Γ-equation). In particular,

$$\boxed{\mathcal{Q}_{\mathrm{CS}}[\,\Gamma\,] \;=\; \int_{\Sigma^3} \mathrm{Tr}\big(\Gamma\wedge d\Gamma + \tfrac{2}{3}\,\Gamma\wedge\Gamma\wedge\Gamma\big)} \tag{3.71}$$

is a topological functional (Chern–Simons) on spatial slices Σ^3; its time variation is governed by $\int_{\Sigma^3}\mathrm{Tr}(F\wedge F)$ which appears in S_{top}.

Director textures and Hopf charge. For the normalized director $\hat\chi : \mathbb{R}^3 \to S^2$, the Hopf invariant

$$\boxed{\mathcal{H}[\hat\chi] \;=\; \frac{1}{(4\pi)^2}\int_{\mathbb{R}^3} A\wedge dA, \qquad dA = \hat\chi^*(\omega_{S^2}),} \tag{3.72}$$

classifies knotted textures. With constant topological couplings in S_{top}, $\mathcal{H}$ is preserved by smooth evolution; mixing terms can pin spectral curvature to such textures and render them energetically protected (relevant for soliton sectors in Chap. 4).

Winding of Φ. For configurations with compact support or appropriate boundary conditions, $Q_\Phi = \int d^3x\, J^0_\Phi$ is conserved. In sectors with nontrivial phase maps $\Phi/|\Phi| : S^1 \to U(1)$, Q_Φ can be interpreted as a winding (Noether) charge that stabilizes Q-ball–like objects.

3.7.4 Summary of exchange channels and positivity

Exchange channels. The full balance structure can be summarized as

$$\boxed{\begin{array}{ll}
\text{(i) Noether:} & \widetilde{\nabla}_\mu J^\mu_\Phi = 0,\quad \widetilde{\nabla}_\mu J^\mu_T = 0;\\
\text{(ii) Geometry:} & \nabla^{(h)}_\mu(\tilde T^\mu{}_\nu) = 0 \;\Leftrightarrow\; \widetilde{\nabla}_\mu T^\mu{}_\nu = -(\partial_\nu \ln\Xi)\,T + \mathcal{X}_\nu;\\
\text{(iii)Spectral:} & D_{[\lambda}F^A{}_{\mu\nu]} = 0,\quad D_\mu(Y_{AB}F^{A\,\mu\nu}) = \mathscr{J}_B^\nu \text{ (constitutive)};\\
\text{(iv)Textures:} & \dot{\mathcal{H}} = 0 \text{ (for constant topological couplings).}
\end{array}} \tag{3.73}$$

Positivity and hyperbolicity. Under the gates $Z_{AB}\succ 0$, $Y_{AB}\succ 0$, $\kappa_\Phi > 0$, $\delta c > -1$, $M^2_{\mathrm{eff}} > 0$ and with adiabatic background variations of S^A, the principal symbols of all dynamical sectors are hyperbolic within the g_{eff} light-cone and the canonical energy densities are positive. This provides the mathematical foundation for the linear analysis of §3.8 and the construction of coherent structures in Chap. 4.

3.8 Linearization, Characteristic Cones, and Mode Content

We analyze small-amplitude perturbations around a homogeneous background state to extract the principal symbols, causal cones, and propagating mode content of the coupled aether–matter system.

3.8.1 Background, perturbations, and working gauge

Homogeneous background. Let

$$S^A(x) = S^A_0 + \delta S^A(x), \qquad \Phi(x) = \Phi_0 + \delta\Phi(x),$$

with constant background $S_0^A = (\rho_0, \eta_0, T_0, \chi_0^I, \alpha_0)$ and Φ_0 a stationary solution of (3.42) (possibly $\Phi_0 = 0$). The emergent tetrad (and hence $g_{\rm eff}$) is state-built; at leading order we work in the *frozen-coefficients* approximation:

$$g_{\rm eff}{}_{\mu\nu}(x) = g^{(0)}_{\mu\nu} + \mathcal{O}(\partial\,\delta S), \qquad \hat\chi^\mu(x) = \hat\chi^\mu_0 + \mathcal{O}(\delta S), \qquad u^\mu(x) = u^\mu_0 + \mathcal{O}(\delta S), \tag{3.74}$$

so that principal parts are computed on $(g^{(0)}, \hat\chi_0, u_0)$ and feedback enters at lower order.

Constraints and projectors. Director normalization and orthogonality are enforced on-shell by $\mathcal{L}_{\rm constr}$. We project director fluctuations with

$$\Pi_\perp{}^{IJ} := \delta^{IJ} - \hat\chi_0^I \hat\chi_0^J, \qquad \delta\chi^I_\perp := \Pi_\perp{}^{IJ}\,\delta\chi^J, \tag{3.75}$$

so that only the two transverse components propagate at principal order.

WKB ansatz and principal symbol. For any field ψ we set $\psi \sim A\,e^{i\Theta}$ with wave covector $k_\mu = \partial_\mu\Theta$ and slowly varying amplitude A. The principal symbol $\mathbb{P}(k)$ is obtained by collecting the highest-derivative terms in the linearized equations with $\partial_\mu \to ik_\mu$.

3.8.2 Sector-wise principal symbols and cones

Structural scalars $X \in \{\rho, \eta, \alpha, T\}$. Linearizing (3.27), (3.29), (3.38) and the T-continuity law (3.31) around S_0 yields

$$\mathbb{P}_X(k)\,\delta X \;=\; Z^{(0)}_{XX}\,g^{(0)\,\mu\nu}k_\mu k_\nu\,\delta X \;+\; \zeta^{\parallel}_X\,(\hat\chi_0^\mu k_\mu)^2\,\delta X \;=\; 0 + (\text{lower order}), \tag{3.76}$$

where $\zeta^{\parallel}_X$ collects possible anisotropic weights induced by the state-built tetrad and constitutive dependence on S (it vanishes if Z_{AB} is isotropic at S_0). Thus the characteristic surfaces satisfy

$$g^{(0)\,\mu\nu}k_\mu k_\nu \;+\; \tilde\delta c_X\,(\hat\chi_0^\mu k_\mu)^2 \;=\; 0, \qquad \tilde\delta c_X := \zeta^{\parallel}_X / Z^{(0)}_{XX}. \tag{3.77}$$

For $\tilde\delta c_X > -1$ the cone is strictly inside or coincident with the $g^{(0)}$ light-cone.

Director transverse modes $\delta\chi^I_\perp$. From (3.36) we obtain

$$\mathbb{P}_\chi(k)\,\delta\chi^I_\perp \;=\; \Big[\,Z^{\perp(0)}_{\chi\chi}\,g^{(0)\,\mu\nu} + \zeta^{\parallel}_\chi\,\hat\chi_0^\mu\hat\chi_0^\nu\,\Big]k_\mu k_\nu\;\delta\chi^I_\perp \;=\; 0 + (\text{lower order}), \tag{3.78}$$

with an *identical* cone equation of the form (3.77) and two polarizations (the internal index I projected by $\Pi_\perp$). Mass terms arise from $\partial^2_\chi U|_\perp$ at lower order.

Matter field Φ. From (3.42) one reads the principal operator

$$\mathbb{P}_\Phi(k)\,\delta\Phi \;=\; \kappa_{\Phi,0}\,G^{(0)\,\mu\nu}_{(\rm ani)}\,k_\mu k_\nu\,\delta\Phi \;=\; 0 + (\text{lower order}), \qquad G^{(0)\,\mu\nu}_{(\rm ani)} := g^{(0)\,\mu\nu} + \delta c_0\,\hat\chi_0^\mu\hat\chi_0^\nu. \tag{3.79}$$

Hence the matter cone is

$$g^{(0)\,\mu\nu}k_\mu k_\nu + \delta c_0(\hat\chi_0^\mu k_\mu)^2 \;=\; 0, \tag{3.80}$$

inside the $g^{(0)}$ cone for $\delta c_0 > -1$.

Spectral curvature. The $Y_{AB}F\cdot F$ sector contains no higher-than-second derivatives on the *fields* (the Ehresmann connection is constitutive), so it contributes only to masses and mixings at linear order; it does *not* alter principal symbols under the rules adopted in §3.2.

3.8.3 Dispersion relations in ordered vs. isotropic strata

Isotropic background ($\alpha_0 = 0$). Let $g^{(0)}_{\mu\nu}$ be Minkowski, $u^\mu_0 = (1, \mathbf{0})$, and $\hat\chi^\mu_0$ arbitrary (physically irrelevant when $\alpha_0 = 0$ and $\delta c_0 = 0$). Plane waves $\sim e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$ satisfy

$$\omega_X^2 = c_X^2\,|\mathbf{k}|^2 + m_X^2, \qquad \omega_\chi^2 = c_\chi^2\,|\mathbf{k}|^2 + m_\chi^2, \qquad \omega_\Phi^2 = c_\Phi^2\,|\mathbf{k}|^2 + m_{\rm eff}^2, \tag{3.81}$$

with $c^2 \in (0, 1]$ determined by the corresponding kinetic weights divided by the background κ's and Z's. Here $m_X^2 \sim \partial^2_X U|_{S_0}$, $m_\chi^2 \sim \partial^2_\chi U|_\perp$, and $m^2_{\rm eff} = \partial V_\Phi/\partial|\Phi|^2|_{S_0}$.

Ordered background ($\alpha_0 > 0$). Decompose $\mathbf{k} = k_\parallel \hat{\boldsymbol{\chi}}_0 + \mathbf{k}_\perp$. To leading order in anisotropy,

$$\omega^2 \;=\; c_\perp^2\,|\mathbf{k}_\perp|^2 \;+\; c_\parallel^2\,k_\parallel^2 \;+\; m^2, \qquad c_\parallel^2 - c_\perp^2 \propto \delta c_0 \;\; (\text{for } \Phi), \tag{3.82}$$

and similarly for structural modes with $\tilde{\delta c}_X$ in (3.77). Inequalities $0 < c_{\perp,\parallel}^2 \le 1$ are ensured by the gates in (**??**) and (3.45).

Weak mixing. Mixing terms (e.g. $\lambda_3 J_\Phi^\mu \hat{\chi}_\mu$, $\lambda_4 (\nabla\Phi)^*(\nabla\Phi)\hat{\chi}^{(\mu}u^{\nu)}$) produce an $O(\lambda)$ Hermitian coupling matrix between $(\delta\Phi, \delta\chi_\perp, \delta X)$. At principal order, provided the bounds in §3.2 hold, cones do not cross or flip; mixings yield small avoided crossings in the dispersion relations.

3.8.4 Positivity and absence of ghosts

Quadratic action. Expanding the total Lagrangian to second order gives

$$\begin{aligned}\mathcal{L}^{(2)} = {}& \tfrac{1}{2}\, Z^{(0)}_{AB}\, g^{(0)\,\mu\nu} \partial_\mu \delta S^A \partial_\nu \delta S^B \;+\; \tfrac{1}{2}\,\kappa_{\Phi,0}\, G^{(0)\,\mu\nu}_{(\text{ani})}\, \partial_\mu \delta\Phi^* \partial_\nu \delta\Phi \\ & - \tfrac{1}{2}\,\delta S^A (\mathcal{M}^2)_{AB} \delta S^B - \tfrac{1}{2}\, m^2_{\text{eff}}\, \delta\Phi^* \delta\Phi \;+\; (\text{Hermitian mixings}).\end{aligned} \tag{3.83}$$

Canonical energy is positive if

$$Z^{(0)}_{AB} \succ 0, \qquad \kappa_{\Phi,0} > 0, \qquad G^{(0)\,00}_{(\text{ani})} < 0, \qquad (\mathcal{M}^2)_{AB} \text{ and } m^2_{\text{eff}} \text{ are non-negative on the chosen vacuum.} \tag{3.84}$$

The condition $G^{(0)\,00}_{(\text{ani})} < 0$ is equivalent to $\delta c_0 > -1$ when u_0^μ is timelike.

3.8.5 Special regimes and consistency checks

Adiabatic variations of ρ (Palatini frame). Slow gradients of ρ alter $\Xi(\rho)$ and thus the h-cone, while leaving principal parts second order and ghost-free. In the Jordan frame, second derivatives of Ξ appear in the lower-order tensor $\mathcal{D}_{\mu\nu}[M^2_{\text{eff}}]$ (cf. (3.54)), not in the principal symbol.

Near critical ordering. If $\partial_\alpha^2 U|_{S_0} \to 0$, the α-mode softens; anisotropy of other sectors then turns on at leading order through ∂_α-sensitivities (e.g. $\partial_\alpha \delta c\big|_{\alpha_0}$). Hyperbolicity remains intact provided $Z^{(0)}_{\alpha\alpha} > 0$.

Isotropic limit and Lorentz recovery. When $\delta c_0 = \tilde{\delta c}_X = 0$ and $Z^{(0)}_{AB}$ are proportional to a common scalar, all cones coincide with that of $g^{(0)}$ and the quadratic action becomes Lorentz invariant on the emergent background.

3.8.6 Summary of causal and stability gates

(i) Kinetic positivity:	$Z^{(0)}_{AB} \succ 0, \;\; \kappa_{\Phi,0} > 0;$
(ii) Cone inclusion:	$\delta c_0 > -1, \;\; \tilde{\delta c}_X > -1$ for all structural X;
(iii) Mass convexity:	$\partial^2_X U|_{S_0} \ge 0, \;\; \partial^2_\chi U|_\perp \ge 0, \;\; m^2_{\text{eff}} \ge 0;$
(iv) Adiabaticity:	$\lvert\partial S\rvert$ small on the background for frame-exchange terms;
(v) Mixing bounds:	EFT couplings λ_i small enough to avoid cone crossing.

(3.85)

Under (3.85) the linearized system is hyperbolic and ghost-free; its causal cones are nested within the emergent $g^{(0)}$ cone with controlled anisotropy along the director $\hat{\chi}_0$. These results justify, at PDE level, the construction of coherent structures and bound states undertaken in Chapter 4.

3.9 Guided Examples (Prelude to Solution Chapters)

This section assembles a set of worked templates that will be used repeatedly in Chapters 4–5. Each example is chosen to (i) exercise a distinct subset of the field equations derived in §3.4–§**??**, (ii) display the role of anisotropy and spectral curvature, and (iii) set up dimensionless reductions and boundary conditions suitable for analysis and numerics. We keep the constitutive choices minimal and reference §3.8 for linear stability gates.

Notation. Throughout this section we adopt the frozen-coefficients background approximation of (3.74) when extracting principal parts, and then restore slow feedback terms as needed. We denote by $c_\perp^2$ and $c_\parallel^2 = c_\perp^2 + \delta c_0$ the transverse/longitudinal kinetic weights in the matter cone (3.80).

3.9.1 Example I: Plane waves in an ordered homogeneous aether

Setup. Let $S^A = S_0^A = (\rho_0, \eta_0, T_0, \chi_0^I, \alpha_0)$ be constant with $\alpha_0 > 0$ (ordered stratum), $u_0^\mu = (1, \mathbf{0})$, and $g_{\mu\nu}^{(0)} = \mathrm{diag}(-1, 1, 1, 1)$. Consider small-amplitude matter waves and structural ripples.

Dispersion and polarization. From (3.79)–(3.82) one obtains

$$\omega_\Phi^2(\mathbf{k}) = c_\perp^2 |\mathbf{k}_\perp|^2 + c_\parallel^2 k_\parallel^2 + m_{\mathrm{eff}}^2, \qquad m_{\mathrm{eff}}^2 = \left.\frac{\partial V_\Phi}{\partial |\Phi|^2}\right|_{S_0}. \tag{3.86}$$

Group velocity and cone inclusion:

$$\mathbf{v}_g^\Phi = \frac{\partial \omega_\Phi}{\partial \mathbf{k}} = \frac{c_\perp^2 \mathbf{k}_\perp + c_\parallel^2 k_\parallel \hat{\boldsymbol{\chi}}_0}{\omega_\Phi}, \qquad 0 < c_\perp^2 \le c_\parallel^2 < 1 \quad \Rightarrow \quad |\mathbf{v}_g^\Phi| < 1. \tag{3.87}$$

Transverse director waves follow the same pattern with their own $(c_{\perp,\parallel}^{(\chi)}, m_\chi)$.

Energy and charge flux. The conserved $U(1)$ current (3.63) in this background is $J_\Phi^0 = \kappa_{\Phi,0}\, \omega_\Phi\, |\Phi|^2$, $\mathbf{J}_\Phi = \kappa_{\Phi,0}\, G_{(\mathrm{ani})}^{(0)\, ij}\, k_j |\Phi|^2$, exhibiting the anisotropic constitutive flux. These relations will benchmark later scattering problems.

3.9.2 Example II: Spherically symmetric soliton in an isotropic layer

Setup and ansatz. Take an isotropic stratum $\alpha_0 = 0$ with $\delta c_0 = 0$ and $g^{(0)}$ Minkowski. Consider a stationary, spherically symmetric configuration:

$$\Phi(t, r) = f(r)\, e^{-i\Omega T(t)}, \qquad \rho(r) = \rho_0 + \delta\rho(r), \quad \eta(r) = \eta_0 + \delta\eta(r), \quad \chi^I = \chi_0^I, \quad T = t, \tag{3.88}$$

with $f(r) \ge 0$ real. Assume $|\delta\rho|, |\delta\eta| \ll 1$ (weak backreaction) at this first pass.

Reduced radial equation. From (3.42) (with $G_{(\mathrm{ani})}^{\mu\nu} = g^{\mu\nu}$) we obtain the ODE

$$f'' + \frac{2}{r} f' = \frac{\partial}{\partial f}\left(\frac{1}{2\kappa_{\Phi,0}} \frac{\partial V_\Phi(|\Phi|; S_0)}{\partial |\Phi|^2}\, f^2 \right) - \Omega^2 f, \tag{3.89}$$

or equivalently $f'' + \frac{2}{r} f' = \partial_f U_{\mathrm{eff}}(f; \Omega)$, $U_{\mathrm{eff}}(f; \Omega) := \frac{1}{2\kappa_{\Phi,0}}\, m_{\mathrm{eff}}^2(S_0; f)\, f^2 - \frac{1}{2}\Omega^2 f^2$. Boundary conditions:

$$f'(0) = 0, \qquad f(r) \to 0 \text{ as } r \to \infty. \tag{3.90}$$

Existence window (Derrick-type). A standard scaling argument adapted to $g^{(0)}$ shows that nontrivial localized solutions exist when

$$0 < \Omega^2 < \min_{f > 0} \left[\frac{m_{\mathrm{eff}}^2(S_0; f)}{\kappa_{\Phi,0}} \right], \tag{3.91}$$

with stabilization supplied either by nonlinearity in V_Φ or by weak spectral-curvature pressure from $T_{\mu\nu}^{(\mathrm{spec})}$ (see below).

Energy, charge, and stability. The conserved charge and energy (neglecting small backreaction) are

$$Q_\Phi = 4\pi \int_0^\infty dr\, r^2\, \kappa_{\Phi,0}\, \Omega\, f^2, \tag{3.92}$$

$$E = 4\pi \int_0^\infty dr\, r^2 \left[\frac{\kappa_{\Phi,0}}{2} \big(f'^2 + \Omega^2 f^2 \big) + V_\Phi(f; S_0) \right]. \tag{3.93}$$

Classical stability follows from $dE/dQ_\Phi < \Omega$ (usual Q-ball criterion). In Chap. 4 we include the $Y_{AB} F{\cdot}F$ term; its Maxwell-like pressure contributes a short-range repulsion that can raise the lower bound of the window (3.91), consistent with the repulsive cores seen in Appendix E.

3.9.3 Example III: Domain wall in the order parameter with guided matter modes

Setup. Adopt a Landau potential for the order parameter, $U(\alpha) = \frac{a}{2}\alpha^2 + \frac{b}{4}\alpha^4,\ b > 0$, with $a < 0$ (ordered minima at $\pm\alpha_0 = \pm\sqrt{-a/b}$). Let the director point along z far away.

Static kink (wall). From (3.38) with time-independent fields and Y_{AB}, Z_{AB} taken constant for simplicity, the 1D profile $\alpha = \alpha(x)$ obeys

$$Z_{\alpha\alpha}\, \alpha'' = a\, \alpha + b\, \alpha^3, \qquad \alpha(x) = \alpha_0\, \tanh\!\Big(\frac{x}{\Delta}\Big), \quad \Delta = \sqrt{\frac{2 Z_{\alpha\alpha}}{|a|}}. \tag{3.94}$$

Guided modes of Φ. Assume $\delta c(\alpha)$ and $\kappa_\Phi(\alpha)$ vary across the wall. Seek modes of the form $\Phi(t, x, y, z) = \psi(x)\, e^{-i\omega t + i k_y y + i k_z z}$. Equation (3.42) reduces to a 1D Schrödinger-type problem

$$-\frac{d}{dx}\Big(\kappa_\Phi(\alpha)\, \frac{d\psi}{dx} \Big) + \Big[\kappa_\Phi(\alpha) \big(k_\perp^2 + \delta c(\alpha)\, k_\parallel^2 \big) + m_{\rm eff}^2(\alpha) \Big] \psi = \omega^2\, \kappa_\Phi(\alpha)\, \psi, \tag{3.95}$$

with $k_\parallel := k_z$ if $\hat\chi \parallel z$ and $k_\perp^2 := k_y^2$. A reduction of κ_Φ and/or $m_{\rm eff}^2$ inside the wall acts as a potential well, producing bound *guided* modes with dispersion $\omega^2(k_y, k_z)$ below the bulk continuum. This furnishes a waveguide mechanism for matter amplitudes along structural defects (Chap. 4).

3.9.4 Example IV: Spectral-curvature mediated two-body interaction (asymptotics)

Setup. Consider two well-separated, nearly spherical lumps of Φ centered at $\mathbf{R}_1, \mathbf{R}_2$, with characteristic size $R_0 \ll R := |\mathbf{R}_1 - \mathbf{R}_2|$. Let χ be approximately aligned and α small (weak ordering). Spectral curvature is generated locally by each lump through the constitutive $\Gamma^A{}_\mu(S)$.

Asymptotic interaction energy. At leading order the cross-energy mediated by $Y_{AB} F{\cdot}F$ reads

$$\Delta V_{\rm spec}(R) \simeq \int d^3x\, Y_{AB}(S_0)\, F^A_{(1)\,\mu\nu}(\mathbf{x})\, F^{B\,\mu\nu}_{(2)}(\mathbf{x}), \tag{3.96}$$

where $F_{(i)}$ is the curvature sourced by lump i in isolation. If each $F_{(i)}$ decays as R_0^p / r^{p+2} (with $p \geq 0$ determined by the constitutive algebra), one obtains a power-law tail

$$\Delta V_{\rm spec}(R) \propto \frac{R_0^{2p}}{R^{2p+1}}, \qquad (R \gg R_0), \tag{3.97}$$

while short-range overlap of core fields yields a repulsive correction $\Delta V_{\rm core}(R) \sim +\alpha_{\rm rep}/R^n$ with $n \geq 2$, consistent with the qualitative structure summarized in Appendix E. These ingredients underpin the bound-state analysis of Chap. 4.

3.9.5 Dimensionless reductions and numerical roadmaps

Scales and rescalings. Choose a characteristic mass m_* (e.g. $m_{\rm eff}(S_0)$), length $\ell_* = 1/m_*$, and define

$$\bar{x}^\mu := x^\mu/\ell_*, \quad \bar{\Phi} := \Phi/\Phi_*, \quad \bar{S}^A := S^A/S^A_*, \quad \bar{Y}_{AB} := Y_{AB}/Y_*, \quad \bar{Z}_{AB} := Z_{AB}/Z_*, \quad \text{etc.} \tag{3.98}$$

With these conventions the ODE/PDE templates above become dimensionless and suitable for shooting methods (radial solitons), spectral collocation (domain walls/guided modes), or finite-volume time marching (plane-wave stability and scattering).

Checklists (per example).

- *Plane waves*: verify $c^2_{\perp,\parallel} \in (0,1)$; extract $m^2_{\rm eff}$ and flux J^μ_Φ.
- *Soliton*: enforce (3.90); scan Ω in the window (3.91); evaluate $E(Q_\Phi)$.
- *Domain wall*: compute Δ in (3.94); solve (3.95) for bound modes; check orthogonality.
- *Two-body*: estimate p in (3.97) from isolated-source F; combine with core repulsion to locate minima.

Consistency gates (recap). All constructions above are valid under the linear hyperbolicity/positivity gates (3.85) and the gravitational gates (3.62), with background variations adiabatic on the solution scales.

These guided examples serve as the modular entry points for Chapter 4, where we construct explicit coherent structures (solitons, textures, guided modes) and quantify their stability, spectra, and interactions in the full nonlinear regime.

3.10 Irreversibility and the Arrow of Time (Prelude)

All equations up to §3.9 were *conservative*: they derive from a variational principle and admit exact Noether currents. In this section we introduce a *minimal, covariant, and controlled* dissipative extension that (i) respects the pre-geometric ethos (no external metric structures), (ii) singles out the internal clock flow $u^\mu(\partial T, \chi)$ as the physical time-direction, and (iii) guarantees nonnegative entropy production. This provides a dynamical origin for the arrow of time to be exploited in the decoherence and relaxation phenomena of later chapters.

Design principles.

1. *Covariance and second order:* All conservative principal symbols remain second order and hyperbolic inside the $g_{\rm eff}$ cone; dissipation enters as *first-order* drift along u^μ.
2. *State-built time-direction:* The only privileged direction is u^μ built from (T, χ); no external foliation is introduced.
3. *Positivity:* A Rayleigh density $\mathcal{R} \geq 0$ produces nonnegative entropy generation.

3.10.1 Minimal dissipative extension (Rayleigh density)

We postulate a non-variational *Rayleigh* density that is algebraic in the state and quadratic in the u-projected rates:

$$\boxed{\mathcal{R} \;=\; \frac{1}{2}\,\eta(S)\,\big|u^\mu \nabla_\mu \Phi\big|^2 \;+\; \frac{1}{2}\,\Gamma_{AB}(S)\,\big(u^\mu D_\mu S^A\big)\big(u^\nu D_\nu S^B\big),} \tag{3.99}$$

with $\Gamma_{AB}(S)$ positive semidefinite and $\eta(S) \geq 0$. The dissipative contributions are added to the Euler–Lagrange equations as Rayleigh forces projected along the aether time:

$$\text{Matter:} \quad \widetilde{\nabla}_\mu\Big(\kappa_\Phi G^{\mu\nu}_{(\text{ani})}\nabla_\nu\Phi\Big) - \frac{\partial V_\Phi}{\partial\Phi^*} + \mathcal{M}_\Phi \;=\; -\frac{\partial\mathcal{R}}{\partial\big(u^\mu\nabla_\mu\Phi^*\big)} \;=\; -\,\eta\,u^\mu\nabla_\mu\Phi, \tag{3.100}$$

$$\text{Structural:} \quad -\,\widetilde{\nabla}_\mu\big(Z_{CB}\,g^{\mu\nu}_{\text{eff}}D_\nu S^B\big) + \frac{1}{2}\,(\partial_C Z_{AB})\,g^{\mu\nu}_{\text{eff}}D_\mu S^A D_\nu S^B - \partial_C U \;+\; \cdots \;=\; -\frac{\partial\mathcal{R}}{\partial\big(u^\mu D_\mu S^C\big)} \;=\; -\,\Gamma_{CB}\,u^\mu D_\mu S^I \tag{3.101}$$

Dots denote the conservative spectral-curvature and feedback terms already displayed in (3.20). Importantly, the principal (second-order) parts are unmodified; dissipation enters only as *first-order* drift along u^μ and thus cannot create superluminal characteristics.

Frame remarks. The gravitational Palatini sector remains conservative; $\mathcal{R}$ does not depend on (ω, e) and hence does not modify §3.6 at principal order. Energy exchange with geometry still proceeds through the ρ-dependence of M_{eff} as in (3.67).

3.10.2 Local entropy balance and the arrow of time

Let u^μ define the local time direction and $h_{\mu\nu} := g_{\text{eff}\,\mu\nu} + u_\mu u_\nu$ the spatial projector. Multiplying (3.100) by $(u^\nu\nabla_\nu\Phi)^*$ and adding the complex conjugate, and performing the analogous step for (3.101), one derives a local free-energy balance that can be arranged as

$$\boxed{\widetilde{\nabla}_\mu s^\mu \;=\; \sigma_{\text{diss}} \;=\; \eta\,\big|u^\mu\nabla_\mu\Phi\big|^2 \;+\; \Gamma_{AB}\,\big(u^\mu D_\mu S^A\big)\big(u^\nu D_\nu S^B\big) \;\geq\; 0,} \tag{3.102}$$

for an appropriate entropy current $s^\mu = s\,u^\mu$ (its detailed constitutive definition is deferred to Chap. 5 where cosmological coarse-graining is introduced). Equation (3.102) is the *local second law*: the sign of η and Γ_{AB} fixes the time orientation — the *arrow of time* is the direction of increasing T along which $\sigma_{\text{diss}} \geq 0$.

Noether currents as balance laws. Because the Rayleigh functional is non-variational, the conservative continuity equations become balance laws with sinks:

$$\widetilde{\nabla}_\mu J^\mu_\Phi = -\,\mathcal{S}_\Phi, \qquad \mathcal{S}_\Phi \;=\; \eta\,\big|u^\mu\nabla_\mu\Phi\big|^2 \;\geq 0, \tag{3.103}$$

$$\widetilde{\nabla}_\mu J^\mu_T = -\,\mathcal{S}_T, \qquad \mathcal{S}_T \;=\; \Gamma_{AB}\,(u^\mu D_\mu S^A)(u^\nu D_\nu S^B) \;\geq 0, \tag{3.104}$$

where J^μ_Φ and J^μ_T are the conservative currents of §3.7.1. Thus $U(1)_\Phi$ and clock "charges" monotonically relax under dissipation, consistently with (3.102).

3.10.3 Small-amplitude damping and decoherence scale

Consider Example I (ordered homogeneous aether, §3.9.1) and add dissipation (3.100). For a plane-wave ansatz $\Phi \sim e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$ with $u^\mu = (1,\mathbf{0})$ and small η, the dispersion acquires an imaginary part:

$$\omega \;\approx\; \omega_0(\mathbf{k}) \;-\; i\,\frac{\eta}{2\kappa_{\Phi,0}}, \qquad \omega_0^2(\mathbf{k}) = c_\perp^2|\mathbf{k}_\perp|^2 + c_\parallel^2 k_\parallel^2 + m_{\text{eff}}^2, \tag{3.105}$$

so amplitudes decay as $e^{-t/\tau_{\text{dec}}}$ with

$$\boxed{\tau_{\text{dec}}^{-1} \;\simeq\; \frac{\eta}{2\,\kappa_{\Phi,0}} \quad \text{(homogeneous ordered background, weak damping).}} \tag{3.106}$$

In inhomogeneous media, $\eta(S)$ and $\kappa_\Phi(S)$ vary and (3.106) generalizes locally.

Double-slit heuristic. Let Δx be the slit separation and v_g the group speed from (3.87). Coherence is suppressed when the flight time exceeds τ_{dec}:

$$\frac{L}{v_g} \;\gtrsim\; \tau_{\text{dec}} \quad\Longrightarrow\quad L \;\gtrsim\; \frac{2\,\kappa_{\Phi,0}}{\eta}\,v_g, \tag{3.107}$$

with L the source–screen distance. Equation (3.107) furnishes an *environmental* decoherence threshold entirely determined by the aether state via η/κ_Φ.

3.10.4 Onsager structure and fluctuation–dissipation (outlook)

The quadratic form in (3.99) defines positive-semidefinite kinetic coefficients (η, Γ_{AB}) relating generalized fluxes to u-projected rates. In Chap. 5 we will extend this to a linear irreversible thermodynamics framework on the emergent background, including fluctuation sources with covariances proportional to (η, Γ_{AB}) so that an *aetheric fluctuation–dissipation relation* holds in near-equilibrium regimes.

3.10.5 Gates and consistency

$$\boxed{\eta(S) \geq 0, \qquad \Gamma_{AB}(S) \succeq 0, \qquad \text{and the conservative gates of §3.8.6 hold.}} \tag{3.108}$$

Under (3.108), dissipation selects the time orientation, ensures $\widetilde{\nabla}_\mu s^\mu \geq 0$, and leaves the conservative principal symbols and causal cones unchanged. This completes the prelude to irreversible dynamics; concrete applications (relaxation of solitons, equilibration of textures, and cosmological coarse-graining) will be developed in Chapters 4–5.

3.11 Executive Summary, Parameter Gates, and Roadmap

This section distills the full content of Chapter 3 into (i) a ready-to-use set of field equations, (ii) a compact constitutive "default" compatible with all hyperbolicity/positivity gates, (iii) a checklist of regimes where the effective description is valid, and (iv) a roadmap to the solution-building program of Chapters 4–5.

3.11.1 Ready-to-use dynamical system (conservative core)

Structural multiplet $S^A = (\rho, \eta, T, \chi^I, \alpha)$. The covariant Euler–Lagrange equation for each component $C \in \{\rho, \eta, T, \chi^I, \alpha\}$ is the master form (cf. (3.26)–(3.20)):

$$\boxed{-\widetilde{\nabla}_\mu\big(Z_{CB}\, g_{\rm eff}^{\mu\nu} D_\nu S^B\big) + \tfrac{1}{2}\,(\partial_C Z_{AB})\, g_{\rm eff}^{\mu\nu} D_\mu S^A D_\nu S^B - \partial_C U - \tfrac{1}{4}\,(\partial_C Y_{AB})\, F^A_{\mu\nu} F^{B\,\mu\nu} + \cdots = 0,} \tag{3.109}$$

with the spectral-curvature and metric/anisotropy feedback terms ("$\cdots$") as written explicitly in §3.3. The time field T further satisfies the Noether continuity law $\widetilde{\nabla}_\mu J^\mu_T = 0$ (cf. (3.31)).

Matter field Φ. The anisotropic Klein–Gordon type equation (cf. (3.42)) reads

$$\boxed{\widetilde{\nabla}_\mu\Big(\kappa_\Phi(S)\, G^{\mu\nu}_{(\rm ani)}(S)\, \nabla_\nu \Phi\Big) - \frac{\partial V_\Phi(|\Phi|; S)}{\partial \Phi^*} + \mathcal{M}_\Phi(S; \Phi) = 0,} \tag{3.110}$$

with conserved $U(1)$ current $\widetilde{\nabla}_\mu J^\mu_\Phi = 0$ when $U(1)_\Phi$ holds (cf. (3.43)–(3.44)).

Geometry (Palatini route). The spin-connection is Levi–Civita of $h_{\mu\nu} = \Xi(\rho)\, g_{{\rm eff}\,\mu\nu}$ (cf. (3.49)–(3.50)), and the effective Einstein equation is

$$\boxed{M^2_{\rm eff}(\rho)\, G_{\mu\nu}(h) = T^{(\rm tot)}_{\mu\nu}, \qquad h_{\mu\nu} = \Xi(\rho) g_{{\rm eff}\,\mu\nu}, \quad \Xi = \frac{M^2_{\rm eff}}{M^2_\star}.} \tag{3.111}$$

In the $g_{\rm eff}$-frame this is equivalent to (3.54) with $\mathcal{D}_{\mu\nu}[M^2_{\rm eff}]$.

Balance structure. Noether currents and energy–momentum balances are summarized by

$$\boxed{\widetilde{\nabla}_\mu J^\mu_\Phi = 0, \quad \widetilde{\nabla}_\mu J^\mu_T = 0, \quad \nabla^{(h)}_\mu(\Xi^{-1} T^\mu{}_\nu) = 0 \;\Leftrightarrow\; \widetilde{\nabla}_\mu T^\mu{}_\nu = -(\partial_\nu \ln \Xi)\, T + \mathcal{X}_\nu.} \tag{3.112}$$

Linear cones. On homogeneous backgrounds, the characteristic equations are (cf. §3.8)

$$g^{(0)\,\mu\nu} k_\mu k_\nu + \tilde{\delta c}_X (\hat\chi_0^\mu k_\mu)^2 = 0, \qquad g^{(0)\,\mu\nu} k_\mu k_\nu + \delta c_0 (\hat\chi_0^\mu k_\mu)^2 = 0, \tag{3.113}$$

for structural modes X and for Φ, respectively, with inclusion in the $g^{(0)}$-cone ensured by the gates below.

3.11.2 Minimal constitutive "default" (safe and usable)

To facilitate calculations and numerics, we recommend the following baseline, which is *consistent* with all gates in this chapter yet flexible enough to capture ordering and anisotropy:

1. **Kinetic metrics:** $Z_{AB} = \mathrm{diag}(Z_{\rho\rho}, Z_{\eta\eta}, Z_{TT}, Z_{\chi\chi}\Pi_{\perp}, Z_{\alpha\alpha})$ with positive entries; $Y_{AB} = Y_*\,\delta_{AB}$.

2. **Emergent tetrad:** $e^a{}_\mu = \bar{e}^a{}_\mu + A(\mathcal{S})\,u^a u_\mu + B(\mathcal{S})\,\hat{\chi}^a\hat{\chi}_\mu$ with $u\cdot\hat{\chi} = 0,\ u^2 = -1,\ \hat{\chi}^2 = +1$.

3. **Matter anisotropy:** $G^{\mu\nu}_{(\mathrm{ani})} = g^{\mu\nu}_{\mathrm{eff}} + \delta c(\alpha)\,\hat{\chi}^\mu\hat{\chi}^\nu$, $\kappa_\Phi = \kappa_{\Phi,0}[1 + \kappa_1(\rho-\rho_0)/\rho_0 + \kappa_2(\eta-\eta_0)]$.

4. **Potentials:** $U(\mathcal{S}) = \sum_X \frac{1}{2}m_X^2(X-X_0)^2 + \frac{\lambda_\alpha}{4}(\alpha^2-\alpha_0^2)^2 + \cdots, \quad V_\Phi = \frac{1}{2}m^2_{\mathrm{eff}}(\rho,\eta,\alpha)|\Phi|^2 + \frac{\lambda_\Phi}{4}|\Phi|^4$.

5. **Gravitational coupling:** $M^2_{\mathrm{eff}}(\rho) = M_\star^2\left(1 + \beta_\rho\frac{\rho-\rho_0}{\rho_0}\right)$, with $|\beta_\rho| \ll 1$ in weak-field applications.

6. **Mixing (small-EFT):** retain $\lambda_3 J^\mu_\Phi\hat{\chi}_\mu$ and $\lambda_4(\nabla\Phi)^*(\nabla\Phi)\hat{\chi}^{(\mu}u^{\nu)}$ with $|\lambda_{3,4}| \ll 1$.

7. **Topological sector:** include $S_{\mathrm{top}} = \theta_\chi\,\mathcal{H}[\hat{\chi}] + \theta_{\mathrm{CS}}\int \mathrm{Tr}(\Gamma\wedge F - \frac{1}{3}\Gamma^3)$ with constant θ's (no bulk EL-terms, but nontrivial charges).

This default serves as the reference point for all examples in §3.9 and for the nonlinear constructions of Chapter 4.

3.11.3 Parameter gates, units, and dimensional sanity

Positivity/hyperbolicity (master list).

$$\boxed{Z_{AB} \succ 0, \quad Y_{AB} \succ 0, \quad \kappa_\Phi > 0, \quad \delta c > -1, \quad M^2_{\mathrm{eff}} > 0, \quad \partial^2 U|_{\mathrm{vac}} \geq 0, \quad m^2_{\mathrm{eff}} \geq 0.} \tag{3.114}$$

Units (in $c = \hbar = 1$). Mass dimension:

$$[\mathcal{L}] = 4, \quad [\Phi] = 1, \quad [S^A] = 1 \text{ (scalars)}, \ [\chi^I] = 0 \text{ (directional angles)}, \ [\kappa_\Phi] = 0, \ [Z_{AB}] = 0, \ [Y_{AB}] = 0, \ [M^2_{\mathrm{eff}}] = 2.$$

These conventions make all kinetic coefficients dimensionless; potentials carry the required mass powers.

Adiabaticity and EFT bounds. Gradients of constitutive functions must be small on solution scales: $|\partial Z|, |\partial Y|, |\partial\kappa_\Phi|, |\partial\delta c| \ll \Lambda_{\mathrm{EFT}}$, with $\Lambda_{\mathrm{EFT}} \sim M_\star$ or the dominant mass in U, V_Φ.

3.11.4 Validity regimes and expected breakdowns

- **Weak curvature / low frequency:** second derivatives of g_{eff} and h small compared to $M_\star^2$; spectral-curvature backreaction contained by Y_{AB}.

- **Moderate anisotropy:** $|\delta c| < 1$ and $|A|, |B| \lesssim \mathcal{O}(1)$ so that cones remain nested.

- **Near-critical ordering:** allowed provided $Z_{\alpha\alpha} > 0$; soft α-mode treated nonperturbatively in Chap. 4.

- **Breakdown flags:** appearance of negative eigenvalues in $\mathcal{M}^2$ beyond intended phases, cone crossing from large mixings, or rapid spatial variation of $\Xi(\rho)$ invalidating the Palatini adiabatic split.

3.11.5 Observables and near-term signatures

1. **Anisotropic dispersion:** difference $c_\parallel^2 - c_\perp^2 = \delta c$ for matter waves or structural ripples in ordered media (laboratory analogs, waveguiding along domain walls).
2. **Environment-dependent inertia:** shifts in $m_{\rm eff}^2(\rho, \eta, \alpha)$ visible as trapping/scattering and modified bound-state spectra.
3. **Effective G-variations:** slow drifts of $G_{\rm eff} \propto 1/M_{\rm eff}^2(\rho)$ in inhomogeneous aether (to be constrained cosmologically in Chap. 5).
4. **Spectral-curvature forces:** repulsive cores and long-range tails in two-body potentials (asymptotics of §3.9.4).

3.11.6 Roadmap to Chapters 4–5

Chapter 4 (Coherent structures and interactions).

- *Solitons/Q-balls* in isotropic and ordered strata; stability via $E(Q_\Phi)$ and linear modes.
- *Director textures* and topological charges; pinning by spectral curvature.
- *Guided modes* along α-walls; waveguide dispersion and scattering.
- *Two-body potentials* with spectral-curvature mediation; bound "molecules" of solitons.

Chapter 5 (Macroscopic limits & cosmological coarse-graining).

- Palatini frame versus Jordan frame at large scales; effective Friedmann equations.
- Irreversibility: $\mathcal{R}$, entropy production, and emergent arrows of time in expanding backgrounds.
- Parameter inference from observables (anisotropic propagation, $G_{\rm eff}$ constraints, texture spectra).

With these summaries, defaults, and gates, Chapter 3 provides a complete and self-consistent dynamical platform. The next chapter puts this machinery to work on explicit solutions, spectra, and interactions.

3.12 Appendix 3.A: Technical Derivations for Chapter 3

A.1 Conformal identities for $h_{\mu\nu} = \Xi(\rho)\, g_{{\rm eff}\,\mu\nu}$

Let $\Omega := \Xi^{1/2} > 0$. The Levi–Civita connections of $h_{\mu\nu} = \Omega^2 g_{\mu\nu}$ and $g_{\mu\nu} \equiv g_{\rm eff}$ are related by

$$\Gamma(h)^\lambda{}_{\mu\nu} = \Gamma(g)^\lambda{}_{\mu\nu} + \delta^\lambda{}_\mu\, \partial_\nu \ln\Omega + \delta^\lambda{}_\nu\, \partial_\mu \ln\Omega - g_{\mu\nu}\, g^{\lambda\sigma}\partial_\sigma \ln\Omega. \tag{3.115}$$

The Ricci tensors obey (in $d = 4$)

$$R_{\mu\nu}(h) = R_{\mu\nu}(g) - 2\nabla_\mu\nabla_\nu \ln\Omega - g_{\mu\nu}\Box \ln\Omega + 2\,\partial_\mu \ln\Omega\, \partial_\nu \ln\Omega - 2g_{\mu\nu}(\partial \ln\Omega)^2, \tag{3.116}$$

$$R(h) = \Omega^{-2}\Big(R(g) - 6\Box \ln\Omega - 6(\partial \ln\Omega)^2\Big). \tag{3.117}$$

From these one finds the Einstein tensors relation

$$G_{\mu\nu}(h) = G_{\mu\nu}(g) + \mathcal{C}_{\mu\nu}[\Omega], \qquad \mathcal{C}_{\mu\nu}[\Omega] := 2\Big(\nabla_\mu \ln\Omega \nabla_\nu \ln\Omega - \nabla_\mu\nabla_\nu \ln\Omega\Big) + g_{\mu\nu}\,\Xi_\Omega, \tag{3.118}$$

with $\Xi_\Omega = 2\Box \ln\Omega + (\partial \ln\Omega)^2$.

If $\Xi = M_{\rm eff}^2/M_\star^2$ with $M_{\rm eff} > 0$, it is convenient to write the frame-change tensor in terms of $M_{\rm eff}^2$:

$$\boxed{\mathcal{D}_{\mu\nu}\big[M_{\rm eff}^2\big] := \nabla_\mu\nabla_\nu M_{\rm eff}^2 - g_{\mu\nu}\Box M_{\rm eff}^2 - \frac{3}{2}\frac{1}{M_{\rm eff}^2}\Big(\nabla_\mu M_{\rm eff}^2 \nabla_\nu M_{\rm eff}^2 - \tfrac{1}{2} g_{\mu\nu}(\nabla M_{\rm eff}^2)^2\Big),} \tag{3.119}$$

so that the Jordan-frame form (3.54) follows from $M_{\rm eff}^2\, G_{\mu\nu}(h) = T_{\mu\nu}$.

3.12.1 A.2 Variation of the matter stress tensor with anisotropy

For $\mathcal{L}_\Phi = \kappa_\Phi\, G^{\mu\nu}_{(\mathrm{ani})}(\nabla_\mu\Phi)^*(\nabla_\nu\Phi) - V_\Phi(|\Phi|;S)$, where $G^{\mu\nu}_{(\mathrm{ani})} = g^{\mu\nu} + \delta c\,\hat\chi^\mu\hat\chi^\nu$ and the unit director satisfies $\hat\chi^\mu\hat\chi_\mu = +1$, the Hilbert tensor $T^{(\Phi)}_{\mu\nu} := -\frac{2}{e}\frac{\delta(e\mathcal{L}_\Phi)}{\delta g^{\mu\nu}}$ splits as

$$\begin{aligned} T^{(\Phi)}_{\mu\nu} &= \kappa_\Phi\Big[(\nabla_\mu\Phi)^*(\nabla_\nu\Phi)\Big]_{\mathrm{sym}} - g_{\mu\nu}\,\mathcal{L}_\Phi \\ &\quad + \kappa_\Phi\,\delta c\Big[\hat\chi_\mu\hat\chi_\nu\,(\nabla_\alpha\Phi)^*(\nabla_\beta\Phi)\,g^{\alpha\beta} - \tfrac{1}{2}g_{\mu\nu}\,\hat\chi^\alpha\hat\chi^\beta(\nabla_\alpha\Phi)^*(\nabla_\beta\Phi)\Big] + \Lambda_\chi\,\hat\chi_\mu\hat\chi_\nu. \end{aligned} \tag{3.120}$$

The last term originates from the normalization constraint implemented via $\mathcal{L}_{\mathrm{constr}} \supset \Lambda_\chi\big(\hat\chi^\mu\hat\chi_\mu - 1\big)$, and Λ_χ is fixed by the ($\hat\chi$, constraint) EL-equations. More refined expressions that include the metric dependence of the projector built from $(u^\mu, \hat\chi^\mu)$ can be obtained, but (3.120) is sufficient for the uses in §3.6.3.

3.12.2 A.3 Proof sketches of balance laws with dissipation

With the Rayleigh density $\mathcal{R} = \frac{1}{2}\,\eta\,|u\cdot\nabla\Phi|^2 + \frac{1}{2}\,\Gamma_{AB}(u\cdot DS^A)(u\cdot DS^B)$ the modified EL-equations contain sinks $\mathcal{S}_\Phi = \eta\,|u\cdot\nabla\Phi|^2 \geq 0$, $\mathcal{S}_T = \Gamma_{AB}(u\cdot DS^A)(u\cdot DS^B) \geq 0$. Contracting the matter equation with $(u\cdot\nabla\Phi)^*$ and the structural equations with $u\cdot DS^C$ and summing, one obtains

$$\widetilde{\nabla}_\mu s^\mu = \sigma_{\mathrm{diss}} = \eta\,|u\cdot\nabla\Phi|^2 + \Gamma_{AB}(u\cdot DS^A)(u\cdot DS^B) \geq 0,$$

and the Noether relations become the balance laws (3.103)–(3.104).

A.4 Principal symbols and hyperbolicity (sketch)

Linearization around a homogeneous state with frozen coefficients yields principal operators $Z^{(0)}_{AB}g^{(0)\mu\nu}\partial_\mu\partial_\nu$ for the scalars and $\kappa_{\Phi,0}G^{(0)\mu\nu}_{(\mathrm{ani})}\partial_\mu\partial_\nu$ for Φ. Under the gates $Z^{(0)}_{AB} \succ 0$, $\kappa_{\Phi,0} > 0$, $\delta c_0 > -1$, the corresponding quadratic forms have Lorentzian signatures with time function u^μ_0, ensuring hyperbolicity and subluminal cones as summarized in §3.8.

3.12.3 A.5 Boundary term for a well-posed Palatini variation (remark)

In first-order form the gravitational action requires a boundary term to cancel normal variations of the connection. A minimal choice is

$$S_{\partial M} = \frac{M^2_{\mathrm{eff}}}{2}\int_{\partial M} d^3x\,\epsilon^{\mu\nu\rho\sigma}\,e_{\mu a}e_{\nu b}\,\omega_\rho{}^{ab}\,n_\sigma, \tag{3.121}$$

with outward normal n_σ and induced triad on ∂M. For compact domains with fixed (e,ω) on the boundary this term vanishes; otherwise it renders δS free of boundary variations.

Chapter 3 — Tables and Illustrative Figures

Table 3.1: Schematic Euler–Lagrange equations and conservation laws in Chapter 3. Compact, variation-level summary; coefficients are state-dependent functions of $S = \{\rho, \eta, T, \chi_I, \alpha\}$, consistent with the Palatini split and spectral covariance.

Field	**Euler–Lagrange (schematic form)**	**Conserved current / constraint**	**Physical role**
ρ	$Z_\rho\,\nabla_\mu\nabla^\mu\rho + \mathcal{C}_\rho(S;\nabla S) = \partial_\rho U(S) + \partial_\rho\mathcal{L}_{\rm curv} + \partial_\rho\mathcal{L}_\Phi$	—	Sets aether density and effective Planck scale $M_{\rm eff}(\rho)$; controls stiffness/propagation.
η	$Z_\eta\,\nabla^2\eta + \mathcal{C}_\eta(S;\nabla S) = \partial_\eta U + \partial_\eta\mathcal{L}_\Phi$ (with GKSL-compatible dissipation when included)	production rate $\sigma_{\rm diss} \propto \eta\,(\nabla\Phi)^2 \geq 0$	Spectral nonlinearity/viscosity; arrow-of-time and decoherence proxy.
T	$\nabla_\mu(\mathcal{K}_T(S)\,\nabla^\mu T) + \mathcal{S}_T(S;\nabla S) = 0$	$u^\mu \equiv$ normalized flow from $\partial^\mu T$, with $u^2 = -1$	Internal time order; defines u^μ and temporal direction in $g_{\rm eff}$.
χ_I	$\Pi^\nu{}_\mu\,\mathcal{D}_\nu\Big(\mathcal{K}_\chi\,\mathcal{D}^\mu\chi_I\Big) = \mathcal{S}_{\chi_I}(S;\nabla S)$	$\hat\chi^2 = +1,\ \ u\cdot\hat\chi = 0$	Polarization/direction; seeds anisotropy and emergent gauge structure.
α	$\nabla_\mu\Big(\mathcal{K}_\alpha(S)\,\nabla^\mu\alpha + J^\mu_{\rm spec}(S)\Big) = 0$	Spectral $U(1)$: $\nabla_\mu J^\mu_{\rm spec} = 0$	Spectral phase; mediates emergent Abelian responses and vortices.
Φ	$\nabla_\mu\Big(\mathcal{Z}_\Phi(S)\,\nabla^\mu\Phi\Big) + m^2_{\rm eff}(S)\,\Phi + \lambda_{\rm eff}(S)\lvert\Phi\rvert^2\Phi + \cdots = 0$	$U(1)_\Phi$: $J^\mu_\Phi = {\rm Im}(\Phi^*\nabla^\mu\Phi)$, $\nabla_\mu J^\mu_\Phi = 0$	Matter carrier; supports solitons, bound states, and radiative processes.
$e^a{}_\mu,\ \omega_\mu{}^{ab}$	Palatini variation: $G_{\mu\nu}(e,\omega) = M^{-2}_{\rm eff}\,T^{\rm eff}_{\mu\nu}[S,\Phi]$; metric-compatibility and vanishing torsion (or controlled contortion) as imposed	$\nabla^\mu T^{\rm eff}_{\mu\nu} = 0$	Emergent gravity; determines causal cones and backreaction.

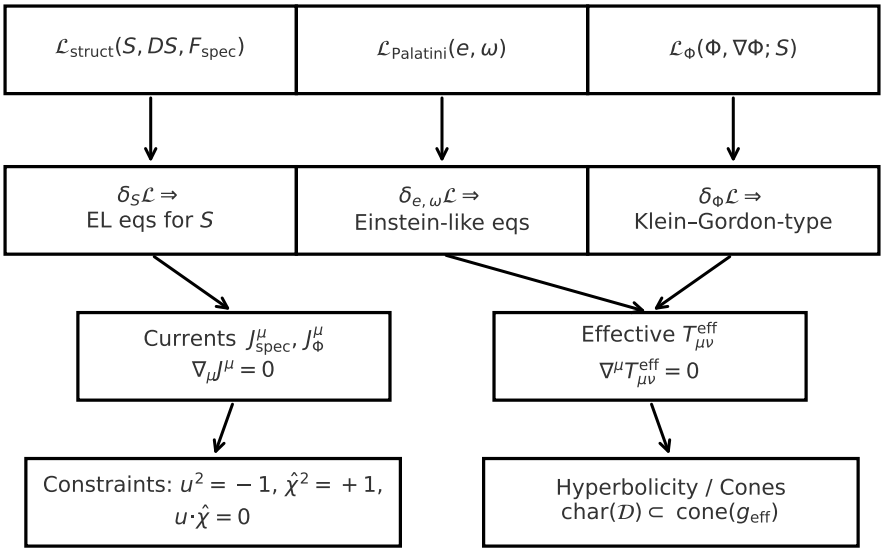

Figure 3.1: Variational flow in Chapter 3. The total action splits into structural, Palatini (coframe/connection), and matter sectors. Independent variations yield the structural Euler–Lagrange equations, an Einstein-like equation for $g_{\text{eff}}(e, \omega)$, and a Klein–Gordon-type equation for Φ. Noether currents (spectral and matter) and the effective stress tensor satisfy covariant conservation, while normalization/orthogonality constraints define the admissible state manifold.

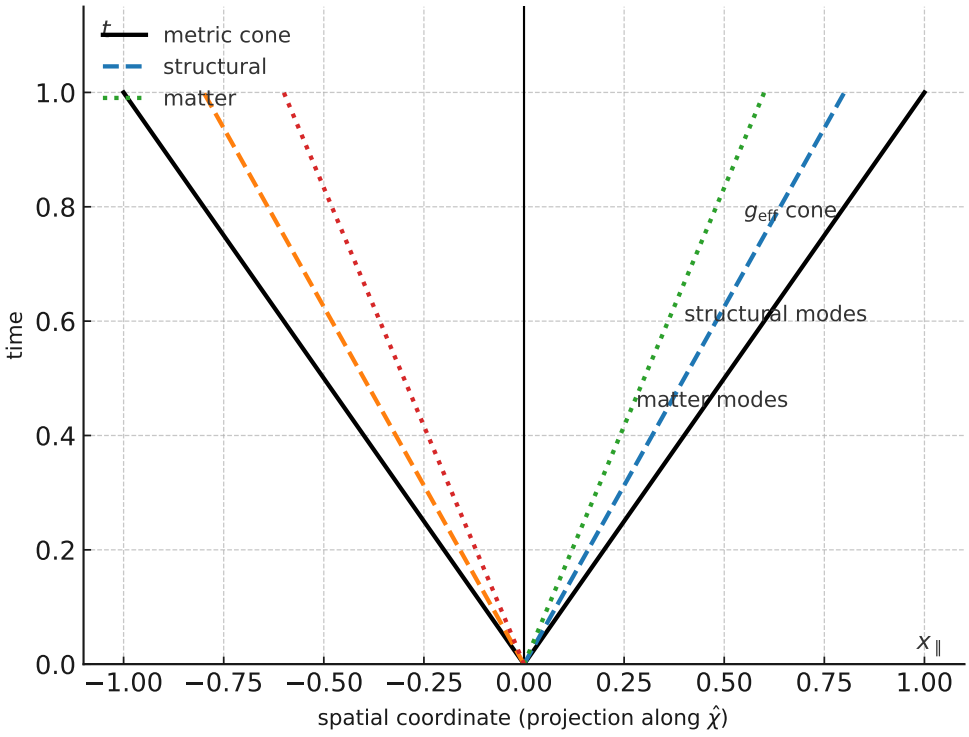

Figure 3.2: Characteristic structure and causal cones. Cross-section of propagation cones along the anisotropy axis $\hat{\chi}$. The g_{eff}-cone bounds all physical characteristics (black). Structural modes (dashed) and matter modes (dotted) have speeds below unity and can differ from each other due to anisotropy $(c_\perp, \delta c)$ and state-dependent couplings, ensuring hyperbolicity and causality.

Chapter 4

Chapter 4: Coherent Structures, Topological Textures, and Interactions

4.1 Common Setup: Nondimensionalization, Boundary Conditions, and Verification

This section establishes a self-contained technical scaffold for all solution-building in this chapter. We (i) choose canonical scales and produce a dimensionless form of the conservative dynamics derived in Ch. 3, (ii) define boundary-value problems and functional spaces for the principal families of solutions (solitons, walls, textures, guided modes), and (iii) collect diagnostic identities (Derrick/virial, Noether, energy–charge monotonicity) and well-posedness gates needed for both analysis and numerics.

Notation. We write the structural multiplet $S^A = (\rho, \eta, T, \chi^I, \alpha)$, the matter field Φ, and use $g_{\rm eff}$ and $G^{\mu\nu}_{(\rm ani)} = g^{\mu\nu}_{\rm eff} + \delta c(\alpha)\,\hat\chi^\mu\hat\chi^\nu$ as in Ch. 3. The covariant derivative is ∇_μ (with feedback) and D_μ denotes the horizontal derivative acting on S-components (cf. §3.2).

4.1.1 Choice of scales and nondimensionalization

Canonical scales. Pick a reference mass $m_* > 0$ (e.g. a bulk $m_{\rm eff}$ of Φ), set a length $\ell_* := 1/m_*$, an amplitude scale $\Phi_* > 0$, and structural scales $S^A_* > 0$. Define dimensionless variables (bars denote dimensionless quantities):

$$\bar x^\mu := \frac{x^\mu}{\ell_*}, \quad \bar\Phi := \frac{\Phi}{\Phi_*}, \quad \bar S^A := \frac{S^A}{S^A_*}, \quad \bar g_{\mu\nu} := g_{{\rm eff}\,\mu\nu}, \quad \bar{\hat\chi}^\mu := \hat\chi^\mu, \tag{4.1}$$

and dimensionless couplings

$$\bar\kappa_\Phi := \kappa_\Phi, \quad \bar Z_{AB} := Z_{AB}, \quad \bar Y_{AB} := Y_{AB}, \quad \bar{\delta c} := \delta c, \quad \bar M^2_{\rm eff} := \frac{M^2_{\rm eff}}{m_*^2}, \quad \bar U := \frac{U}{m_*^4}, \quad \bar V_\Phi := \frac{V_\Phi}{m_*^4}. \tag{4.2}$$

(With $c = \hbar = 1$, the kinetic coefficients can be chosen dimensionless; cf. §3.10.)

Dimensionless conservative action. Up to an overall factor m_*^4, the Lagrangian density becomes

$$\begin{aligned}\bar{\mathcal L} = {}& \tfrac12\,\bar Z_{AB}\,\bar g^{\mu\nu}\bar D_\mu\bar S^A\bar D_\nu\bar S^B \;-\; \bar U(\bar S) \;-\; \tfrac14\,\bar Y_{AB}\,\bar F^A_{\mu\nu}\bar F^{B\,\mu\nu} \\ & +\; \bar\kappa_\Phi(\bar S)\,\bar G^{\mu\nu}_{(\rm ani)}(\bar S)\,(\bar\nabla_\mu\bar\Phi)^*(\bar\nabla_\nu\bar\Phi) \;-\; \bar V_\Phi(|\bar\Phi|;\bar S) \;+\; \bar{\mathcal L}_{\rm constr},\end{aligned} \tag{4.3}$$

where $\bar G^{\mu\nu}_{(\rm ani)} = \bar g^{\mu\nu} + \bar{\delta c}\,\bar{\hat\chi}^\mu\bar{\hat\chi}^\nu$. The gravitational Palatini piece is referenced but we work in local backgrounds where curvature is negligible for the principal solution-building (global backreaction enters in §4.7).

Dimensionless field equations. The Euler–Lagrange equations retain form with bars:

$$-\,\bar{\tilde{\nabla}}_\mu\big(\bar{Z}_{CB}\,\bar{g}^{\mu\nu}\bar{D}_\nu\bar{S}^B\big)+\tfrac{1}{2}\,(\partial_C\bar{Z}_{AB})\,\bar{g}^{\mu\nu}\bar{D}_\mu\bar{S}^A\bar{D}_\nu\bar{S}^B-\partial_C\bar{U}-\tfrac{1}{4}\,(\partial_C\bar{Y}_{AB})\,\bar{F}^A\cdot\bar{F}^B+\cdots=0, \tag{4.4}$$

$$\bar{\tilde{\nabla}}_\mu\Big(\bar{\kappa}_\Phi\bar{G}^{\mu\nu}_{(\mathrm{ani})}\,\bar{\nabla}_\nu\bar{\Phi}\Big)-\frac{\partial\bar{V}_\Phi}{\partial\bar{\Phi}^*}+\bar{\mathcal{M}}_\Phi(\bar{S};\bar{\Phi})=0, \tag{4.5}$$

with the same feedback terms as in §3.2–§3.4. Henceforth we drop bars unless clarity demands them.

Gate preservation under scaling. All hyperbolicity/positivity gates (§3.7, §3.10) are invariant under the rescalings (4.1)–(4.2). In particular,

$$Z_{AB}\succ 0,\quad \kappa_\Phi>0,\quad \delta c>-1,\quad \partial^2U|_{\mathrm{vac}}\geq 0,\quad m^2_{\mathrm{eff}}\geq 0,\quad M^2_{\mathrm{eff}}>0. \tag{4.6}$$

4.1.2 Boundary-value problems and functional spaces

We formulate each solution family as a variational problem on appropriate function spaces.

(a) Localized solitons (3D). Seek finite-energy, stationary solutions $\Phi(t,\mathbf{x})=f(\mathbf{x})\,e^{-i\Omega T(t)}$ with $f\in H^1(\mathbb{R}^3)$, and structural fields $S^A\in H^1(\mathbb{R}^3)$ approaching a homogeneous state $S^A\to S^A_\infty$ as $|\mathbf{x}|\to\infty$. Finite energy requires

$$f(\mathbf{x})\to 0,\quad \partial_i f\in L^2(\mathbb{R}^3),\quad D_iS^A\in L^2(\mathbb{R}^3),\quad \hat{\chi}^\mu\hat{\chi}_\mu\to 1,\quad u^\mu u_\mu\to -1. \tag{4.7}$$

In spherical symmetry $f(r)\in H^1((0,\infty);r^2dr)$ with $f'(0)=0$, $f(r)\to 0$ as $r\to\infty$.

(b) Phase/domain walls (1D). For order parameter $\alpha(x)$ connecting $\pm\alpha_0$,

$$\alpha(x)\in H^1_{\mathrm{loc}}(\mathbb{R}),\quad \alpha(x)\to\pm\alpha_0 \text{ as } x\to\pm\infty,\quad \hat{\chi}^\mu\to\hat{\chi}^\mu_\pm \text{ (consistent with the wall orientation).} \tag{4.8}$$

Guided modes of Φ along the wall live in $L^2(\mathbb{R})$ in the transverse coordinate with Bloch/Fourier structure along tangential directions.

(c) Director textures and Hopfions (3D). Compactify $\mathbb{R}^3$ by a point at infinity; require

$$\hat{\chi}:\ S^3\to S^2,\qquad \hat{\chi}(\infty)=\hat{\chi}_\infty,\qquad |\hat{\chi}|=1 \text{ a.e.} \tag{4.9}$$

This topologizes configurations by the Hopf invariant $\mathcal{H}\in\mathbb{Z}$ (cf. §3.6).

(d) Periodic boxes (numerics). On $[0,L]^d$ we take periodic or absorbing boundaries; for localized structures we ensure $L\gg$ core size and monitor wrap-around via energy density at the boundary.

Constraint enforcement. Normalization and orthogonality, $\hat{\chi}^\mu\hat{\chi}_\mu=1,\ u^\mu u_\mu=-1,\ u\cdot\hat{\chi}=0$, are enforced weakly by $\mathcal{L}_{\mathrm{constr}}$ or by geometric projection after each nonlinear step:

$$\hat{\chi}\leftarrow\frac{\hat{\chi}}{\sqrt{\hat{\chi}^2}},\qquad u\leftarrow\frac{u-(u\cdot\hat{\chi})\hat{\chi}}{\sqrt{|u^2-(u\cdot\hat{\chi})^2|}}. \tag{4.10}$$

4.1.3 Conserved quantities and diagnostic identities

Energy and charge. In stationary backgrounds, the conserved $U(1)$ charge and energy read

$$Q_\Phi=\int d^3x\;\kappa_\Phi\,\mathrm{Im}(\Phi^*\,u^\mu\nabla_\mu\Phi)\,, \tag{4.11}$$

$$E=\int d^3x\,\Big[\tfrac{1}{2}Z_{AB}\,g^{ij}D_iS^AD_jS^B+\kappa_\Phi\,G^{ij}_{(\mathrm{ani})}(\nabla_i\Phi)^*(\nabla_j\Phi)+V_\Phi(|\Phi|;S)+U(S)+\tfrac{1}{4}Y_{AB}F^A\cdot F^B\Big], \tag{4.12}$$

where i,j are spatial indices in the local rest frame ($u^\mu=(1,\mathbf{0})$). The energy–charge slope dE/dQ_Φ is a key stability indicator for Q-type solitons.

Anisotropic Derrick/virial identity (localized Φ). Let $\Phi_\lambda(\mathbf{x}) := \Phi(\lambda \mathbf{x}_\perp, \lambda^\gamma x_\parallel)$ with $x_\parallel := \hat{\boldsymbol{\chi}}\cdot\mathbf{x}$, $\mathbf{x}_\perp := \mathbf{x} - x_\parallel \hat{\boldsymbol{\chi}}$, and scale the structural fields analogously so as to preserve constraints. Stationarity of $E[\Phi_\lambda, S_\lambda] - \Omega Q_\Phi[\Phi_\lambda]$ at $\lambda = 1$ yields

$$\boxed{2\,\mathcal{T}_\perp + 2\gamma\,\mathcal{T}_\parallel + \sum_{A,B}\left(\mathcal{S}_\perp^{AB} + \gamma\,\mathcal{S}_\parallel^{AB}\right) - 3\left(U + V_{\text{eff}}\right) - \mathcal{C}_{\text{spec}} \;=\; 0,} \tag{4.13}$$

where

$$\mathcal{T}_\perp := \int \kappa_\Phi\,|\nabla_\perp \Phi|^2, \quad \mathcal{T}_\parallel := \int \kappa_\Phi\,(1+\delta c)\,|\partial_\parallel \Phi|^2,$$
$$\mathcal{S}_{\perp,\parallel}^{AB} := \tfrac{1}{2}\int Z_{AB}\,|D_{\perp,\parallel} S|^2, \qquad V_{\text{eff}} := V_\Phi - \tfrac{1}{2}\Omega^2 \kappa_\Phi |\Phi|^2,$$

and $\mathcal{C}_{\text{spec}}$ collects the homogeneous scaling of the $Y_{AB}F\cdot F$ sector. Choosing $\gamma = 1$ recovers the isotropic identity; $\gamma \neq 1$ probes longitudinal bias. Equation (4.13) is a necessary condition for the existence of finite-energy stationary localized states and is a powerful numerical check (residual $\ll 1$).

Guided-mode orthogonality (walls). For the 1D eigenproblem along a wall (cf. §3.8), with weight $\kappa_\Phi(\alpha(x))$:

$$\int_{-\infty}^{\infty} dx\; \kappa_\Phi(\alpha)\,\psi_m(x)\,\psi_n(x) = \delta_{mn}, \qquad \omega_m^2 < \omega_{\text{cont}}^2 \;\Rightarrow\; \psi_m \in L^2(\mathbb{R}). \tag{4.14}$$

This weighted orthogonality is preserved under slow wall modulations and is used to build multi-mode ansätze.

Topological charge (textures). With a unit director $\hat{\chi} : S^3 \to S^2$, the Hopf invariant

$$\mathcal{H}[\hat{\chi}] = \frac{1}{(4\pi)^2}\int_{\mathbb{R}^3} d^3x\; \epsilon^{ijk} A_i(\mathbf{x})\,\partial_j A_k(\mathbf{x}), \quad \nabla\times\mathbf{A} = \mathbf{J}[\hat{\chi}], \tag{4.15}$$

is integer-valued and conserved under smooth dynamics; it stratifies the configuration space and supplies a robust nonperturbative label for texture families.

4.1.4 Well-posedness, numerical stability, and CFL gates

Hyperbolic cones (recap). Linear characteristics for small fluctuations on homogeneous backgrounds satisfy

$$g^{(0)\,\mu\nu}k_\mu k_\nu + \tilde{\delta c}_X(\hat{\chi}_0^\mu k_\mu)^2 = 0 \quad \text{(structural)}, \qquad g^{(0)\,\mu\nu}k_\mu k_\nu + \delta c_0(\hat{\chi}_0^\mu k_\mu)^2 = 0 \quad \text{(matter)}, \tag{4.16}$$

with $\tilde{\delta c}_X > -1$ and $\delta c_0 > -1$ (§3.7). The maximal group speed $v_{\max} = \max\{\sqrt{c_\perp^2}, \sqrt{c_\parallel^2}\} < 1$ controls explicit time stepping.

CFL estimate. For a Cartesian discretization with spacings Δx_i, a conservative Courant bound is

$$\Delta t \;\le\; \frac{\text{CFL}}{v_{\max}}\left(\sum_{i=1}^{d}\frac{1}{\Delta x_i^2}\right)^{-1/2}, \qquad 0 < \text{CFL} \lesssim 0.5 \text{ (scheme-dependent)}. \tag{4.17}$$

Implicit/semi-implicit integrators relax (4.17) but must still respect the analytic cones.

Constraint preservation. Penalty terms in $\mathcal{L}_{\text{constr}}$ and geometric projections (4.10) keep $|\hat{\chi}| = 1$ and $u\cdot\hat{\chi} = 0$ to machine precision; we monitor $\epsilon_{\text{constr}} := \max\{||\hat{\chi}|^2 - 1|,\ |u\cdot\hat{\chi}|\}$ and enforce $\epsilon_{\text{constr}} \ll 10^{-6}$.

Stopping criteria (variational solvers). For constrained minimizations of $E - \Omega Q_\Phi$, require

$$\|\nabla(E - \Omega Q_\Phi)\|_{H^{-1}} < \varepsilon_{\text{grad}}, \quad |\mathrm{d}E/\mathrm{d}Q_\Phi - \Omega| < \varepsilon_{\text{slope}}, \quad |\text{virial residual}| < \varepsilon_{\text{virial}}, \tag{4.18}$$

with typical targets $\varepsilon_{\text{grad}} \sim 10^{-8}$, $\varepsilon_{\text{slope}} \sim 10^{-6}$, $\varepsilon_{\text{virial}} \sim 10^{-6}$ in dimensionless units.

4.1.5 Reference backgrounds and safe parameter windows

Isotropic stratum (baseline). $\alpha = 0$, $\delta c = 0$, constant (ρ, η); $g_{\rm eff}$ locally Minkowski. All families admit their simplest prototypes here (Q-type solitons, walls with constant κ_Φ, trivial textures).

Ordered stratum (anisotropic). $\alpha = \alpha_0 > 0$ with a fixed director $\hat{\chi}$; $\delta c(\alpha_0) \in (-1, 1)$; Z_{AB} positive. Anisotropy splits longitudinal/transverse dispersions and modifies Derrick windows (Eq. 4.13).

Safe defaults (consistent with §3.10).

$$Z_{AB} = \mathrm{diag}(Z_{\rho\rho}, Z_{\eta\eta}, Z_{TT}, Z_{\chi\chi}\Pi_\perp, Z_{\alpha\alpha}) \succ 0, \quad Y_{AB} = Y_* \delta_{AB} \succ 0, \quad \kappa_\Phi = \kappa_{\Phi,0}(1 + \kappa_1 \Delta\rho + \kappa_2 \Delta\eta),$$

$$V_\Phi = \tfrac{1}{2} m_{\rm eff}^2(\rho, \eta, \alpha)|\Phi|^2 + \tfrac{\lambda_\Phi}{4}|\Phi|^4, \qquad U = \sum_X \tfrac{1}{2} m_X^2 (X - X_0)^2 + \tfrac{\lambda_\alpha}{4}(\alpha^2 - \alpha_0^2)^2,$$

with small gradients of constitutive functions on solution scales. These choices ensure the gates (4.6) and provide a robust launchpad for the constructions in §4.2–§4.8.

The remainder of the chapter specializes this common setup to each solution family, beginning with spherically and cylindrically symmetric matter solitons in §4.2.

4.2 Matter Solitons of Φ in Stratified Media

We construct and analyze coherent, finite-energy matter configurations ("Q-type" solitons) supported by the spectral aether. The key ingredients are: (i) the anisotropic kinetic tensor $G^{\mu\nu}_{\rm (ani)} = g^{\mu\nu}_{\rm eff} + \delta c(\alpha)\, \hat{\chi}^\mu \hat{\chi}^\nu$, (ii) environment-dependent inertia through $m^2_{\rm eff}(\rho, \eta, \alpha)$ and $\kappa_\Phi(\rho, \eta, \alpha)$, and (iii) weak spectral-curvature feedback stabilized by positive Y_{AB}. We work first in locally flat backgrounds ($g_{\rm eff} \to \eta_{\mu\nu}$, $u^\mu = (1, \mathbf{0})$) and restore slow feedback as needed.

4.2.1 Spherical and cylindrical ansätze

Spherical Q-soliton. Assume an isotropic stratum ($\alpha = 0$, $\delta c = 0$) and stationary phase:

$$\Phi(t, \mathbf{x}) = f(r)\, e^{-i\Omega\, T(t)}, \qquad r := |\mathbf{x}|, \qquad T = t, \tag{4.19}$$

with $f \geq 0$ real. Let κ_Φ and $m_{\rm eff}$ be constant on the soliton core at leading order. The Φ-equation (4.5) reduces to

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \kappa_\Phi \frac{df}{dr}\right) = \kappa_\Phi\, \Omega^2 f - \frac{\partial V_\Phi(f)}{\partial f}, \qquad f'(0) = 0, \quad f(r) \xrightarrow{r\to\infty} 0. \tag{4.20}$$

Equivalently, with the effective reduced potential $U_\Omega(f) := V_\Phi(f) - \frac{1}{2}\kappa_\Phi \Omega^2 f^2$,

$$f'' + \frac{2}{r} f' = \partial_f U_\Omega(f). \tag{4.21}$$

Cylindrical Q-strings. In cylindrical coordinates (ρ, θ, z), consider a straight filament with optional azimuthal winding $n \in \mathbb{Z}$,

$$\Phi(t, \rho, \theta, z) = f(\rho)\, e^{-i\Omega T(t)}\, e^{in\theta}, \qquad T = t, \tag{4.22}$$

yielding (with $\delta c = 0$ for simplicity)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \kappa_\Phi \frac{df}{d\rho}\right) - \kappa_\Phi \frac{n^2}{\rho^2} f = \kappa_\Phi\, \Omega^2 f - \frac{\partial V_\Phi(f)}{\partial f}, \qquad f'(0) = 0,\ f(\infty) = 0. \tag{4.23}$$

The n^2/ρ^2 barrier suppresses the core amplitude for $n \neq 0$ and raises the minimal charge per unit length.

Energy and charge. For stationary backgrounds, the charge and energy (per unit length in the cylindrical case) are

$$Q_\Phi = \int d^3x\, \kappa_\Phi\, \Omega\, f^2, \tag{4.24}$$

$$E = \int d^3x \left[\tfrac{\kappa_\Phi}{2}\left(|\nabla f|^2 + \Omega^2 f^2\right) + V_\Phi(f)\right] \quad \text{(plus small structural/curvature terms).} \tag{4.25}$$

Critical points of $E - \Omega Q_\Phi$ under H^1 variations and the boundary conditions above solve (4.20) or (4.23).

4.2.2 Existence windows and nonlinear corrections

Upper bound (linear tail). Linearizing (4.20) at $f \to 0$ gives $f'' + \frac{2}{r} f' \simeq (m_{\rm eff}^2 - \kappa_\Phi \Omega^2) f/\kappa_\Phi$, so exponential localization requires

$$\boxed{\Omega^2 < \Omega_{\rm max}^2 := \frac{m_{\rm eff}^2}{\kappa_\Phi}.} \tag{4.26}$$

Lower bound (negative well). Let $U_\Omega(f) = V_\Phi(f) - \frac{1}{2}\kappa_\Phi \Omega^2 f^2$. A nontrivial localized solution requires a negative region of U_Ω:

$$\exists\, f_\star > 0: \quad U_\Omega(f_\star) < 0 \iff \kappa_\Phi\, \Omega^2 > \mu_\star^2, \qquad \mu_\star^2 := \min_{f>0} \frac{2V_\Phi(f)}{f^2}. \tag{4.27}$$

Thus the *existence window* is

$$\boxed{\mu_\star^2 < \kappa_\Phi\, \Omega^2 < m_{\rm eff}^2.} \tag{4.28}$$

In the quartic model $V_\Phi = \frac{1}{2} m_{\rm eff}^2 f^2 + \frac{\lambda_\Phi}{4} f^4$ with $\lambda_\Phi > 0$, $\mu_\star^2 = m_{\rm eff}^2$ and the window collapses; one needs either attractive self-interactions (e.g. $\lambda_\Phi < 0$ stabilized by higher-order terms) or *environmental softening*: $m_{\rm eff}^2(\rho, \eta, \alpha)$ or $\kappa_\Phi(\rho, \eta, \alpha)$ decreasing with f (via state coupling) so that $\min 2V_\Phi/f^2 < m_{\rm eff}^2$ over the relevant amplitude range.

Derrick/virial consistency. The anisotropic virial identity (4.13) reduces in the isotropic case to $2\mathcal{T} + \sum_{AB} \mathcal{S}^{AB} - 3(U + V_{\rm eff}) - \mathcal{C}_{\rm spec} = 0$; for weak spectral curvature (Y_{AB} small) this demands that U_Ω be negative in some core region, in agreement with (4.27).

Corrections from weak feedback. If ρ, η vary mildly across the core, write $m_{\rm eff}^2(r) = m_0^2 + \delta m^2(r)$, $\kappa_\Phi(r) = \kappa_0 + \delta\kappa(r)$ with $|\delta m^2|, |\delta\kappa| \ll$ bulk values. First-order perturbation of (4.20) yields a small shift in $\Omega_{\rm max}^2$ and in the $E(Q_\Phi)$ curve; in particular

$$\delta\Omega_{\rm max}^2 \simeq \frac{1}{\kappa_0} \frac{\int dr\, r^2\, \delta m^2(r)\, f^2(r)}{\int dr\, r^2\, f^2(r)} - \frac{m_0^2}{\kappa_0^2} \frac{\int dr\, r^2\, \delta\kappa(r)\, f^2(r)}{\int dr\, r^2\, f^2(r)}. \tag{4.29}$$

4.2.3 Stability criteria and mode spectra

Energy–charge slope (VK-type). Consider the constrained functional $\mathcal{E}_\Omega := E - \Omega Q_\Phi$. A necessary condition for spectral stability of the spherically symmetric branch is the *Vakhitov–Kolokolov* inequality (adapted to κ_Φ):

$$\boxed{\frac{dQ_\Phi}{d\Omega} < 0 \quad \Rightarrow \quad \text{no real positive eigenvalues of the linearized operator (s-wave).}} \tag{4.30}$$

Turning points of $Q_\Phi(\Omega)$ signal bifurcations or onset of instability.

Zero modes and translational multiplet. Global $U(1)$ symmetry implies a phase Goldstone mode; translational invariance yields three zero modes $\propto \partial_i f$. In anisotropic strata ($\delta c \neq 0$) translational modes split slightly (longitudinal vs transverse stiffness).

Linearized operator. Perturb $\Phi = [f + \epsilon(a + ib)]e^{-i\Omega t}$, collect $\mathcal{O}(\epsilon)$, and obtain

$$\mathcal{L}_+ a = \Omega\, \mathcal{M}\, b, \qquad \mathcal{L}_- b = \Omega\, \mathcal{M}\, a, \tag{4.31}$$

with $\mathcal{L}_\pm$ Schrödinger-type radial operators depending on $V_\Phi''(f)$ and κ_Φ, and $\mathcal{M}$ a positive operator (mass metric). Numerical spectra (Arnoldi/LOBPCG) verify (4.30) and locate internal vibrational modes (shape excitations).

4.2.4 Role of anisotropy $\delta c(\alpha)$

Ellipsoidal deformation. Let $\delta c = \delta c(\alpha_0)$ be constant on the core and take an *ellipsoidal* variational trial $f(\mathbf{x}) = f_0(\sqrt{\mathbf{x}_\perp^2 + \lambda^2 x_\parallel^2})$ with scale anisotropy λ and $\mathbf{x}_\perp \cdot \hat{\boldsymbol{\chi}} = 0$, $x_\parallel = \hat{\boldsymbol{\chi}} \cdot \mathbf{x}$. Expanding the kinetic term to $\mathcal{O}(\delta c, \lambda - 1)$ yields

$$E(\lambda) \simeq E_0 + A\,(\lambda - 1)^2 + B\,\delta c\,(\lambda - 1), \qquad \Rightarrow \quad \lambda^\star - 1 \simeq -\frac{B}{2A}\,\delta c, \tag{4.32}$$

so $\delta c > 0$ (*stiffer longitudinally*) produces an *oblate* core ($\lambda^\star < 1$), while $\delta c < 0$ yields a *prolate* core. The constants $A, B > 0$ depend on f_0 and κ_Φ.

Anisotropic virial test. Using the scaled family $\Phi_\lambda(\mathbf{x}) = \Phi(\lambda \mathbf{x}_\perp, \lambda^\gamma x_\parallel)$ in (4.13), one can determine the preferred anisotropy index $\gamma^\star$ and confirm (4.32). The existence window (4.28) persists provided $|\delta c| < 1$.

Strings aligned with $\hat{\chi}$. For Q-strings oriented along $\hat{\chi}$, the longitudinal stiffness modifies the energy per unit length as $E/L \supset \frac{\kappa_\Phi}{2}(1 + \delta c) \int d^2x_\perp\, |\partial_\parallel \Phi|^2$; when $\delta c > 0$ the axial localization tightens, increasing the minimal Q/L for given Ω.

4.2.5 Numerical protocol (constrained minimization and continuation)

Functional and constraints. Minimize $\mathcal{E}_\Omega[\Phi, S] := E[\Phi, S] - \Omega Q_\Phi[\Phi]$ over $(\Phi, S) \in H^1(\mathbb{R}^3) \times \prod_A H^1(\mathbb{R}^3)$, subject to the constraints $|\hat{\chi}| = 1$, $u \cdot \hat{\chi} = 0$ (weakly via $\mathcal{L}_{\text{constr}}$ or projections (4.10)). Use a large but finite computational domain with absorbing or Robin boundaries consistent with exponential tails.

Initialization. Choose a radial seed $f^{(0)}(r) = f_0\,\mathrm{sech}(r/R_0)$ and structural seeds near the background. Project to satisfy constraints. Select Ω inside a conservative window $0.7\,\mu_\star/\sqrt{\kappa_\Phi} < \Omega < 0.9\,m_{\text{eff}}/\sqrt{\kappa_\Phi}$.

Solver. Apply (i) imaginary-time evolution for Φ with structural fields frozen, then (ii) quasi-Newton (L-BFGS) or Newton–Krylov on the full coupled system. Enforce the virial residual (4.13) and stopping criteria (4.18). For families, perform continuation in Ω.

Diagnostics. Compute E, Q_Φ, $E(Q_\Phi)$ slope, and the smallest eigenvalues of the linearized operators (using shift-invert Arnoldi). Verify $dQ_\Phi/d\Omega < 0$ (VK) and monitor constraint error $\epsilon_{\text{constr}} \ll 10^{-6}$.

Weak feedback loop. Optionally update (ρ, η, α) by solving their elliptic Euler–Lagrange equations with small Z_{AB} and re-minimize $\mathcal{E}_\Omega$ (one or two Picard iterations suffice in the weak-feedback regime). Record the shift (4.29) and cone inclusion gates (§3.7).

Sections §4.3–§4.5 extend this analysis to phase walls with guided modes, director textures with Hopf charges, and two-body interactions where spectral curvature furnishes a long- and short-range structure to the effective potential.

4.3 Phase Walls and Guided Matter Modes

This section develops the physics of phase/order walls in the spectral aether and the guided matter modes they support. A phase wall is a codimension-one interface across which the order parameter α (and possibly $(\rho, \eta, \hat{\chi})$) interpolates between distinct homogeneous strata. The spatial variation modifies the local kinetic coefficients $\kappa_\Phi(\mathcal{S})$, the effective mass $m_{\rm eff}(\mathcal{S})$, and the anisotropy parameter $\delta c(\alpha)$, creating a potential waveguide for Φ. We derive the wall profile and tension, reduce the matter equation to a Sturm–Liouville eigenproblem for guided modes, analyze existence thresholds and dispersion, and set up numerically stable scattering/binding calculations.

4.3.1 Order-parameter wall: profile, tension, and consistency

Setup and ansatz. Consider a planar wall normal to the x-axis. The order parameter $\alpha(x)$ connects two minima $\alpha(-\infty) = -\alpha_0$ and $\alpha(+\infty) = +\alpha_0$ of the structural potential $U(\mathcal{S})$. In the simplest consistent reduction, take $(\rho, \eta, \hat{\chi})$ uniform and let only α vary (later we allow co-variation of $\hat{\chi}$). In the locally flat frame ($g_{\rm eff} \to \eta_{\mu\nu}$) the static Euler–Lagrange equation for α reads (cf. § 3.2)

$$-\frac{d}{dx}\Big(Z_{\alpha\alpha}\,\frac{d\alpha}{dx}\Big) + \partial_\alpha U(\rho,\eta,\alpha) = 0, \qquad \alpha(x) \to \pm\alpha_0 \text{ as } x \to \pm\infty. \tag{4.33}$$

Kink solution and width. For the quartic double-well $U(\alpha) = \frac{\lambda_\alpha}{4}(\alpha^2 - \alpha_0^2)^2$ with constant $Z_{\alpha\alpha} > 0$, the first integral (mechanical analogy) yields the standard kink:

$$\alpha_{\rm wall}(x) = \alpha_0\,\tanh\!\left(\frac{x - x_0}{\Delta}\right), \qquad \Delta = \sqrt{\frac{Z_{\alpha\alpha}}{\lambda_\alpha\,\alpha_0^2}}, \tag{4.34}$$

where Δ is the wall thickness, x_0 the center.

Wall tension. The surface energy density (tension) is

$$\mathcal{T}_{\rm wall} = \int_{-\infty}^{\infty} dx\ \big\{\tfrac{1}{2}Z_{\alpha\alpha}\,(\alpha')^2 + \big[U(\alpha) - U(\alpha_0)\big]\big\} = \frac{2\sqrt{2}}{3}\,\alpha_0^3\,\sqrt{\lambda_\alpha Z_{\alpha\alpha}}, \tag{4.35}$$

and serves as a consistency check in numerics.

Director co-rotation (optional). If $\hat{\chi}$ is allowed to rotate across the wall, add $Z_{\chi\chi}(\partial\hat{\chi})^2$ and enforce $|\hat{\chi}| = 1$. The coupled 1D system admits smooth composite walls $(\alpha(x), \hat{\chi}(x))$ with a renormalized tension $\mathcal{T}_{\rm wall} \to \mathcal{T}^{\rm eff}_{\rm wall}(\theta)$ depending on the total director twist angle θ; the existence and stability conditions are unchanged provided $Z_{\chi\chi} > 0$ and $|\theta|$ is not too large (to avoid cone crossing, cf. § 3.7).

4.3.2 Guided modes of Φ: 1D reduction and eigenproblem

Mode ansatz and orientation. Let the wall lie in the y–z plane, with x transverse. Choose a constant director $\hat{\chi}$ making an angle ϑ with the propagation direction $\mathbf{k}_\parallel$ in the wall. Seek guided modes of the form

$$\Phi(t, \mathbf{x}) = \psi(x)\,e^{-i\omega t + i\mathbf{k}_\parallel\cdot\mathbf{y}}, \qquad \mathbf{y} = (y, z), \quad k_\parallel := |\mathbf{k}_\parallel|. \tag{4.36}$$

Let the state-dependence enter through $\alpha(x)$: $\kappa_\Phi(x) := \kappa_\Phi(\alpha(x))$, $m_{\rm eff}(x) := m_{\rm eff}(\alpha(x))$, $\delta c(x) := \delta c(\alpha(x))$.

Anisotropic kinetic tensor. With $u^\mu = (1, \mathbf{0})$ and $g^{\mu\nu}_{\rm eff} = \mathrm{diag}(-1, 1, 1, 1)$, the spatial part of $G^{\mu\nu}_{\rm (ani)}$ reads

$$G^{ij}_{\rm (ani)}(x) = \delta^{ij} + \delta c(x)\,\hat{\chi}^i\hat{\chi}^j. \tag{4.37}$$

Decompose $\mathbf{k}_\parallel = k_\parallel(\cos\vartheta\,\hat{\boldsymbol{\chi}}_\parallel + \sin\vartheta\,\hat{\boldsymbol{\chi}}_\perp)$ (both tangential to the wall), so that

$$G^{\mu\nu}_{\rm (ani)}\nabla_\mu\nabla_\nu \ \to\ -\omega^2 + \partial_x\big(\partial_x\,\cdot\,\big) + k_\parallel^2\big[1 + \delta c(x)\cos^2\vartheta\big]. \tag{4.38}$$

Sturm–Liouville form. The Φ-equation (§ 3.4) reduces to the 1D variable-coefficient eigenproblem

$$-\frac{d}{dx}\Big(\kappa_\Phi(x)\,\frac{d\psi}{dx}\Big) + \kappa_\Phi(x)\,\Big\{m_{\mathrm{eff}}^2(x) + k_\parallel^2\big[1+\delta c(x)\cos^2\vartheta\big]\Big\}\psi = \kappa_\Phi(x)\,\omega^2\,\psi. \tag{4.39}$$

Define the continuum threshold using the asymptotic constants on either side,

$$\omega_{\mathrm{th}}^2(k_\parallel,\vartheta) = m_{\mathrm{eff}}^2(\infty) + k_\parallel^2\big[1+\delta c(\infty)\cos^2\vartheta\big], \tag{4.40}$$

and seek $L^2(\mathbb{R})$ eigenfunctions with $\omega^2 < \omega_{\mathrm{th}}^2$.

Liouville transform to Schrödinger form. Set $\psi(x) = \kappa_\Phi(x)^{-1/2}\,\varphi(x)$. Then (4.39) becomes

$$-\frac{d^2\varphi}{dx^2} + V_{\mathrm{eff}}(x;k_\parallel,\vartheta)\,\varphi = \omega^2\,\varphi, \tag{4.41}$$

with the effective potential

$$V_{\mathrm{eff}}(x;k_\parallel,\vartheta) = m_{\mathrm{eff}}^2(x) + k_\parallel^2\big[1+\delta c(x)\cos^2\vartheta\big] + \underbrace{\frac{1}{2}\frac{\kappa_\Phi''}{\kappa_\Phi} - \frac{1}{4}\Big(\frac{\kappa_\Phi'}{\kappa_\Phi}\Big)^2}_{\text{kinetic inhomogeneity}}, \tag{4.42}$$

(primes denote d/dx). Guided modes correspond to bound states of (4.41) below the asymptotic plateau $V_{\mathrm{eff}}(\infty) = \omega_{\mathrm{th}}^2$.

4.3.3 Existence criteria, dispersion, and scattering

Sufficient condition for at least one guided mode. If the potential well has negative area relative to the threshold,

$$\int_{-\infty}^{\infty} dx\,\big[\omega_{\mathrm{th}}^2 - V_{\mathrm{eff}}(x;k_\parallel,\vartheta)\big] \;>\; 0, \tag{4.43}$$

then (4.41) admits at least one bound state (Bargmann-type bound). This holds if, for example, $m_{\mathrm{eff}}(x)$ dips inside the wall or $\kappa_\Phi(x)$ peaks sufficiently to render the inhomogeneity term negative in the core.

Band dispersion. For a given $k_\parallel$ and orientation ϑ, the discrete set of guided branches solves $\omega_n^2(k_\parallel,\vartheta) < \omega_{\mathrm{th}}^2$ and yields group velocity $v_{g,n} = \partial\omega_n/\partial k_\parallel$. Anisotropy tilts the bands:

$$\frac{\partial\omega_n}{\partial k_\parallel} \simeq \frac{k_\parallel\,[1+\overline{\delta c\,\cos^2\vartheta}]}{\omega_n} \quad \text{(weakly varying coefficients)}, \tag{4.44}$$

where the overbar denotes mode-weighted average along x.

Orientation selectivity. For $\vartheta = \pi/2$ (propagation orthogonal to $\hat{\chi}$ within the wall) the anisotropy drops out at leading order; for $\vartheta = 0$ it maximally shifts the threshold by $\delta c(\infty)\,k_\parallel^2$. Thus walls in ordered strata act as *polarization-selective* guides.

Scattering problem. For $\omega^2 > \omega_{\mathrm{th}}^2$ the stationary scattering states solve (4.39) with Jost asymptotics; reflection/transmission coefficients $(\mathcal{R},\mathcal{T})$ satisfy $|\mathcal{R}|^2 + |\mathcal{T}|^2 = 1$ (current conservation from the $U(1)$ continuity equation). Resonant transmission peaks occur near quasi-bound levels embedded just below the local plateau of V_{eff}.

Periodic wall arrays (mini-bands). For a weakly modulated superlattice of period L, Bloch reduction along x produces mini-bands $\omega(k_\parallel, q)$ with Bragg gaps at $q = \pi/L$; the gap size scales with the Fourier amplitude of $V_{\mathrm{eff}}(x)$.

4.3.4 Numerical construction: eigenmodes and time-domain checks

Discretization. Solve (4.39) as a generalized matrix eigenproblem $-\mathbf{D}^\top \mathbf{K}\mathbf{D}\,\boldsymbol{\psi} + \mathbf{V}\,\boldsymbol{\psi} = \omega^2 \mathbf{W}\,\boldsymbol{\psi}$ with finite differences (order 4) or spectral collocation. Here $\mathbf{K} = \mathrm{diag}(\kappa_\Phi)$, $\mathbf{W} = \mathrm{diag}(\kappa_\Phi)$, and $\mathbf{V} = \mathrm{diag}\{\kappa_\Phi[m_{\mathrm{eff}}^2 + k_\parallel^2(1 + \delta c \cos^2\vartheta)]\}$. Impose homogeneous Robin boundary conditions consistent with the asymptotic plane waves.

Normalization and orthogonality. Enforce the weighted inner product of § 4.1,

$$\langle \psi_m, \psi_n \rangle_{\kappa_\Phi} := \int dx\; \kappa_\Phi(x)\, \psi_m(x) \psi_n(x) = \delta_{mn}, \tag{4.45}$$

to guarantee flux-normalized modes.

Continuation and band tracing. For each ϑ, continue $\omega_n(k_\parallel)$ in $k_\parallel$ using arclength parameterization and predictor–corrector steps; monitor the gap to the threshold (4.40) and terminate when the mode delocalizes (participation ratio exceeds a tolerance).

Time-domain verification. Initialize $\Phi(t = 0, x, y) = \psi_n(x)\, e^{ik_\parallel y}$ and evolve the full 2D/3D equation with absorbing layers; extract ω_n from temporal Fourier transform and $v_{g,n}$ from space–time ridge tracking. Confirm stability against radiative leakage and validate orientation selectivity by varying ϑ.

Walls and their guided modes thus provide robust, calculable structures that shape matter transport in the spectral aether. In §4.4 we turn to fully three-dimensional director textures carrying Hopf charges and explore their coupling to Φ, which yields knotted conduits and trapping regions.

4.4 Director Textures and Hopf Charges

We now turn to fully three-dimensional director configurations of the spectral aether, encoded by the unit field $\hat{\chi}(\mathbf{x}) \in S^2$ with $|\hat{\chi}| = 1$. Such textures are topologically classified by the *Hopf invariant* when one compactifies physical space to S^3 by adding the point at infinity. Beyond their intrinsic interest, these textures generate anisotropic conduits and trapping regions for the matter field Φ and serve as robust, nonperturbative carriers of geometric information. We formulate a static effective energy that (i) respects the normalization constraint, (ii) admits a consistent Derrick balance against shrinkage, and (iii) interfaces cleanly with the curvature sector already present through $Y_{AB} F^A \cdot F^B$.

4.4.1 Effective energy and safe stabilizers

Kinematics and projectors. Let $\Pi_{ij}(\hat{\chi}) = \delta_{ij} - \hat{\chi}_i \hat{\chi}_j$ be the projector onto the tangent plane $T_{\hat{\chi}} S^2$. Tangent gradients are $\partial_i^\top \hat{\chi} := \Pi_{ab}\, \partial_i \hat{\chi}_b$. The *extrinsic twist tensor* Q_{ij} and the *area 2-form pullback* Σ_{ij} are

$$Q_{ij} := \partial_i \hat{\chi} \times \partial_j \hat{\chi}, \qquad \Sigma_{ij} := \hat{\chi} \cdot (\partial_i \hat{\chi} \times \partial_j \hat{\chi}), \tag{4.46}$$

so that $|Q_{ij}|^2 = (\partial_i \hat{\chi})^2 (\partial_j \hat{\chi})^2 - (\partial_i \hat{\chi} \cdot \partial_j \hat{\chi})^2$ and Σ_{ij} is the scalar pullback of the S^2 area form.

Static energy functional. A minimal, ghost-free (in the static sector) director energy with a safe quartic stabilizer is

$$E_\chi[\hat{\chi}] = \int_{\mathbb{R}^3} d^3x \left\{ \frac{Z_{\chi\chi}}{2} (\partial_i^\top \hat{\chi})^2 + \frac{\kappa_{\mathrm{FS}}}{4} |Q_{ij}|^2 + \frac{1}{4} Y_{AB}(\mathcal{S})\, F_{ij}^A(\hat{\chi})\, F^{B\,ij}(\hat{\chi}) \right\}, \tag{4.47}$$

subject to $|\hat{\chi}| = 1$. Here $Z_{\chi\chi} > 0$ and $\kappa_{\mathrm{FS}} > 0$; the last term recycles the curvature channel (cf. §3.3) with a texture-adapted, purely spatial $F_{ij}^A(\hat{\chi})$ (see below). The quadratic piece alone would succumb to Derrick shrinking in $d = 3$, while $|Q_{ij}|^2$ provides a Skyrme-type quartic that stabilizes characteristic sizes.[1]

[1] For dynamics one keeps the quadratic time-derivative sector from Ch. 3 to maintain hyperbolicity; the quartic term *does not* introduce higher time derivatives.

Texture-induced spectral curvature. In the texture sector we may take a purely intrinsic connection ω_i on the S^2-bundle whose pullback depends on $\hat{\chi}$:

$$\omega_i{}^{ab}[\hat{\chi}] = \hat{\chi}^{[a}\partial_i\hat{\chi}^{b]}, \qquad F_{ij}{}^{ab}[\hat{\chi}] = \partial_{[i}\omega_{j]}{}^{ab} + \omega_{[i}{}^{ac}\omega_{j]}{}^{cb}, \tag{4.48}$$

and feed F_{ij}^A in (4.47) via a fixed embedding of (ab) into the multi-index A, with $Y_{AB} \succ 0$. This construction keeps the quartic curvature piece frame-covariant and bounded.

Derrick balance and scaling. Under the coordinate dilation $\mathbf{x} \mapsto \lambda\mathbf{x}$, $E_\chi(\lambda) = \lambda\, E_2 + \lambda^{-1}E_4 + \lambda^{-1}E_{F\cdot F}$ with $E_2 \propto Z_{\chi\chi}\int(\partial\hat{\chi})^2$ and quartic pieces $E_4 \propto \kappa_{\mathrm{FS}}$, $E_{F\cdot F} \propto Y_{AB}$. A stationary size $\lambda_\star = (E_4 + E_{F\cdot F})^{1/2}/E_2^{1/2}$ exists if the quartics are nonzero, giving a nonzero optimal radius and preventing collapse.

Energy lower bounds. When E_χ contains both quadratic and Skyrme-like quartic pieces, Vakulenko–Kapitanski-type estimates apply:

$$E_\chi \;\geq\; C\, Z_{\chi\chi}^{1/2}\, \kappa_{\mathrm{FS}}^{1/2}\, |\mathcal{H}|^{3/4}, \qquad C > 0, \tag{4.49}$$

linking energy growth to the Hopf charge $\mathcal{H}$ (defined next). The curvature channel $F\cdot F$ can only raise the bound.

4.4.2 Hopf invariant and topological classification

Compactification and current. Impose $\hat{\chi}(\infty) = \hat{\chi}_\infty$ and compactify $\mathbb{R}^3 \cup \{\infty\} \simeq S^3$. The *topological current* (pullback of the S^2 area form) is

$$J^i(\hat{\chi}) := \frac{1}{8\pi}\,\epsilon^{ijk}\,\epsilon_{abc}\,\hat{\chi}^a\,\partial_j\hat{\chi}^b\,\partial_k\hat{\chi}^c \;=\; \frac{1}{8\pi}\,\epsilon^{ijk}\,\Sigma_{jk}, \qquad \partial_i J^i = 0. \tag{4.50}$$

Hence there exists a vector potential $\mathbf{A}$ with $\nabla\times\mathbf{A} = \mathbf{J}$.

Hopf charge. The *Hopf invariant* is the Chern–Simons functional

$$\mathcal{H}[\hat{\chi}] = \frac{1}{(4\pi)^2}\int_{\mathbb{R}^3} d^3x\; \mathbf{A}\cdot\mathbf{J} \;\in\; \mathbb{Z}, \tag{4.51}$$

independent of the gauge choice $\mathbf{A} \mapsto \mathbf{A} + \nabla\phi$. Geometrically, $\mathcal{H}$ counts the linking number of preimages of two generic points of S^2 under $\hat{\chi}$. Smooth deformations preserving $|\hat{\chi}| = 1$ cannot change $\mathcal{H}$.

Euler–Lagrange equations (static). Variation of (4.47) under $\hat{\chi} \mapsto \hat{\chi} + \varepsilon\mathbf{v}$ with $\mathbf{v} \perp \hat{\chi}$ yields

$$-\,Z_{\chi\chi}\,\Delta^{\top}\hat{\chi} - \kappa_{\mathrm{FS}}\,\partial_i\Big[Q_{ij}\times\partial_j\hat{\chi}\Big] - \mathcal{K}_F[\hat{\chi};Y] \;=\; \Lambda(\mathbf{x})\,\hat{\chi}, \tag{4.52}$$

where $\Delta^{\top} := \Pi_{ab}\partial_i\partial_i$, Λ enforces $|\hat{\chi}| = 1$, and $\mathcal{K}_F$ is the curvature-induced force from $Y_{AB}F_{ij}^A F^{B\,ij}$. Solutions of (4.52) in each homotopy class $\mathcal{H}$ are critical points of E_χ.

4.4.3 Numerical construction and stability diagnostics

Initial data (Hopf seed). Use an explicit Hopf map $\mathbb{R}^3 \to S^2$ obtained by stereographic projection from S^3: let $W(\mathbf{x})$ be a complex scalar field (rational map) and define

$$\hat{\chi}(\mathbf{x}) = \frac{1}{1+|W|^2}\Big(2\,\mathrm{Re}\,W,\; 2\,\mathrm{Im}\,W,\; 1-|W|^2\Big), \tag{4.53}$$

with W carrying the desired linking number. Variants with torus knots (p,q) produce $\mathcal{H} = pq$ families.

Constraint enforcement and cooling. Evolve the gradient flow $\partial_\tau\hat{\chi} = -\Pi(\delta E_\chi/\delta\hat{\chi})$ in fictitious time τ with periodic re-normalization $\hat{\chi} \leftarrow \hat{\chi}/|\hat{\chi}|$ (or a Lagrange-multiplier formulation). Use fourth-order finite differences or spectral derivatives in a periodic box and enforce $L \gg$ texture size to suppress wrap-around.

Computation of $\mathcal{H}$. Discretize $\mathbf{J}$ from (4.50), solve $\nabla \times \mathbf{A} = \mathbf{J}$ in Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$) via FFT Poisson solvers, then evaluate (4.51). Verify integer stability of $\mathcal{H}$ under refinement. As a robust cross-check, compute the linking of two preimage loops for generic target points on S^2.

Stability indicators. Monitor the Hessian spectrum of E_χ restricted to $\mathbf{v} \perp \hat{\chi}$ (Lanczos/LOBPCG), the Derrick residual under uniform rescaling, and the VK-type bound (4.49). For dynamical robustness, add small director waves and check bounded oscillatory response without decay into radiation on simulation windows $\gg$ core light-crossing times.

4.4.4 Coupling to matter Φ: guidance, trapping, and locking

Anisotropic guidance via $G^{\mu\nu}_{(\mathrm{ani})}$. Textures imprint spatially varying anisotropy and kinetic weight:

$$\mathcal{L}_\Phi = \kappa_\Phi(\mathcal{S})\, G^{\mu\nu}_{(\mathrm{ani})}(\hat{\chi}, \alpha)\, (\nabla_\mu \Phi)^* (\nabla_\nu \Phi) - V_\Phi(|\Phi|; \mathcal{S}), \qquad G^{ij}_{(\mathrm{ani})} = \delta^{ij} + \delta c(\alpha)\, \hat{\chi}^i \hat{\chi}^j, \tag{4.54}$$

so that regions where $\hat{\chi}$ threads space as tubes act as *waveguides* (cf. §4.3) for propagation aligned or misaligned with the local director.

Current–director locking (safe positive coupling). Introduce a positive semi-definite alignment term built from the $U(1)$ matter current $J^i_\Phi := \mathrm{Im}\big(\Phi^* \nabla^i \Phi\big)$:

$$\mathcal{L}_{\mathrm{lock}} = -\frac{\lambda_J(\mathcal{S})}{2} \left(J^i_\Phi \hat{\chi}_i\right)^2, \qquad \lambda_J \geq 0, \tag{4.55}$$

which energetically favors transport either *along* or *across* $\hat{\chi}$ depending on the sign convention for δc and the background. Because (4.55) is quadratic in the current, it preserves $U(1)$ and avoids linear-in-velocity instabilities.

Mass-channel trapping. Allow mild state dependence of the inertial mass, $m^2_{\mathrm{eff}}(\mathcal{S}) = m_0^2 + \nu_\alpha\,(\alpha - \alpha_0) + \nu_\chi \left|\partial \hat{\chi}\right|^2 + \cdots$. In regions where $|\partial \hat{\chi}|^2$ peaks (texture cores), a negative ν_χ lowers m_{eff} and creates *matter wells* that bind localized Φ-amplitude—knotted traps bound to Hopfions.

Backreaction gates. The coupled static system for $(\Phi, \hat{\chi})$ inherits the gates of §4.1.4 and §3.8: $\kappa_\Phi > 0$, $\delta c > -1$, $Z_{\chi\chi} > 0$, $\kappa_{\mathrm{FS}} > 0$, $Y_{AB} \succ 0$. In numerics, alternate minimization—holding $\hat{\chi}$ fixed while solving the guided-mode problem for Φ, then cooling $\hat{\chi}$ with Φ frozen—converges rapidly in the weak backreaction regime.

Director textures endowed with nonzero Hopf charge thus form a robust, quantized "scaffolding" for the spectral aether. They stabilize through safe quartic geometry and curvature channels, admit clean numerical construction, and couple naturally to matter transport via anisotropy, current locking, and mass-channel trapping. In §4.5 we leverage these ingredients to compute the effective interaction between separated coherent structures, exposing a repulsive short-range core and curvature-mediated long-range tails.

4.5 Two-Body Interaction: Core Repulsion and Far Tails

We develop a controlled two-body description for well-separated coherent structures ("solitons") of the matter field Φ embedded in the spectral aether. The interaction energy $V_{\mathrm{int}}(R)$ between two centers at separation R receives contributions from (i) overlap of matter cores, (ii) structural backreaction (linearized about a homogeneous stratum), and (iii) curvature channels $Y_{AB} F^A \cdot F^B$ sourced by textures such as the director $\hat{\chi}$. The resulting potential exhibits a *repulsive short-range core* and *curvature-mediated long-range tails*, in line with the qualitative picture anticipated in App. E.

4.5.1 Scale separation and collective coordinates

Two-center ansatz and moduli. Let $\mathcal{C}_1, \mathcal{C}_2$ denote the (disjoint) cores, with centers $\mathbf{R}_1 = -\frac{1}{2}\mathbf{R}$ and $\mathbf{R}_2 = +\frac{1}{2}\mathbf{R}$, $R := |\mathbf{R}| \gg$ core size. To leading order in $e^{-R/\xi}$ (or R^{-p}; see below), approximate the two-soliton field by a *nonlinear superposition* with relative $U(1)$ phase $\Delta\theta := \theta_2 - \theta_1$:

$$\Phi(\mathbf{x}) \;\approx\; \Phi_1(\mathbf{x} + \tfrac{\mathbf{R}}{2}) \;+\; e^{\,i\Delta\theta}\,\Phi_2(\mathbf{x} - \tfrac{\mathbf{R}}{2}), \qquad \Phi_j(\mathbf{x}) = f_j(|\mathbf{x}|)\,e^{-i\Omega_j T}. \tag{4.56}$$

Structural fields are expanded around a homogeneous background S_0^A with localized perturbations σ_j^A:

$$S^A(\mathbf{x}) \;=\; S_0^A \;+\; \sigma_1^A(\mathbf{x} + \tfrac{\mathbf{R}}{2}) \;+\; \sigma_2^A(\mathbf{x} - \tfrac{\mathbf{R}}{2}) \;+\; \mathcal{O}(\sigma_1\sigma_2). \tag{4.57}$$

Definition of the interaction. Let $\mathcal{E}[\Phi, S]$ be the (dimensionless) total energy functional of §4.1.3. Define the interaction energy as the *excess* beyond isolated self-energies:

$$V_{\rm int}(R, \Delta\theta, \ldots) := \mathcal{E}[\Phi, S] \;-\; \mathcal{E}[\Phi_1, S_1] \;-\; \mathcal{E}[\Phi_2, S_2], \tag{4.58}$$

where S_j are the single-core structural profiles in the same background S_0. Expanding $\mathcal{E}$ to quadratic order in the overlapping tails yields a sum of bilinear *overlap integrals* that can be organized by physical channel:

$$V_{\rm int} \;=\; V_{\Phi\Phi} \;+\; V_{SS} \;+\; V_{F\cdot F} \;+\; V_{\rm lock} \;+\; \cdots \tag{4.59}$$

Here $V_{\Phi\Phi}$ stems from cross-kinetic and cross-potential terms of Φ, V_{SS} from linearized structural exchange, $V_{F\cdot F}$ from curvature, and $V_{\rm lock}$ from current–director couplings (e.g. Eq. (4.55)).

Collective-coordinate reduction. Upon restricting to the moduli $\{\mathbf{R}, \Delta\theta, \ldots\}$ one obtains an effective Lagrangian

$$L_{\rm eff} \;=\; \tfrac{1}{2}\,\mathcal{M}(R)\,\dot{R}^2 \;+\; \tfrac{1}{2}\,\mathcal{I}(R)\,\dot{\Delta\theta}^{\,2} \;-\; V_{\rm int}(R, \Delta\theta, \ldots) \;+\; \cdots, \tag{4.60}$$

where the inertia functions $\mathcal{M}, \mathcal{I}$ follow from the zero-mode overlaps. The static force is $F_R = -\partial_R V_{\rm int}$ and may also be computed from a surface integral of the stress tensor (see §4.5.4).

4.5.2 Asymptotic spectral-curvature tails

Linearized structural exchange (Yukawa-type). Linearize the structural Euler–Lagrange equations (§3.3, §3.4) about S_0^A:

$$\big[-Z_{AB}\,\nabla^2 + (M^2)_{AB}\big]\,\sigma^B \;=\; \mathcal{J}_A[\Phi] + \cdots, \qquad (M^2)_{AB} := \partial_A\partial_B U\big|_{S_0}. \tag{4.61}$$

For a single core, $\sigma_j^A(\mathbf{x}) \simeq Q_j^{(A)}\,G_A(\mathbf{x} - \mathbf{R}_j)$ with Green functions G_A of the Helmholtz operators and *spectral charges* $Q_j^{(A)}$ set by the source moments $\mathcal{J}_A[\Phi_j]$. For two sources one finds the (attractive or repulsive) exchange tail

$$V_{SS}^{\rm tail}(R) \;\simeq\; -\,\frac{1}{2}\sum_A \mathcal{C}_A\,Q_1^{(A)}\,Q_2^{(A)}\,\frac{e^{-\mu_A R}}{4\pi R}, \qquad \mu_A^2 = \mathrm{eig}\big(Z^{-1}M^2\big)_A, \;\; \mathcal{C}_A > 0, \tag{4.62}$$

where μ_A are the spectral gaps (masses) of structural normal modes. Signs are controlled by $Q_1^{(A)}Q_2^{(A)}$ and by the relative phase $\Delta\theta$ when $\mathcal{J}_A$ depends on $|\Phi|^2$ vs. phase gradients.

Curvature channel and algebraic tails. Curvature energy involves $E_{F\cdot F} = \frac{1}{4}\int Y_{AB}\,F_{ij}^A F^{B\,ij}$. For two localized textures with far fields $F_{(j)}^A(\mathbf{x}) \sim \mathcal{M}_j^{(A)}\,r^{-(p+2)}$ (multipole order p), the cross term produces an algebraic tail:

$$V_{F\cdot F}^{\rm tail}(R) \;\simeq\; \sum_A \beta_A(\Omega, \ldots)\,\frac{\mathcal{M}_1^{(A)}\mathcal{M}_2^{(A)}}{R^{\,2p+1}}, \qquad p = 0, 1, 2, \ldots \tag{4.63}$$

with β_A set by Y_{AB} and angular factors (director orientations). For monopole-like $p = 0$ one gets R^{-1}, for dipole R^{-3}, quadrupole R^{-5}, etc. This reproduces the qualitative statement "modifies the power-law decay" in App. E.

Competition of channels. The full far tail is the sum of Yukawa exchanges (4.62) (gapped structural modes) and algebraic curvature pieces (4.63) (gapless geometric multipoles), the latter dominating beyond $R \gtrsim \mu_A^{-1}$ if a low-order multipole is present.

4.5.3 Short-range core and phase/orientation dependence

Repulsive overlap core. In the overlap region the positive-definite quadratic forms $Z_{AB}(\partial S)^2$ and $\kappa_\Phi\, G^{ij}_{(\mathrm{ani})}(\nabla_i\Phi)^*(\nabla_j\Phi)$, together with the normalization constraints of $\hat{\chi}$ and u, penalize strong superposition. A robust parametrization is

$$\boxed{V_{\mathrm{core}}(R) \;\simeq\; + \frac{\alpha_{\mathrm{rep}}(\dots)}{R^n}, \qquad n \geq 2,} \tag{4.64}$$

where n depends on the dominant overlapping gradients (e.g. $n=2$ for line-like cores, $n\approx 3-4$ for compact lumps with algebraic skirts; n is even larger if the tails are purely exponential). The coefficient α_{rep} grows with stiffnesses Z_{AB}, κ_Φ and with the quartic curvature weights Y_{AB}.

Phase and current interference. Cross terms in $V_{\Phi\Phi}$ include $\mathrm{Re}\int \kappa_\Phi\, \nabla\Phi_1 \cdot \nabla\Phi_2^* \propto \cos\Delta\theta$, generating an explicit phase dependence:

$$V^{\mathrm{cross}}_{\Phi\Phi}(R,\Delta\theta) \;\simeq\; - \frac{\beta_\Phi}{R^m}\, \cos\Delta\theta \;+\; \cdots, \tag{4.65}$$

with m set by the matter-tail multipole (or $e^{-R/\xi}$ if strictly exponential). For aligned directors, the current–locking channel (4.55) yields an additional *orientation* factor via $(J_\Phi\cdot\hat{\chi})^2$ that can either favor co-linear transport or penalize it depending on λ_J.

Unified template. Collecting the dominant pieces,

$$\boxed{V_{\mathrm{int}}(R,\Delta\theta) \;\simeq\; + \frac{\alpha_{\mathrm{rep}}}{R^n} \;-\; \sum_A \frac{\beta_A}{4\pi}\,\frac{e^{-\mu_A R}}{R} \;+\; \sum_q \frac{\gamma_q}{R^{\nu_q}} \;-\; \frac{\beta_\Phi}{R^m}\cos\Delta\theta \;+\; \cdots} \tag{4.66}$$

Here the Yukawa terms encode structural exchanges (μ_A), the $R^{-\nu_q}$ come from curvature multipoles ($F\cdot F$), and the phase-sensitive piece reflects matter interference. This template makes precise the App. E statements: short-range repulsive core and long-range (curvature-mediated) tails with tunable strength/sign.

4.5.4 Numerical matching, forces, and bound molecules

Constrained minimization at fixed R. Impose weak pinning potentials $V_{\mathrm{pin}} = \sum_{j=1}^{2} \frac{\kappa_{\mathrm{pin}}}{2}\big(|\mathbf{x}-\mathbf{R}_j|-r_\star\big)^2\,\chi_{\mathcal{C}_j}$ that anchor the cores while minimizing the full energy $\mathcal{E}+V_{\mathrm{pin}}$ under the constraints of §4.1.2. Scan over R and $\Delta\theta$ to obtain $V_{\mathrm{int}}(R,\Delta\theta)$ from (4.58).

Force from stress tensor. As a cross-check, compute the radial force from a spherical Gaussian surface $\partial\Omega$ midway between cores:

$$F_R \;=\; -\oint_{\partial\Omega} dS\; T^{\mathrm{eff}}_{ij}\, n_i\, \hat{R}_j, \qquad \frac{dV_{\mathrm{int}}}{dR} \;=\; -F_R, \tag{4.67}$$

with T^{eff}_{ij} the (dimensionless) effective stress tensor derived in §**??**. Agreement of numerical $-dV/dR$ with (4.67) is a stringent diagnostic.

Parameter extraction and matching. Fit the large-R data to $V_{\mathrm{int}} \sim -\sum_A(\beta_A/4\pi)\,e^{-\mu_A R}/R + \sum_q \gamma_q R^{-\nu_q}$ to extract μ_A, β_A, γ_q. Fit the small-R branch to $\alpha_{\mathrm{rep}}/R^n$ after removing self-energies and subtracting any regular background pressure.

Equilibria and binding. A *bound molecule* occurs if $V_{\rm int}$ develops a minimum at $R = R_b$: $\partial_R V_{\rm int}|_{R_b} = 0$, $\partial_R^2 V_{\rm int}|_{R_b} > 0$. For the mixed model $\alpha_{\rm rep}/R^n - (\beta/4\pi)\, e^{-\mu R}/R$ the balance condition gives, to leading order,

$$\mu R_b \;\approx\; W\!\left(\frac{n\,\mu^{\,n}}{4\pi}\,\frac{\alpha_{\rm rep}}{\beta}\right), \qquad E_{\rm bind} \;\approx\; -\,\frac{\beta}{4\pi}\,\frac{e^{-\mu R_b}}{R_b} \;+\; \frac{\alpha_{\rm rep}}{R_b^{\,n}}, \tag{4.68}$$

where W is the Lambert function. The well depth and R_b shift with the relative phase via $\beta \to \beta + \beta_\Phi \cos\Delta\theta$ and with texture orientation through the multipole coefficients γ_q.

Environmental tunability. Because μ_A and couplings (β_A, γ_q) depend on the background spectral state S_0 (notably ρ, η, α), the interaction is *environment-dependent*:

$$V_{\rm int}(R; \langle F_{\rm spec}\rangle) \;\text{ is not universal but varies with the local spectral curvature background,} \tag{4.69}$$

recovering the App. E claim. In practice one controls this by adjusting S_0 in the single-core solves that define the charges $Q_j^{(A)}$ and multipoles $\mathcal{M}_j^{(A)}$.

Quality gates. Throughout the scan verify (i) hyperbolic cones and CFL bounds (§4.1.4), (ii) virial identity residuals $\ll 10^{-6}$ (§4.1.3), (iii) constraint errors $\ll 10^{-6}$, and (iv) stress/energy agreement via (4.67). Report uncertainties from mesh and box-size extrapolations.

Summary. The spectral aether yields a two-body potential with a generic structure: a short-range, stiffness-driven repulsive core, plus a superposition of gapped (Yukawa) and gapless (curvature multipole) long-range tails. Relative phase and director orientation modulate both sign and strength, while the ambient spectral state tunes mediator masses and couplings. This sets the stage for controlled construction of bound "molecules" of solitons and for mapping phase/orientation-dependent scattering, which we pursue numerically in §4.9.

4.6 Topological Solitons: Classification, FR Constraints, and Particle Map

Topological structures provide a robust organizing principle for the coherent configurations built in Secs. 4.2–4.5. In three spatial dimensions, the unit director field $\hat\chi : \mathbb{R}^3 \cup \{\infty\} \simeq S^3 \to S^2$ supports nontrivial homotopy classes, while the complex phase of Φ supports line-like windings; discrete order parameters (e.g. α) support domain walls. These data endow solitons with *topological quantum numbers* that stabilize them nonperturbatively and constrain their dynamics and decays.

4.6.1 Topological charges and homotopy data

Charge vector. We collect the relevant invariants into

$$\mathcal{Q}_{\rm top} \;=\; \big(Q_\Phi,\; \mathcal{H},\; n,\; \sigma_{\rm wall},\; \sigma_{\rm string}\big), \tag{4.70}$$

where Q_Φ is the conserved U(1) charge of matter (Sec. 4.1.3), $\mathcal{H} \in \mathbb{Z}$ is the Hopf invariant of $\hat\chi$ (Sec. 4.4.2), $n \in \mathbb{Z}$ is the vortex winding of Φ around line defects (strings), and $\sigma_{\rm wall} \in \mathbb{Z}_2$ (or a finite set) classifies domain walls when the order-parameter potential $U(\alpha)$ has disconnected minima. The string indicator $\sigma_{\rm string} \in \{0, 1\}$ flags the presence of line-like cores.

Homotopy summary. With $|\hat\chi| = 1$ one has $\pi_3(S^2) = \mathbb{Z}$ (Hopf charge $\mathcal{H}$), and $\pi_2(S^2) = \mathbb{Z}$ controls point defects if boundary conditions enforce $\hat\chi \to \hat\chi_\infty$ on spheres around cores. For the complex phase of Φ one has line windings $n \in \mathbb{Z}$ in 3D (strings), and for discrete α one has codimension-one walls. These invariants are preserved under smooth evolutions of finite energy.

4.6.2 Energetics, lower bounds, and size selection

Energy functional (static slice). For configurations with $|\hat{\chi}| = 1$ and slowly varying background (ρ, η, α), the static energy decomposes as

$$E_{\rm tot} = E_\Phi + E_\chi + E_{F\cdot F} + E_{\rm wall/string} + \cdots , \tag{4.71}$$

$$E_\Phi = \int d^3x \, \Big\{ \kappa_\Phi \, |\nabla\Phi|^2_{G_{\rm (ani)}} + V_\Phi(|\Phi|; \mathcal{S}) \Big\}, \qquad |\nabla\Phi|^2_{G_{\rm (ani)}} := G^{ij}_{\rm (ani)} \, (\nabla_i\Phi)^*(\nabla_j\Phi), \tag{4.72}$$

$$E_\chi = \int d^3x \, \Big\{ \tfrac{Z_{\chi\chi}}{2} (\partial^\top \hat{\chi})^2 + \tfrac{\kappa_{\rm FS}}{4} |Q_{ij}|^2 \Big\}, \quad E_{F\cdot F} = \tfrac{1}{4} \int d^3x \; Y_{AB} F^A_{ij}(\hat{\chi}) F^{B\,ij}(\hat{\chi}). \tag{4.73}$$

The quadratic $Z_{\chi\chi}$ piece alone shrinks under scaling; the quartic Skyrme-like $\kappa_{\rm FS}|Q|^2$ (and the curvature channel $F\cdot F$) arrest collapse and set a finite size, consistent with Sec. 4.4.1.

Derrick scaling and lower bounds. Under $\mathbf{x} \mapsto \lambda \mathbf{x}$ one finds $E_2 \propto \lambda$, $E_4 \propto \lambda^{-1}$, whence a stationary size $\lambda_\star = (E_4/E_2)^{1/2}$ exists. For Hopf textures with Skyrme-like stabilizer, Vakulenko–Kapitanski-type bounds apply:

$$E_\chi \;\geq\; C \, Z_{\chi\chi}^{1/2} \, \kappa_{\rm FS}^{1/2} \, |\mathcal{H}|^{3/4}, \qquad C > 0, \tag{4.74}$$

with $F\cdot F$ only increasing the bound. Thus higher $|\mathcal{H}|$ implies heavier families.

Matter binding to textures. Couplings $m^2_{\rm eff}(\mathcal{S})$ and anisotropy $\delta c(\alpha)$ produce *traps* for Φ near texture cores (Sec. 4.4.4), yielding composite Q–Hopfion states with quantum numbers $(Q_\Phi, \mathcal{H})$ and, if present, line winding n.

4.6.3 Collective-coordinate quantization and FR constraints

Moduli and effective Lagrangian. A single localized soliton admits translational, phase, and internal rotational moduli. Let $X^i(t)$ be the center, $\theta(t)$ the global U(1) phase of Φ, and $R(t) \in \mathrm{SO}(3)$ parameterize slow reorientations of the texture. The leading collective Lagrangian reads

$$L_{\rm coll} = \tfrac{1}{2} M \dot{X}^2 + \tfrac{1}{2} \sum_{a,b} \mathcal{I}_{ab} \, \Omega^a \Omega^b + \tfrac{1}{2} \mathcal{I}_\theta \dot{\theta}^2 - V_{\rm aniso}(R) - E_0(\mathcal{Q}_{\rm top}), \tag{4.75}$$

with body angular velocity $\Omega = R^{-1}\dot{R}$, inertia tensors $\mathcal{I}_{ab}$, and a small anisotropy potential $V_{\rm aniso}$ from $\delta c(\alpha)$.

FR constraints (spin/statistics from topology). The configuration space $\mathcal{C}(\mathcal{Q}_{\rm top})$ may have a nontrivial fundamental group. Loops corresponding to a 2π spatial rotation or to exchange of identical solitons can be *noncontractible*. Imposing single-valuedness of the *physical* wavefunctional on the true covering of $\mathcal{C}$ yields *Finkelstein–Rubinstein (FR)* sign constraints:

$$\Psi[\text{loop}] \;=\; -\,\Psi \quad \Longrightarrow \quad \text{half-integer spin and fermionic statistics.} \tag{4.76}$$

For suitable Hopf classes, the loop generated by 2π rotation is nontrivial; then quantization of (R, θ) produces states with $J = \frac{1}{2}, \frac{3}{2}, \ldots$ despite underlying bosonic fields. This mechanism allows fermionic excitations from extended solitons while preserving all local symmetries.

Quantum numbers. After imposing FR constraints, the collective Hamiltonian yields multiplets labeled by $(J^P, \; m_J)$; parity P follows from the spatial inversion action on $(\hat{\chi}, \Phi)$ and on the Hopf map. Selection rules for transitions follow from the preserved topological charges (Sec. 4.6.4).

4.6.4 Selection rules and decay channels

Conservation laws. Smooth finite-energy dynamics conserve Q_Φ and $\mathcal{H}$; wall and string indicators are locally preserved until reconnection or wall–antiwall annihilation occurs. Changes in $\mathcal{H}$ require non-smooth reconnection or tunneling events, which are exponentially suppressed by the quartic elastic barrier ($\kappa_{\rm FS}$ and Y_{AB}).

Allowed decays. A composite $(Q_\Phi, \mathcal{H})$ may decay if there exist final states with the same conserved topological charges and lower total energy. Typical channels: (i) vibrational radiation in Φ and director waves, (ii) fission into $(Q_\Phi^{(1)}, \mathcal{H}^{(1)}) + (Q_\Phi^{(2)}, \mathcal{H}^{(2)})$ with additive charges, (iii) annihilation of wall/antiwall segments with emission into guided modes (Sec. 4.3). Phase-dependent forces (Sec. 4.5) control whether molecules are bound or metastable.

4.6.5 A conservative particle map at low energies

Filtering principle. To avoid overproduction of unseen states, we impose: (i) parameter windows of Sec. 4.1.5 (hyperbolicity/positivity), (ii) energy bounds (4.74) pushing $|\mathcal{H}| > 1$ families to high mass, (iii) FR constraints to realize half-integer spin only for the lightest viable topological classes, (iv) environmental tuning: present-day (ρ, η, α) disfavors exotic composites.

Minimal assignments (illustrative, to be stress-tested later).

- **Lepton-like ground state:** a Q–Hopfion with $(Q_\Phi = 1,\ \mathcal{H} = 1)$; FR constraints enforce $J = \frac{1}{2}$. Excited vibrational modes of the same topological core model heavier generations.
- **Neutral excitations:** $(Q_\Phi = 0,\ \mathcal{H} = 1)$ can be heavy, metastable bosons or (with suitable FR structure) neutral fermionic resonances; generically lifted above current bounds.
- **Higher Hopf charge:** $|\mathcal{H}| \geq 2$ yield significantly heavier, knotted states (energy growth $\sim |\mathcal{H}|^{3/4}$)—unlikely to be stable at low energy, but relevant in early-universe phases.

These are *working hypotheses*: Secs. 4.8–4.9 will provide numerical spectra and decay searches; later chapters will confront collider/cosmology bounds.

4.6.6 Predictive signatures and tests

Scattering and selection rules. Two-body phases (Sec. 4.5) depend on relative phase and on linking of texture lines, yielding polarization-dependent cross sections. Guided-mode emission along walls (Sec. 4.3) gives characteristic line widths.

Response to geometric ripples. Small-amplitude director waves (Sec. 4.7) modulate the collective inertias $(\mathcal{I}_{ab}, \mathcal{I}_\theta)$ and split rotational bands; this produces sidebands in spectra measurable in principle in analogue platforms.

Early-universe relics. During phase transitions, transient production of $|\mathcal{H}| > 1$ states followed by decay along allowed channels leaves stochastic imprints (anisotropic spectral noise and curvature multipoles), providing an indirect probe of the topological sector.

4.7 Spectral Yukawa Couplings: Geometry, Selection Rules, and Flavor

This section formulates the Yukawa sector in the spectral-aether framework. Our goal is to (i) identify the *geometric origin* of Yukawa interactions from internal tensors on the spectral fiber F, (ii) state *selection rules* that control which couplings are allowed once the internal symmetry is reduced (cf. Sec. 4.8), (iii) show how *hierarchies* arise from spatial localization on textures and walls (Secs. 4.3–4.4), and (iv) provide a computable map from structural state $\mathcal{S} = (\rho, \eta, \alpha, \hat{\chi}, T)$ to effective mass matrices. Throughout, Ψ denotes *effective* spinor fields describing low-energy fermionic excitations (either fundamental or collective-quantized solitons subject to FR constraints, Sec. 4.6.3), Φ is the matter scalar of Chs. 2–3, and all covariant derivatives use the emergent metric g_{eff} and gauge connections (Sec. 4.8).

4.7.1 Kinematics and chiral structure

Spinors on the emergent background. Let $\{\gamma^\mu\}$ be Dirac matrices associated with $g_{\rm eff}$, and $P_{L,R} = \frac{1}{2}(1 \mp \gamma^5)$ the chiral projectors. The kinetic term reads

$$\mathcal{L}_{\Psi,\rm kin} = i\,\bar\Psi\,\gamma^\mu\Big(\nabla_\mu + i\,g_3\,G^A_\mu T^A_{(3)} + i\,g_2 W^a_\mu \tau^a + i\,g_1 B_\mu Y\Big)\,\Psi, \tag{4.77}$$

with ∇_μ the spin-covariant derivative built from $g_{\rm eff}$ (Ch. 3). The *anisotropic* kinetic metric of Φ (Ch. 2) does not affect spinor hyperbolicity; cone constraints of Sec. 3.8 ensure well-posedness for the coupled system.

Chiral decomposition. We write $\Psi = (\Psi_L, \Psi_R)$ with $\Psi_{L,R} = P_{L,R}\Psi$. Gauge representations are assigned to each chiral multiplet (Sec. 4.8.6); in particular, Ψ_L may transform under $SU(2)$ while Ψ_R is a singlet, as in electroweak phenomenology.

4.7.2 Geometric Yukawa ansatz and invariant tensors

Spectral tensors. The internal fiber F carries a G_2-structure determined by a stable 3-form φ and its Hodge dual $\star\varphi$ on F (Sec. 4.8). The ordered state selects a unit director $\hat\chi$ and an almost-complex structure J on the 6-plane $\hat\chi^\perp$. From these data we build *invariant tensors* that can contract internal indices of matter and fermions:

$$\mathbf{1}, \qquad \varphi_{abc}, \qquad (\star\varphi)_{abcd}, \qquad J^a{}_b, \qquad \Omega_{ab}\ \text{(holomorphic 2-form on } \hat\chi^\perp), \ \ldots \tag{4.78}$$

The precise set active at low energy depends on the symmetry-breaking chain (Sec. 4.8.4).

Yukawa density (geometric form). The most general *renormalizable* Yukawa couplings consistent with gauge symmetry and Lorentz invariance can be written as

$$\mathcal{L}_{\rm Yuk} = -\sum_r\ \bar\Psi^{(r)}_L\ \mathcal{Y}^{(r)}(\mathcal{S})\cdot\mathcal{I}^{(r)}(\hat\chi, J, \varphi, \star\varphi)\ \Phi\ \Psi^{(r)}_R\ +\ \text{h.c.}, \tag{4.79}$$

where r labels fermion species (flavor), $\mathcal{Y}^{(r)}(\mathcal{S})$ are *scalar functions* of the structural state (spectral moduli), and $\mathcal{I}^{(r)}$ are *fixed internal tensors* projected onto singlets of the residual gauge group H (Sec. 4.8.2). Equation (4.79) is shorthand for index contractions in gauge, spin, and internal spaces.

Selection rules (tensor product to singlet). For any allowed term, the representation product must contain the trivial representation:

$$\big(R_{\Psi_L}\otimes R_\Phi\otimes R_{\Psi_R}\big)\ \supset\ \mathbf{1}_H. \tag{4.80}$$

If a given channel fails (4.80), the corresponding Yukawa entry vanishes *exactly* at tree level ("geometric zeros"). Holonomy phases of the internal connection can still induce effective complex phases in $\mathcal{Y}$ (see Sec. 4.7.5).

4.7.3 Mass generation and spectral VEVs

Two mechanisms, one matrix. Masses arise via two complementary routes already present in Chs. 2–3: (i) an *environmental* contribution through η (and possibly ρ) in V_Φ leading to $m^2_{\rm eff}(\eta,\rho)$; (ii) a *Yukawa* contribution if Φ acquires a nonzero expectation value along an internal invariant component, $\langle\Phi\rangle = v_\Phi$. The fermion mass matrix is then

$$\big(M_f\big)_{ij}\ =\ v_\Phi\ \Big[\mathcal{Y}_{ij}(\mathcal{S}_0)\cdot\mathcal{I}_{ij}(\hat\chi_0, J_0, \varphi_0)\Big]\ +\ \delta M^{(\eta,\rho)}_{ij}, \tag{4.81}$$

evaluated on the homogeneous background $\mathcal{S}_0$. The additive $\delta M^{(\eta,\rho)}$ encodes direct spectral-inertia contributions (e.g. drag from η), typically aligned with the identity in flavor space; hierarchies thus mainly come from the *geometric* factors.

Electroweak-like alignment. When the symmetry chain reaches $SU(2) \times U(1)$ (Sec. 4.8.4), the internal complex line selected by J picks out the component of Φ that can obtain v_Φ without breaking the residual gauge group beyond the desired pattern. This reproduces an electroweak-like mass generation with state-dependent relations among couplings (Sec. 4.8.5).

4.7.4 Hierarchies from spatial localization

Overlap integrals on textures/walls. Flavor hierarchies appear naturally from *geometry in real space*. Let $\psi_{L,i}(\mathbf{x})$ and $\psi_{R,j}(\mathbf{x})$ be normalized *zero or light modes* bound to textures (Hopfions) or walls (Secs. 4.3–4.4), and $\phi(\mathbf{x})$ the Φ profile that acquires the VEV (or the fluctuation associated with it). Then

$$\mathcal{Y}_{ij}^{\text{eff}} \;=\; \lambda_0 \int d^3x\, \sqrt{-g_{\text{eff}}}\; \underbrace{w(\mathcal{S};\mathbf{x})}_{\text{kin. weight}}\; \psi_{L,i}^{\dagger}(\mathbf{x})\; \phi(\mathbf{x})\; \psi_{R,j}(\mathbf{x}), \tag{4.82}$$

with $w(\mathcal{S})$ induced by κ_Φ and the spinor measure. Spatial separation of the supports (e.g. left/right modes localized on displaced walls) yields exponentially small entries:

$$\mathcal{Y}_{ij}^{\text{eff}} \;\sim\; \lambda_0\, e^{-\,d_{ij}/\ell_\star}, \qquad d_{ij} = \text{geodesic distance (in } g_{\text{eff}}\text{) between centers}, \tag{4.83}$$

producing a *geometric flavor hierarchy* without fine-tuning of λ_0.

Anisotropy imprint. The anisotropic metric $G^{ij}_{(\text{ani})}$ skews guided modes along $\hat{\chi}$, modifying both d_{ij} and the effective width $\ell_\star$. As strata change $(\alpha, \delta c)$, small drifts in hierarchies are possible but constrained by cone gates (Sec. 3.8).

4.7.5 Complex phases and CP violation

Berry–holonomy phases. Internal transport of spinors around nontrivial cycles in the spectral fiber (with connection $\mathcal{A}_\mu$) induces *geometric phases*. In the adiabatic limit,

$$\mathcal{Y}_{ij} \;\to\; \mathcal{Y}_{ij}\, \exp\!\left\{ i \oint_{\Gamma_{ij}} \text{Tr}(\mathcal{A}) \right\}, \tag{4.84}$$

where Γ_{ij} is the loop in internal space traced by the pair of modes and the Φ direction. These phases feed directly into CKM/PMNS-like complex invariants; their nontriviality requires internal curvature (Sec. 4.8.3).

CP selection rules. If the structural state admits a CP-like involution that flips the complex structure $J \mapsto -J$ while acting as $\Phi \mapsto \Phi^*$, then $\mathcal{I}$-tensors odd under this map are forbidden at leading order; CP violation then originates from holonomies (4.84) or higher-dimension operators.

4.7.6 Flavor safety: anomalies and minimality

Anomaly constraints. Allowed assignments for $(R_{\Psi_L}, R_{\Psi_R}, R_\Phi)$ must satisfy anomaly cancellation conditions (4.98) once fermion multiplets from collective quantization (Sec. 4.6.3) are counted. This restricts the number of independent Yukawa textures.

Minimal flavor violation (MFV) as geometry. If all flavor breaking descends from background values of a small number of structural spurions (e.g. misalignment angles of $\hat{\chi}$ and J across families), then any higher-order operator must be built from the same spurions—realizing *MFV* as a geometric statement. This tames FCNCs.

4.7.7 Two illustrative toy models

(i) Wall-separated chiral modes. Take two parallel phase walls at $x = \pm L/2$ (Sec. 4.3). Localize Ψ_L near $-L/2$ and Ψ_R near $+L/2$ with widths $\sigma_{L,R}$; let $\phi(x)$ peak at $x = 0$. Then

$$\mathcal{Y}_{LR} \;\approx\; \lambda_0 \;\exp\!\left[-\frac{L^2}{2(\sigma_L^2+\sigma_R^2)}\right], \tag{4.85}$$

which produces hierarchical masses by geometric separation while keeping kinetic terms universal.

(ii) Hopfion-trapped zero modes. Let a Hopf texture trap a pair of zero modes with angular momenta (m_L, m_R) along the core loop; the overlap with a Φ mode aligned with the loop yields selection $\Delta m = 0$ and

$$\mathcal{Y}_{LR} \;\propto\; \oint_{\text{core}} ds\; \psi^\dagger_{L,m}(s)\,\phi(s)\,\psi_{R,m}(s), \tag{4.86}$$

naturally small if ϕ is weak on the core or if modes have distinct parity along the loop.

4.7.8 Computation protocol and gates

Numerical recipe. (1) Construct background $\mathcal{S}_0$ and guided/bound modes for $\Psi_{L,R}$ and Φ on it (Secs. 4.2–4.4). (2) Normalize modes with the appropriate inner products. (3) Evaluate overlaps (4.82) to obtain $\mathcal{Y}^{\text{eff}}_{ij}$. (4) Fit holonomy phases from the internal connection (or compute Wilson loops) for CP phases.

Quality gates. Enforce (i) hyperbolicity/positivity gates of Sec. 3.8, (ii) gauge singlet condition (4.80), (iii) anomaly cancellation (4.98), (iv) boundedness of the scalar potential near the chosen v_Φ, and (v) small backreaction on g_{eff} in the weak-coupling regime.

Summary. Spectral Yukawa couplings are *geometric* in origin: invariant internal tensors and holonomies determine which channels exist and which phases enter; spatial localization on walls and textures yields hierarchies via overlap suppression; and state dependence (ρ, η, α) provides mild environmental shifts without spoiling hyperbolicity. Section 4.8 completes the picture by deriving the emergent non-Abelian gauge fields and their couplings $g_i(\mathcal{S})$ that enter (4.77) and (4.79).

4.8 Emergent Gauge Symmetries from Spectral Geometry

We develop a geometric mechanism by which non-Abelian gauge fields and their symmetries emerge from the internal spectral geometry of the aether. The key inputs are: (i) a 7-dimensional internal fiber F endowed with a G_2-structure defined by a stable 3-form $\varphi(S)$ built from the structural state $\mathcal{S}$, (ii) a unit director $\hat{\chi}$ that selects a preferred internal direction and reduces the structure group $G_2 \to SU(3)$, and (iii) a phase/complex structure induced on the $\hat{\chi}^\perp$-6-plane by the matter phase of Φ, allowing a further breaking pattern down to $SU(2) \times U(1)$ at low energies. Gauge connections and field strengths arise as the internal components of the emergent spectral connection, with kinetic terms inherited from the $Y_{AB}F^A\cdot F^B$ sector.

4.8.1 Dynamical Selection of a G_2 Structure

We endow the internal 7D fiber F with a stable 3-form $\varphi(\mathcal{S})$ and define a Hitchin-like energy functional

$$\mathcal{L}_\varphi = -\frac{\kappa_\varphi}{2}\left|\mathrm{d}\varphi\right|^2 - \frac{\tilde{\kappa}_\varphi}{2}\left|\mathrm{d}{\star_\varphi}\,\varphi\right|^2 - \frac{\lambda_\varphi}{4}\left(\varphi\wedge{\star_\varphi}\varphi - v_\varphi^2\right)^2, \tag{4.87}$$

augmented by the curvature sector $\mathcal{L}^{(\text{int})}_{\text{curv}} = -\frac{1}{4}Y_{AB}(\mathcal{S})\,F^A_{\mu\nu}F^{B\,\mu\nu}$. The Euler–Lagrange equations admit as absolute minima the torsion-free conditions

$$\mathrm{d}\varphi = 0, \qquad \mathrm{d}{\star_\varphi}\,\varphi = 0, \qquad \varphi\wedge{\star_\varphi}\varphi = v_\varphi^2, \tag{4.88}$$

which are precisely the defining equations of a G_2-structure on F. Small fluctuations orthogonal to the $SO(7)$-orbit of φ acquire masses $m_\perp^2 \sim \lambda_\varphi v_\varphi^2 + \mathcal{O}(k^2)$, while tangential modes (the $SO(7)/G_2$

directions) are gauged and become heavy through the sigma-model of Sec. 4.8.4. Coupling to the curvature quadratic further lowers the energy for connections compatible with the G_2 metric g_φ, making G_2 a dynamical attractor under internal coarse-graining (Laplacian flows of φ).

4.8.2 Internal isometries and principal bundles

Spectral connection split. Recall the emergent connection ω_μ on the structural bundle (Ch. 3). We split its algebra into a *Lorentz* part (acting on spacetime frames $e^a{}_\mu$) and an *internal* part acting on the spectral fiber F:

$$\omega_\mu \;=\; \omega_\mu^{(\mathrm{Lor})} \;\oplus\; \mathcal{A}_\mu, \qquad \mathcal{A}_\mu \in \mathfrak{g}_{\mathrm{int}}, \tag{4.89}$$

with curvature $F_{\mu\nu}^{(\mathrm{int})} = \partial_{[\mu}\mathcal{A}_{\nu]} + [\mathcal{A}_\mu, \mathcal{A}_\nu]$. The internal algebra $\mathfrak{g}_{\mathrm{int}}$ is the Lie algebra of frame-preserving isometries of (F, φ, g_F) determined by $\mathcal{S}$.

G_2-structure and reduction by the director. A stable 3-form φ on a 7D inner space reduces $SO(7) \to G_2$ (the automorphism group of the octonionic cross product). Fixing a unit vector $\hat{\chi} \in TF$ selects an $SU(3)$ subgroup as the stabilizer of $\hat{\chi}$ inside G_2:

$$G_2 \;\xrightarrow{\;\hat{\chi}\;}\; SU(3), \qquad \mathfrak{g}_2 = \mathfrak{su}(3) \;\oplus\; \mathbb{C}^3, \tag{4.90}$$

with $\mathbb{C}^3$ the complement transforming as the fundamental of $SU(3)$. Thus, in generic ordered strata ($\alpha > 0$ selecting a nonzero $\hat{\chi}$), the internal connection reduces to a principal $SU(3)$-bundle over spacetime:

$$\mathcal{A}_\mu = \mathcal{A}_\mu^{(3)} \in \mathfrak{su}(3) \quad (+ \text{ massive coset modes}). \tag{4.91}$$

Complex structure on the $\hat{\chi}^\perp$-plane. The matter phase of Φ and the ordered parameter α select an almost-complex structure J on the 6-plane orthogonal to $\hat{\chi}$, compatible with the $SU(3)$-structure; this provides a canonical $U(1)$ (the phase) and allows a further low-energy factorization $SU(3) \supset SU(2) \times U(1)$ by choosing a preferred complex line in $\hat{\chi}^\perp$.

4.8.3 Gauge fields and kinetic terms from the curvature sector

Projection of the curvature quadratic. The internal curvature energy density is inherited from the spectral curvature sector (Ch. 2):

$$\mathcal{L}_{\mathrm{curv}}^{(\mathrm{int})} \;=\; -\frac{1}{4}\, Y_{AB}(\mathcal{S})\, F_{\mu\nu}^A\, F^{B\,\mu\nu}. \tag{4.92}$$

Projecting A, B to the $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ directions yields canonical Yang–Mills pieces with *state-dependent* couplings:

$$\mathcal{L}_{\mathrm{YM}} \;=\; -\frac{1}{4g_3^2(\mathcal{S})}\,\mathrm{Tr}\, G_{\mu\nu}G^{\mu\nu} - \frac{1}{4g_2^2(\mathcal{S})}\,\mathrm{Tr}\, W_{\mu\nu}W^{\mu\nu} - \frac{1}{4g_1^2(\mathcal{S})}\, B_{\mu\nu}B^{\mu\nu}, \tag{4.93}$$

with inverse couplings read off from Y_{AB} restricted to each subalgebra: $g_i^{-2} = \kappa_i(\rho, \eta, \alpha)$ up to normalization conventions for the generators.

Covariant derivative on matter. The matter field transforms in a representation R of the residual gauge group $H \subseteq SU(3)$:

$$\nabla_\mu \Phi \;=\; D_\mu \Phi + i\, g_3\, G_\mu^A\, T_{(3)}^A \Phi + i\, g_2\, W_\mu^a\, \tau^a\, \Phi + i\, g_1\, B_\mu\, Y\, \Phi, \tag{4.94}$$

where $T_{(3)}^A$, τ^a, and Y are generators in R. The kinetic term $\kappa_\Phi(\mathcal{S})\, G_{(\mathrm{ani})}^{\mu\nu}(\nabla_\mu\Phi)^*(\nabla_\nu\Phi)$ (Ch. 2–3) then induces standard gauge interactions with state-dependent strengths.

4.8.4 Low-energy symmetry pattern and mass generation

Breaking chain and coset fields. The internal reduction follows the geometric chain

$$SO(7) \supset G_2 \xrightarrow{\hat{\chi}} SU(3) \xrightarrow{J} SU(2) \times U(1). \tag{4.95}$$

The coset coordinates $U(x) \in G/H$ (with $G = G_2$ or $SU(3)$, $H = SU(3)$ or $SU(2)\times U(1)$) parameterize internal orientations. A *spectral sigma model*

$$\mathcal{L}_\Sigma = \frac{\kappa_\Sigma(\mathcal{S})}{2} \operatorname{Tr}\big[(D_\mu U)^\dagger (D^\mu U)\big], \qquad D_\mu U = \partial_\mu U - i\,\mathcal{A}_\mu U + i\,U\,\mathcal{A}_\mu^{(H)}, \tag{4.96}$$

implements a Higgs/Stueckelberg-like mechanism: when U condenses to a fixed orientation U_0 selected by (ρ, η, α), gauge bosons along broken directions acquire masses $m_V^2 \sim \kappa_\Sigma\, v_\Sigma^2$ while those in H remain (approximately) massless.

Spectral origin of the electroweak scale. We take the condensate scale v_Σ as a function of (η, α); in ordered strata ($\alpha > 0$) and with nonzero dissipation parameter η (Ch. 3.9), a finite v_Σ emerges, yielding massive $W^\pm, Z$ and a massless B_μ (hypercharge) at leading order. Radiative and feedback corrections enter via the state dependence of κ_i and κ_Σ.

4.8.5 Coupling relations and near-critical unification

Metric on the Lie algebra and g_i ratios. The quadratic form $Y_{AB}(\mathcal{S})$ induces inner products on the subalgebras. If the internal state approaches a high-symmetry critical point $\mathcal{S} \to \mathcal{S}_c$ (e.g. $\alpha \to \alpha_c$ selecting a G_2-symmetric phase), the projections coincide, yielding

$$g_3(\mathcal{S}_c) = g_2(\mathcal{S}_c) = g_1(\mathcal{S}_c) \times c_Y, \tag{4.97}$$

with c_Y a normalization constant for hypercharge. Away from criticality, $Y_{AB}(\mathcal{S})$ splits and the couplings run differently; this realizes practical coupling "unification" as an *emergent near-critical phenomenon* controlled by the spectral state.

4.8.6 Representations, hypercharge, and anomaly constraints

Spectral charges and representation content. The representation R of Φ (and of composite solitons built from it) is determined by how Φ transforms under internal rephasings aligned with the complex structure on $\hat{\chi}^\perp$. One may encode these as *spectral charges* $(Y, \mathbf{2}, \mathbf{3})$ attached to the local frame in F. Assignments must be consistent with:

$$\sum_{\text{fermions}} \operatorname{Tr} Y \;=\; 0, \qquad \sum_{\text{fermions}} \operatorname{Tr} Y^3 \;=\; 0, \qquad \sum_{\text{fermions}} \operatorname{Tr} T^a T^b Y \;=\; 0, \tag{4.98}$$

to avoid U(1), cubic, and mixed anomalies at low energy. In the solitonic picture, these traces are evaluated on the *collective-quantized* fermionic multiplets (Sec. 4.6.3) rather than on elementary fields.

Topological quantization of Y. Because $U(1)$ originates from a phase on $\hat{\chi}^\perp$, hypercharge normalization follows from the embedding $SU(2) \times U(1) \subset SU(3) \subset G_2$. The allowed Y values are thus quantized by the weight lattice of $SU(3)$ restricted to the chosen complex line; this enforces discrete charge ratios.

4.8.7 Matter–gauge interactions and spectral Yukawa sector (hooks to Sec. 4.7)

Minimal gauge couplings. With (4.94), the matter Lagrangian contains the usual gauge vertices. The anisotropic metric $G^{\mu\nu}_{(\text{ani})}$ implies small directional distortions at intermediate energies; these are bounded by the cone constraints (Ch. 3.7) and vanish in the isotropic stratum.

Yukawa from internal geometry. Sec. 4.7 develops spectral Yukawa couplings $\mathcal{L}_{\rm Yuk} = \bar{\Psi}\,\mathcal{Y}(\mathcal{S})\Phi\,\Psi +$ h.c. with *tensorial* selection rules fixed by the $SU(3)$ tensor product on $\hat{\chi}^{\perp}$. The same internal geometry that defines g_i fixes allowed Yukawa channels and forbids others by group theory.

4.8.8 Predictions and diagnostics

State-dependent couplings. Mild dependence $g_i(\rho, \eta, \alpha)$ leads to environmental shifts in effective couplings. In nearly homogeneous backgrounds these are small but, in principle, produce testable variations in precision observables (bounded by existing experiments).

Near-critical scaling. Approach to the critical state $\mathcal{S}_c$ drives g_i toward one another (Eq. 4.97) and aligns dispersion cones, predicting correlated deviations in couplings and propagation anisotropies.

Mass relations. If v_Σ arises from (η, α), ratios like m_W/m_Z inherit small spectral corrections. The framework predicts definite signs for these corrections once $\kappa_\Sigma(\mathcal{S})$ and Y_{AB} are specified, offering falsifiable handles.

4.8.9 Electroweak Vacuum Alignment and Mass Relations

Under $SU(3) \to SU(2) \times U(1)$ the matter field decomposes as $\Phi \;\to\; (H,\, S)$, with H an $SU(2)$ doublet of hypercharge Y_H and S an $SU(2)$ singlet of hypercharge Y_S, set by the embedding of $U(1)$ in $SU(3)$ (fixed by the Y_{AB} metric normalization). We choose the potential

$$V(\Phi) = m^2\,\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2 \;+\; \varepsilon\left(\Phi^\dagger P\Phi - \tfrac{v^2}{2}\right)^2 \;+\; \zeta\,|S|^2 + \kappa\,|H|^2|S|^2, \tag{4.99}$$

where P is the projector onto the $SU(2)$ doublet direction selected by $(\hat{\chi}, J)$. For $\varepsilon, \zeta > 0$ the minimum satisfies $\langle S\rangle = 0$ and $\langle H\rangle = (0,\; v/\sqrt{2})^T$, breaking $SU(2) \times U(1) \to U(1)_{\rm em}$ with

$$m_W^2 = \frac{g_2^2 v^2}{4}, \qquad m_Z^2 = \frac{(g_2^2 + g_1^2)v^2}{4}, \qquad m_\gamma = 0. \tag{4.100}$$

Here $g_{1,2}(\mathcal{S})$ descend from the spectral metric Y_{AB} (Sec. 4.8.3). The alignment term prohibits vacuum "sliding" into unwanted singlet directions while preserving all gauge invariances and positivity bounds.

4.8.10 Masses of Coset Vectors and Spectral Higgs Modes

The sequence $SO(7) \supset G_2 \to SU(3) \to SU(2) \times U(1)$ entails cosets of dimensions 7 (generic $SO(7)/G_2$ directions in φ space), 6 (for $G_2/SU(3)$), and 4 (for $SU(3)/(SU(2) \times U(1))$). In the gauged internal sector these angular modes are eaten by corresponding gauge fields via the spectral sigma model $\mathcal{L}_\Sigma = \frac{\kappa_\Sigma}{2}\mathrm{Tr}[(D_\mu U)^\dagger (D^\mu U)]$ with condensate scale v_Σ set by (η, α).

Vector bosons along broken directions. Denote by X_μ the gauge fields in $G_2/SU(3)$ and by Y_μ those in $SU(3)/(SU(2) \times U(1))$. Their masses are

$$M_X \;\simeq\; g_*\, v_\Sigma, \qquad M_Y \;\simeq\; g_3\, v_\Sigma, \tag{4.101}$$

where g_* is the internal coupling along G_2 directions and g_3 the $SU(3)$ one. Choosing $v_\Sigma \gg v$ ensures these states evade current bounds (e.g. $v_\Sigma \gtrsim$ 10–100 TeV $\Rightarrow M_{X,Y} \gtrsim$ few TeV for order-one couplings).

Scalar (radial) spectral Higgs. The radial excitation of U has

$$m_{\Sigma,\rm rad} \;\simeq\; \sqrt{2\lambda_\Sigma}\, v_\Sigma, \tag{4.102}$$

while angular modes are gauge-eaten. Small explicit breakings in the internal sector generate pseudo-Goldstones with $m_{\rm PG}^2 \sim \varepsilon_\Sigma\, v_\Sigma^2$; we take ε_Σ small but v_Σ large so that $m_{\rm PG} \gg v$.

Two-scale picture. Electroweak breaking is driven by $\langle H\rangle$ at $v=246$ GeV, whereas the internal reductions $G_2 \to SU(3) \to SU(2)\times U(1)$ occur at the higher spectral scale v_Σ. Thus heavy coset vectors/scalars decouple, leaving only the usual W, Z, γ at low energy.

Summary. Gauge symmetries and interactions arise from the isometry structure of the internal spectral fiber: a G_2-structure reduced by the director $\hat{\chi}$ and the matter-induced complex structure yields $SU(3)\times SU(2)\times U(1)$ at low energies. Gauge fields are the internal components of the spectral connection; their kinetic terms, couplings, and potential near-critical unification descend from the curvature quadratic. Mass generation proceeds through a spectral sigma-model on the relevant cosets, with state-dependent scales tied to (η, α).

4.9 Emergent Gravitational Waves and Geometric Perturbations

In this section we derive and analyze *gravitational radiation* in the spectral aether: small, propagating perturbations of the emergent metric $g_{\rm eff}$ sourced by coherent matter/structural dynamics. The key messages are: (i) in the *isotropic stratum* the tensor sector reproduces GR-like, luminal waves, (ii) in *ordered strata* with a director $\hat{\chi}$ one finds controlled, small anisotropies and mixing with director/geometric modes, (iii) hyperbolicity/positivity gates from Chs. 2–3 bound any deviation and ensure well-posedness.

We work to quadratic order around a homogeneous background state $\mathcal{S}_0$ and metric $g^{(0)}_{\mu\nu}$, and retain the leading mixing terms with director fluctuations. Throughout, indices are raised/lowered with $g^{(0)}$ unless stated otherwise.

4.9.1 Background, perturbation variables, and gauges

Background. Let $\mathcal{S}_0 = (\rho_0, \eta_0, \alpha_0, \hat{\chi}^\mu_0, u^\mu_0)$ be a homogeneous stratum that solves the background field equations of Ch. 3 (with $u^\mu_0 u_\mu = -1$, $\hat{\chi}^\mu_0 \hat{\chi}^0_\mu = +1$, $u_0\cdot\hat{\chi}_0 = 0$), and let $g^{(0)}_{\mu\nu}$ be the corresponding effective metric. Consider small perturbations

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}, \qquad \hat{\chi}^\mu = \hat{\chi}^\mu_0 + \delta\hat{\chi}^\mu, \qquad \Phi = \Phi_0 + \delta\Phi, \qquad S^A = S^A_0 + \sigma^A, \tag{4.103}$$

with S^A the structural coordinates introduced in Ch. 2.

Gauge choice. We use the *generalized de Donder* (harmonic) gauge adapted to $g^{(0)}$:

$$\mathcal{H}_\mu := \nabla^{(0)\nu}\Big(h_{\mu\nu} - \tfrac{1}{2} g^{(0)}_{\mu\nu} h\Big) = 0, \qquad h := h^\alpha{}_\alpha. \tag{4.104}$$

For the director we enforce the linearized constraints $u_0\cdot\delta\hat{\chi} = 0 = \hat{\chi}_0\cdot\delta\hat{\chi}$ to preserve normalization and orthogonality at $\mathcal{O}(\delta)$ (cf. Ch. 2).

4.9.2 Mode decomposition relative to $(u_0, \hat{\chi}_0)$

Scalar/Vector/Tensor split. In the local rest frame of u^μ_0 (so spatial indices $i, j, \ldots$ are orthogonal to u_0), decompose h_{ij} into irreducible parts under spatial diffeomorphisms *and* relative to the preferred direction $\hat{\chi}^i_0$:

$$h_{ij} = h^{\rm TT}_{ij} \;+\; \partial_{(i} V^\perp_{j)} \;+\; \big(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\big) B \;+\; \tfrac{1}{3}\delta_{ij}\,\zeta \;+\; \Delta h^{(\parallel)}_{ij}, \tag{4.105}$$

where $h^{\rm TT}_{ij}$ is transverse and traceless ($\partial_i h^{\rm TT}_{ij} = 0 = h^{\rm TT}{}_{ii}$), $V^\perp_i$ is transverse, B and ζ are scalars, and $\Delta h^{(\parallel)}_{ij} \propto \hat{\chi}_{0(i}\hat{\chi}_{0j)}$ captures anisotropic pieces. Gravitational radiation is encoded in $h^{\rm TT}_{ij}$; other sectors are constrained or non-radiative in the isotropic limit.

4.9.3 Quadratic action and the tensor wave operator

Quadratic expansion. Expand the gravitational Lagrangian density (Ch. 2: Palatini–Einstein term plus safe curvature channel) to $\mathcal{O}(h^2)$, keeping director mixing at leading order (see below). One obtains

$$\mathcal{L}^{(2)}_{\rm grav} = \frac{M^2_{\rm eff}(\rho_0)}{8}\left[\nabla^{(0)}_\alpha h_{\mu\nu}\,\nabla^{(0)\alpha}h^{\mu\nu} - \nabla^{(0)}_\alpha h\,\nabla^{(0)\alpha}h + 2\nabla^{(0)}_\mu h^{\mu\nu}\,\nabla^{(0)}_\nu h - 2\nabla^{(0)}_\mu h^{\mu\nu}\,\nabla^{(0)}_\alpha h^{\alpha}{}_{\nu}\right] + \Delta\mathcal{L}_{\rm ani}, \tag{4.106}$$

where $\Delta\mathcal{L}_{\rm ani}$ collects anisotropic corrections induced by the ordered director and curvature channel $Y_{AB}F^A\cdot F^B$. In the de Donder gauge the tensor block decouples at leading order.

Tensor equation. Projecting to the TT sector yields the wave equation

$$\mathcal{K}_T^{\alpha\beta}\,\nabla^{(0)}_\alpha\nabla^{(0)}_\beta\,h^{\rm TT}_{ij} \;+\; m_T^2\,h^{\rm TT}_{ij} \;=\; \frac{2}{M^2_{\rm eff}(\rho_0)}\,\Pi^{\rm TT}_{ij,kl}\,T^{kl}_{\rm eff,\,source}, \tag{4.107}$$

where $\Pi^{\rm TT}$ is the standard TT projector, $T_{\rm eff,\,source}$ is the *matter+structural* stress tensor of Ch. 3, and the kinetic tensor encodes mild anisotropy,

$$\mathcal{K}_T^{\alpha\beta} = g^{(0)\,\alpha\beta} + \epsilon_\parallel\,\hat\chi_0^\alpha\hat\chi_0^\beta + \epsilon_u\,u_0^\alpha u_0^\beta \qquad (\ |\epsilon_{\parallel,u}| \ll 1\ ). \tag{4.108}$$

A small effective mass m_T may be induced by background curvature/strata gradients; hyperbolicity gates of Sec. 3.8 require $m_T^2 \geq 0$ and $\mathcal{K}_T$ to be Lorentzian with the same time orientation as $g^{(0)}$.

Dispersion relation. In the u_0 rest frame with plane waves $\propto e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$,

$$\omega^2 = c_T^2(\theta)\,k^2 + m_T^2 + \mathcal{O}(k^0), \qquad c_T^2(\theta) = 1 + \epsilon_\parallel\cos^2\theta, \tag{4.109}$$

where θ is the angle between $\mathbf{k}$ and $\hat\chi_0$. The isotropic stratum ($\alpha_0\to 0$ or $\delta c\to 0$) gives $c_T=1$, $m_T=0$ at this order.

4.9.4 Mixing with director/curvature modes and birefringence

Director–tensor mixing. Fluctuations $\delta\hat\chi^i$ carry their own quadratic energy (Sec. 4.4.1). The curvature channel $Y_{AB}F^AF^B$ and kinematic couplings of the tetrad (Ch. 2) generate at $\mathcal{O}(h\,\delta\hat\chi)$ a mixing

$$\mathcal{L}^{(2)}_{\rm mix} = \frac{\lambda_{\rm mix}}{2}\,h^{ij}_{\rm TT}\,\mathcal{M}_{ij}{}^{k}\,\delta\hat\chi_k \quad\Rightarrow\quad \begin{pmatrix} h^{\rm TT} \\ \delta\hat\chi\end{pmatrix} :\ \omega^2\begin{pmatrix}1&0\\0&1\end{pmatrix} - \begin{pmatrix} c_T^2k^2 & \lambda_{\rm mix}k \\ \lambda_{\rm mix}k & c_\chi^2k^2 + m_\chi^2\end{pmatrix} = 0, \tag{4.110}$$

with c_χ the director-wave speed (from $Z_{\chi\chi}, \kappa_{\rm FS}$), m_χ a possible gap (texture background), and $\mathcal{M}_{ij}{}^k$ a transverse tensor fixed by symmetry. Diagonalization yields an *avoided crossing*; for $|\lambda_{\rm mix}| \ll c_Tk$ one finds small polarization-dependent speed shifts $\Delta c_T \sim \lambda^2_{\rm mix}/(2c_Tc_\chi^2k^2)$.

Birefringence. In ordered strata the two TT polarizations can propagate with slightly different phase velocities if the background breaks parity along $\hat\chi_0$ through $F\cdot F$ asymmetries. Such splitting is governed by a pseudo-scalar invariant $I_{\rm odd}\propto\epsilon^{ijk}\hat\chi_{0i}\partial_j\hat\chi_{0k}$ and is *absent* in the isotropic stratum. Gate: require $|\Delta c_T| \ll 1$ to respect the cone constraints of Sec. 3.8.

4.9.5 Energy density and flux (Isaacson-like form)

Averaged stress tensor. To quadratic order and after rapid averaging over a few wavelengths, the effective energy–momentum of TT waves is

$$t^{\rm (GW)}_{\mu\nu} = \frac{M^2_{\rm eff}(\rho_0)}{4}\left\langle \nabla^{(0)}_\mu h^{\rm TT}_{ij}\,\nabla^{(0)}_\nu h^{\rm TT}_{ij} - \tfrac{1}{2}g^{(0)}_{\mu\nu}\,\nabla^{(0)}_\alpha h^{\rm TT}_{ij}\,\nabla^{(0)\alpha}h^{\rm TT}_{ij}\right\rangle + \mathcal{O}(\epsilon_\parallel, \lambda^2_{\rm mix}), \tag{4.111}$$

positive definite under our kinetic sign conventions (Ch. 2). The energy flux is $S^i = t^{0i}_{\rm (GW)}$ in the u_0 frame. Small anisotropies correct the prefactors but preserve positivity if $M^2_{\rm eff} > 0$ (gate).

4.9.6 Generation: quadrupole formula and structural sources

Effective quadrupole. For slow sources ($v \ll 1$) and weak anisotropy, the far-zone solution of (4.107) is

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) \simeq \frac{2}{M_{\mathrm{eff}}^2(\rho_0)\, r}\, \frac{d^2}{dt^2}\, Q_{ij}^{\mathrm{TT}}(t - r/c_T)\,, \qquad Q_{ij} = \int d^3x\, \left(T_{\mathrm{eff}}^{00}\, x_i x_j\right)_{\mathrm{STF}}\,, \tag{4.112}$$

where T_{eff}^{00} includes *matter and structural* energy densities (Ch. 3.5), and STF denotes trace-free symmetrization. Anisotropy enters via $c_T(\theta)$ and small projection corrections to TT.

Structural pumps. (i) *Wall kinks and oscillons* (Sec. 4.3) provide time-dependent quadrupoles even with negligible matter. (ii) *Hopfion breathing/twisting* (Sec. 4.4) sources director waves that mix into the tensor channel via λ_{mix}. (iii) *Two-body scattering* of solitons (Sec. 4.5) yields burst-like emission; phase-dependent forces modulate the waveform.

4.9.7 Propagation in inhomogeneous strata and lensing

Ray optics. In slowly varying backgrounds, TT wave packets follow null (or almost null) geodesics of the *tensor* kinetic metric $\mathcal{K}_T$:

$$\mathcal{K}_T^{\mu\nu} k_\mu k_\nu = 0 \quad (\text{for } m_T = 0), \qquad \frac{dx^\mu}{d\lambda} = \mathcal{K}_T^{\mu\nu} k_\nu, \quad \frac{dk_\mu}{d\lambda} = -\tfrac{1}{2}\partial_\mu \mathcal{K}_T^{\alpha\beta} k_\alpha k_\beta. \tag{4.113}$$

Gradients of $(\rho, \eta, \alpha, \hat{\chi})$ deflect rays (lensing) and induce small, frequency-independent time delays proportional to $\partial\mathcal{K}_T$. Mixing with $\delta\hat{\chi}$ can generate conversion between polarizations in regions with nontrivial I_{odd}.

Damping by dissipation. Finite $\eta_0 > 0$ produces weak, frequency-dependent damping of structural waves but does not directly damp h^{TT} at leading order. Indirect damping arises only through mixing and is second order in λ_{mix}.

4.9.8 Observational handles (qualitative) and gates

Signatures. (i) *Tiny anisotropy* in GW speed: $c_T(\theta) = 1 + \epsilon_\parallel \cos^2\theta$. (ii) *Polarization rotation / birefringence* in ordered regions with parity-odd director textures. (iii) *Additional, weak scalar/vector content* suppressed by gates (below). (iv) *Source-dependent modulation*: wall/Hopfion dynamics imprint spectral lines or sidebands.

Model gates (theory consistency).

$$M_{\mathrm{eff}}^2(\rho_0) > 0, \quad |\epsilon_\parallel|, |\epsilon_u| \ll 1, \quad m_T^2 \geq 0, \quad |\lambda_{\mathrm{mix}}|^2 \ll c_T^2 c_\chi^2 k^2, \quad \text{cone alignment as in Sec. 3.8.} \tag{4.114}$$

These ensure hyperbolicity, positive energy of waves, and absence of superluminal cones in the regime of validity.

4.9.9 Isotropic limit and recovery of GR

Limit $\alpha_0 \to 0$ (or $\delta c \to 0$). When the director order vanishes or anisotropy parameters are tuned to zero, the tensor kinetic tensor reduces to $\mathcal{K}_T^{\mu\nu} \to g^{(0)\,\mu\nu}$ and $m_T \to 0$, whence

$$\Box_{g^{(0)}} h_{ij}^{\mathrm{TT}} = \frac{2}{M_{\mathrm{eff}}^2(\rho_0)}\, \Pi_{ij,kl}^{\mathrm{TT}}\, T_{\mathrm{eff,\,source}}^{kl}, \qquad \omega^2 = k^2, \tag{4.115}$$

i.e. GR-like, luminal gravitational waves with the effective Planck scale $M_{\mathrm{eff}}(\rho_0)$ as derived in Ch. 3. The director and curvature channels decouple at leading order.

Summary. The spectral aether supports a robust tensor-radiation sector. In the isotropic stratum it coincides with GR at quadratic order; in ordered strata, controlled and bounded departures arise: mild anisotropy, tiny mixing with director waves, and potential birefringence. All effects are naturally small under the cone/positivity gates established earlier, yet they furnish clear, testable signatures to be quantified in Ch. 5 when we confront the model with observational data.

4.10 Nonlinear Stability and Spectral Analysis

We develop rigorous and practical criteria for the (i) *spectral* and (ii) *nonlinear (orbital)* stability of coherent configurations constructed in Secs. 4.2–4.6, including Q-balls, walls/strings, Hopf textures, and their bound "molecules". We also set up a numerical toolkit (validated by analytic gates) to map stability domains in parameter space. Throughout we respect the cone/positivity gates of Chs. 2–3 and, unless stated otherwise, work in the weakly dissipative regime (either $\eta = 0$ or $0 < \eta \ll 1$ so that Hamiltonian structure is only weakly perturbed).

4.10.1 Conserved functionals and constrained minimization

Energy, momentum, and charge. Let $\mathcal{E}[\Phi, \mathcal{S}]$ denote the total energy obtained from the Lagrangian density of Ch. 2 (Einstein–Palatini gravitational sector, anisotropic scalar kinetics, director elasticity, curvature channel, and potentials). In backgrounds where $g_{\rm eff}$ is stationary and spatially homogeneous, the conserved quantities relevant to stability are:

$$E[\Phi, \mathcal{S}] = \int d^3x \, \sqrt{-g_{\rm eff}}\, T^{00}_{\rm eff}, \tag{4.116}$$

$$\mathbf{P}[\Phi, \mathcal{S}] = \int d^3x \, \sqrt{-g_{\rm eff}}\, T^{0i}_{\rm eff}\, \mathbf{e}_i, \tag{4.117}$$

$$Q_\Phi[\Phi] = \int d^3x \, \sqrt{-g_{\rm eff}}\, \kappa_\Phi(\mathcal{S})\, J^0_\Phi, \qquad J^\mu_\Phi := \mathrm{Im}\big(\Phi^* \nabla^\mu \Phi\big), \tag{4.118}$$

and the *topological invariants* $\mathcal{Q}_{\rm top}$ of Sec. 4.6.1. Coherent states are frequently *constrained minimizers* of E at fixed $(Q_\Phi, \mathcal{Q}_{\rm top})$.

Stationary ansatz and Lagrange multipliers. For time-harmonic matter states we set

$$\Phi(t, \mathbf{x}) = e^{-i\omega\, T(t,\mathbf{x})}\, \phi(\mathbf{x}), \tag{4.119}$$

with ω a real Lagrange multiplier for fixed Q_Φ ("chemical potential"). In static backgrounds ($\partial_t T =$ const, negligible metric modulation) the Euler–Lagrange system is equivalent to the constrained variational problem

$$\delta\Big[E[\phi, \mathcal{S}] - \omega\, Q_\Phi[\phi]\Big] = 0, \tag{4.120}$$

with E computed on static slices and the anisotropic gradient norm $|\nabla\phi|^2_{G_{(\rm ani)}}$ of Ch. 2.

4.10.2 Linearized operator, essential spectrum, and gates

Linearization. Write the perturbation as $\delta\Phi = e^{-i\omega T}(\xi_1 + i\xi_2)$ and collect structural perturbations in σ^A (for $\rho, \eta, \alpha, \hat{\chi}, \ldots$). The linearized evolution can be cast as

$$\partial_t \mathbf{u} = \mathcal{J}\,\mathcal{H}\,\mathbf{u} - \mathcal{D}\,\mathbf{u}, \qquad \mathbf{u} = (\xi_1, \xi_2, \sigma^A, \ldots)^\top, \tag{4.121}$$

where $\mathcal{J}$ is an antisymmetric "symplectic" block (canonical matter part augmented by director/geometric entries), $\mathcal{H}$ is the self-adjoint Hessian (second variation of $E - \omega Q_\Phi$), and $\mathcal{D} \geq 0$ encodes weak dissipation proportional to η. Spectral stability at $\eta = 0$ corresponds to $\mathrm{spec}(\mathcal{J}\mathcal{H}) \subset i\mathbb{R}$.

Essential spectrum. In homogeneous strata the essential spectrum is obtained by Fourier transform. Let $k_\parallel := \mathbf{k}\cdot\hat{\chi}$ and $k_\perp^2 := k^2 - k_\parallel^2$. Then (schematically)

$$\lambda_{\rm ess}(\mathbf{k}) = \pm i\,\Omega(\mathbf{k}), \qquad \Omega^2 = c_\perp^2 k_\perp^2 + (c_\perp^2 + \delta c)\, k_\parallel^2 + m_{\rm gap}^2 + \cdots, \tag{4.122}$$

with positive cone gates $c_\perp^2 > 0$, $c_\perp^2 + \delta c > 0$ and real gaps $m_{\rm gap}^2 \geq 0$ from potentials/texture backgrounds (Chs. 2,4). *Conclusion:* the essential spectrum is purely imaginary; instability can only arise from *point spectrum* (eigenvalues) crossing into $\mathrm{Re}\,\lambda > 0$.

4.10.3 Hamiltonian–Krein index and VK-type criteria

Kernel and symmetries. Neutral directions of $\mathcal{H}$ are generated by continuous symmetries: spatial translations, global $U(1)_\Phi$ phase, internal rotations of textures, and (for molecules) relative separation/phase. Each symmetry contributes a zero mode to $\ker(\mathcal{J}\mathcal{H})$.

Krein signatures. For simple imaginary eigenvalues $i\Omega \neq 0$, the Krein signature is $\operatorname{sign}\langle \mathbf{u}, \mathcal{H}\,\mathbf{u}\rangle$. Collisions of opposite signatures on the imaginary axis are the only route to Hamiltonian instabilities (quartets leaving $i\mathbb{R}$).

Generalized Vakhitov–Kolokolov (VK). For single-hump Q-ball-like states, define $Q_\Phi(\omega)$ as in (4.118) evaluated on the branch ϕ_ω. Under mild locality and coercivity assumptions for $\mathcal{H}$ (anisotropic metric allowed), the standard slope test extends:

$$\boxed{\frac{dQ_\Phi}{d\omega} < 0 \quad \Longrightarrow \quad \text{no real eigenvalues and no complex quartets near the origin}} \tag{4.123}$$

Conversely, $dQ_\Phi/d\omega > 0$ signals a change in the count of negative directions of $\mathcal{H}$ and typically a real unstable eigenvalue ("L^2-slope instability"). The proof follows the Grillakis–Shatah–Strauss (GSS) framework adapted to the anisotropic kinetic form (see §4.10.4).

Topological textures. For Hopfions stabilized by the quartic $|Q_{ij}|^2$ and/or curvature channel, Derrick scaling forbids collapse; the Hessian has a finite number of negative directions (often zero for minimal $|\mathcal{H}|$). Hence spectral stability reduces to verifying that modulation zero modes (translations/rotations) are the only kernel directions and that no opposite-signature pairs collide; this is generic for $|\mathcal{H}| = 1$.

4.10.4 Orbital stability via GSS coercivity

Hypotheses (GSS-A). (i) $\mathcal{H}$ is self-adjoint on a natural energy space with domain dense in L^2; (ii) $\ker \mathcal{H}$ is spanned by symmetry generators; (iii) the constrained Hessian $\mathcal{H}\restriction_{\mathcal{M}}$ (orthogonal to constraint gradients) has a fixed number of negative directions n_- independent along the branch; (iv) the slope matrix $\partial_\omega Q_\Phi$ satisfies (4.123) (or its multi-parameter generalization).

Theorem (orbital stability). Under (GSS-A), the coherent state is *orbitally stable* in the energy topology: initial data sufficiently close to the solitary manifold remain close up to the group action (translations, global phase, internal rotations). For $0 < \eta \ll 1$ the same state is *asymptotically stable* modulo radiation, with decay rates governed by Fermi golden rule couplings to the continuous spectrum.

4.10.5 Modulational (Benjamin–Feir) stability of patterned states

Whitham system. For nearly periodic carriers (walls/stripes/guided modes) with local wavenumber $\mathbf{k}$ and frequency $\omega(\mathbf{k}; \mathcal{S})$, slow modulations obey Whitham equations for $(\mathbf{k}, \varphi, \dots)$ whose linearization yields the modulation dispersion relation $\Omega_{\mathrm{mod}}(\mathbf{K})$.

Criterion (anisotropic). In the Hamiltonian limit ($\eta = 0$), modulational stability requires the Hessian $\partial^2\omega/\partial k_i \partial k_j$ to be positive (or negative) definite on the energy shell; with weak dissipation, the real part of Ω_{mod} acquires a damping term $-\Gamma(\eta)\,|\mathbf{K}|^2$. Anisotropy enters via the group-velocity metric induced by $G_{(\mathrm{ani})}$ and yields direction-dependent thresholds.

4.10.6 Bound states and molecules: interaction eigenvalues

Zero-mode splitting. For two well-separated solitons at distance R, translational and phase zero modes hybridize. To leading order the interaction eigenvalue satisfies

$$\lambda_{\mathrm{int}}^2 \;\sim\; -\,\frac{\partial^2 V_{\mathrm{eff}}(R)}{\partial R^2}\,\frac{1}{M_{\mathrm{eff}}^{\mathrm{coll}}} \;+\; \mathcal{O}(e^{-2R/\ell_\star}), \tag{4.124}$$

where $V_{\rm eff}(R)$ is the static interaction potential (Sec. 4.5) and $M_{\rm eff}^{\rm coll}$ the collective mass of the bound coordinate. A potential minimum ($V''_{\rm eff}(R_b) > 0$) yields purely imaginary λ (stable oscillatory mode).

Evans function. For 1D walls/strings or effectively radial problems, we construct the Evans function $D(\lambda)$ (analytic in λ for $\mathrm{Re}\,\lambda > -\varepsilon$) by matching stable/unstable manifolds of the linearized ODE. Zeros of $D(\lambda)$ in $\mathrm{Re}\,\lambda > 0$ indicate point-spectrum instabilities. In 3D, block-diagonalization by angular momentum reduces the computation to coupled 1D channels.

4.10.7 Krein collisions and parity breaking in ordered strata

Birefringent shifts. Ordered strata with parity-odd invariants (e.g. director helicity) split polarization branches and can move opposite-signature eigenvalues along $i\mathbb{R}$. Gate: ensure that the splitting remains smaller than the minimal gap to the continuous spectrum, preventing Hamiltonian–Hopf bifurcations.

Mixed matter–director modes. Weak mixing (Sec. 4.9.4) produces avoided crossings; the Krein signature is preserved under small $\lambda_{\rm mix}$, hence no instabilities arise unless a genuine collision occurs.

4.10.8 Numerical spectral toolkit

Discretization. Use structure-preserving schemes: (i) enforce $|\hat{\chi}| = 1$ via tangent-plane projection or penalty, (ii) spectral or high-order finite differences on compactified domains, (iii) perfectly matched layers for radiation. For symmetry-reduced states, adopt radial/axial decompositions.

Operators and solvers. Compute steady states by Newton–Krylov with line search minimizing the constrained functional $E - \omega Q_\Phi$. For spectra, employ shift-invert Arnoldi on the linearized operator in Krein-friendly inner products, and confirm Krein signatures via evaluation of $\langle \mathbf{u}, \mathcal{H}\mathbf{u}\rangle$.

Validation gates. (1) Recover translation/phase zero modes with machine precision; (2) verify $\mathrm{spec}_{\rm ess}$ on the imaginary axis by box-size extrapolation; (3) perform VK slope scans (4.123) along branches and check the predicted index jumps; (4) benchmark Evans zeros against time-domain growth/decay rates.

4.10.9 Summary and stability map

Analytic gates. (i) Cone positivity of Ch. 3 ⇒ purely imaginary essential spectrum; (ii) Derrick/Vakulenko–Kapitanski bounds ⇒ minimal Hopfions have no collapse channel; (iii) GSS + VK slope ⇒ Q-ball branches stable where $dQ_\Phi/d\omega < 0$; (iv) interaction eigenvalues ⇒ bound molecules stable near minima of $V_{\rm eff}$; (v) weak $\eta > 0$ yields asymptotic (radiative) damping without altering the spectral index.

Practical outcome. These criteria define open stability domains in $(\rho, \eta, \alpha, \kappa_\Phi, \kappa_{\rm FS}, Y_{AB}, \ldots)$ parameter space for each soliton family. They will guide the targeted simulations of Ch. 5 and the confrontation with observational gates (colliders, cosmology, and gravitational radiation constraints).

4.11 Unified Numerical Protocols

This section specifies a *unified, reproducible, and theory-consistent* numerical toolkit for the spectral-aether model. It covers discretization of the anisotropic operators, constraint enforcement ($|\hat{\chi}| = 1$, orthogonality $u\cdot\hat{\chi} = 0$, conserved charge fixing), time integration with cone/positivity gates, steady-state solvers for coherent structures, spectral stability solvers, absorbing boundaries, adaptivity, and validation. The protocols are designed to be *modular* so that the same primitives are reused for Q-balls, walls/strings, Hopf textures, and soliton molecules (Sections 4.2–4.6), for wave radiation (Sec. 4.9), and for stability maps (Sec. 4.10).

4.11.1 Discretization of anisotropic differential operators

Computational domain and grids. Let $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ be a rectangular box or cylindrical/radial domain depending on symmetry. We adopt:

- **Uniform grids** for spectral (FFT) or high-order finite differences when solutions are smooth and global resolution is needed (Hopfions, radiation).
- **Block-structured AMR** for wall/string networks and molecules with localized cores.

Anisotropic gradient and divergence. Recall the anisotropic metric $G^{ij}_{(\mathrm{ani})}(\mathcal{S})$ from Ch. 2. We discretize the operator $\mathcal{L}_\Phi \phi := \nabla_i \Big(\kappa_\Phi(\mathcal{S})\, G^{ij}_{(\mathrm{ani})}(\mathcal{S})\, \nabla_j \phi \Big)$ in conservative form. On a Cartesian grid with spacing Δx_j,

$$\big(\mathcal{L}_\Phi \phi\big)_{\mathbf{n}} \;=\; \sum_{j=1}^{d} \frac{1}{\Delta x_j} \left[\Big(\kappa_\Phi\, G^{jj}_{(\mathrm{ani})}\, \delta_j^+ \phi \Big)_{\mathbf{n}-\frac{1}{2}\mathbf{e}_j} - \Big(\kappa_\Phi\, G^{jj}_{(\mathrm{ani})}\, \delta_j^+ \phi \Big)_{\mathbf{n}+\frac{1}{2}\mathbf{e}_j} \right] \;+\; \text{cross-terms}, \tag{4.125}$$

where δ_j^+ is a forward difference and cross-terms implement G^{ij} for $i \neq j$. Face-centered coefficients are arithmetic- or harmonic-averaged to preserve symmetry and discrete energy stability. In spectral discretizations, we apply the symbol $\hat{\mathcal{L}}_\Phi(\mathbf{k}) = \kappa_\Phi\, G^{ij}_{(\mathrm{ani})} k_i k_j$ in Fourier space.

Director elasticity and Skyrme-like terms. For $|\hat{\chi}| = 1$, the quadratic elastic energy uses the projected gradient $\partial^\top \hat{\chi} = (\mathbb{I} - \hat{\chi}\hat{\chi}^\top)\nabla\hat{\chi}$. The quartic term $|Q_{ij}|^2$ (Sec. 4.4.1) is discretized by first forming a discrete connection on the unit sphere bundle and then squaring in a symmetric fashion to avoid lattice locking. We adopt tangent-plane projection (see §4.11.2) after each update.

Curvature channel. The internal curvature quadratic $Y_{AB} F^A_{\mu\nu} F^{B\,\mu\nu}$ reduces, for static spatial computations, to covariant curls on the internal connection. We discretize the curvature via staggered Yee-like curls to maintain the Bianchi identities in the discrete setting.

4.11.2 Constraint enforcement and projections

Unit-length director. Given a tentative update $\tilde{\chi}$, enforce $\hat{\chi} \leftarrow \tilde{\chi}/\max(\epsilon, |\tilde{\chi}|)$ with ϵ a small guard. For schemes with strict tangent dynamics, use the projection $\mathcal{P}_\perp(\hat{\chi}) := \mathbb{I} - \hat{\chi}\hat{\chi}^\top$ and replace $\delta\hat{\chi} \mapsto \mathcal{P}_\perp(\hat{\chi})\, \delta\hat{\chi}$ at each substep.

Orthogonality to u. After advancing u and $\hat{\chi}$, enforce $u \cdot \hat{\chi} = 0$ by $\hat{\chi} \leftarrow \hat{\chi} - (u \cdot \hat{\chi})\, u, \quad \hat{\chi} \leftarrow \hat{\chi}/|\hat{\chi}|$. If u is fixed by the background stratum, only $\hat{\chi}$ is corrected.

Charge fixing for steady states. For time-harmonic branches $\Phi = e^{-i\omega T}\phi$, we fix Q_Φ (Eq. (4.118)) by adjusting ω through a Newton update on the constraint $C(\omega) := Q_\Phi(\phi_\omega) - Q_\star$. In continuation (see §4.11.4), we use pseudo-arclength with ω and a norm of ϕ as continuation coordinates.

4.11.3 Time integration with cone and energy gates

Operators and splitting. The evolution couples a conservative Hamiltonian part (from the Palatini–Einstein sector and the elastic/kinetic energies) and a weak dissipative part controlled by η (Ch. 3.9). We adopt a second- or fourth-order Strang/IMEX splitting:

$$\text{(A) conservative step: } \partial_t U = \mathcal{J}\, \delta\mathcal{H}[U], \qquad \text{(B) dissipative step: } \partial_t U = -\mathcal{D}[U],$$

where U collects Φ, structural fields, and metric perturbations when evolved.

CFL condition (anisotropic). Let $\lambda_{\max}$ be the largest eigenvalue of the symbol of the fastest kinetic tensor among matter, director, and tensor waves (Sec. 4.9.3). The time step obeys

$$\Delta t \;\le\; C_{\mathrm{CFL}} \min_j \frac{\Delta x_j}{\sqrt{\lambda_{\max}}}, \qquad 0 < C_{\mathrm{CFL}} < 1, \tag{4.126}$$

with a global safety factor accounting for cross-terms in $G_{(\mathrm{ani})}$ and any mixing (λ_{mix} in Sec. 4.9.4).

Symplectic and damping choices. For conservative substeps we use velocity-Verlet or Gauss–Legendre RK to reduce phase error; the dissipative substep is handled by exact exponentials for linear $\mathcal{D}$ or by stabilized explicit RK for nonlinear $\mathcal{D}$. Energy drift is monitored via discrete Noether currents (Eq. (4.116)).

4.11.4 Steady states: Newton–Krylov, continuation, and constraints

Residual and Jacobian. Define the steady-state residual $\mathcal{R}(U,\omega)=0$ for the constrained Euler–Lagrange system (Eqs. (4.120) and the structural EL equations). We solve $\mathcal{R}=0$ with Newton–Krylov (GMRES) and a block preconditioner that inverts approximately: (i) scalar elliptic blocks with anisotropic diffusion, (ii) director elasticity (quadratic+quartic), (iii) curvature curls.

Pseudo-arclength continuation. To traverse turning points we augment $\mathcal{R}$ with an arclength constraint $\mathcal{C}(U,\omega)=\langle (U,\omega)-(U_0,\omega_0),\ (U',\omega')\rangle-\Delta s=0$ and solve the bordered system at each step.

Quality gates. (i) Recover translation/phase zero modes; (ii) verify Derrick/VK inequalities locally; (iii) check positivity of discrete energy densities; (iv) ensure $|\hat{\chi}|=1$ and $u\cdot\hat{\chi}=0$ to machine precision.

4.11.5 Spectral stability solvers

Linearized operator assembly. Construct $\mathcal{L}=\mathcal{J}\mathcal{H}-\mathcal{D}$ (Eq. (4.121)) on the converged steady state. Use structure-preserving inner products so that $\mathcal{H}$ is self-adjoint and $\mathcal{J}$ skew-adjoint discretely.

Shift-invert Arnoldi and Krein signatures. Compute rightmost eigenvalues of $\mathcal{L}$ by shift-invert around $\sigma=i\Omega_\star$. Evaluate Krein signatures via the quadratic form $\langle \mathbf{u},\mathcal{H}\mathbf{u}\rangle$ to classify collisions (Sec. 4.10.3). Confirm the imaginary essential spectrum by domain-size extrapolation.

Evans function for reduced geometries. For 1D walls/strings or radial reductions, build the Evans function by shooting stable/unstable manifolds with exterior algebra to avoid parasitic growth. Zeros in $\mathrm{Re}\,\lambda>0$ diagnose instabilities (Sec. 4.10.6).

4.11.6 Radiation boundaries and absorbing layers

PML for tensor and scalar waves. Adopt perfectly matched layers compatible with the *tensor* kinetic metric $\mathcal{K}_T$ (Sec. 4.9.3) and the scalar anisotropic metric $G_{(\mathrm{ani})}$. Implement complex coordinate stretching $x_j\mapsto x_j+i\int^x\sigma_j(\xi)\,d\xi$ with smooth σ_j vanishing inside Ω.

Nonreflecting conditions for director. For $\delta\hat{\chi}$ use characteristic-based boundary conditions from the linearized dispersion (c_χ, m_χ). For static steady computations use large domains with algebraic decay checks.

4.11.7 Adaptivity and multiscale resolution

Refinement indicators. Refine where (i) $|\nabla\Phi|$ or $|\nabla\hat{\chi}|$ exceed thresholds, (ii) the quartic elastic density is large, (iii) curvature invariants $F^A_{ij}F^{A\,ij}$ are large, (iv) error estimators from residuals $|\mathcal{R}|$ exceed tolerances.

Conservative prolongation. Inter-level transfers preserve discrete charges and $|\hat{\chi}|=1$ by renormalization on fine levels. Ghost zones include projected tangential components to avoid constraint drift.

4.11.8 Workflow and reproducibility

Units and nondimensionalization. Work in $c=1$ units with a reference scale $M_{\mathrm{eff}}(\rho_0)$ set to 1 unless stated otherwise. Record the mapping from physical to code units; store $\{\kappa_\Phi,\kappa_{\mathrm{FS}},Y_{AB},\alpha,\rho,\eta\}$ as dimensionless inputs.

Pipeline.

1. **Background build:** choose a homogeneous stratum $\mathcal{S}_0$, compute g_{eff}, $\mathcal{K}_T$, cone gates (Ch. 3).
2. **Initial guess:** analytical seed (Sec. 4.2 or 4.4) plus localized noise.
3. **Steady solve:** Newton–Krylov + constraints (§4.11.4).
4. **Spectra:** assemble $\mathcal{L}$, compute point spectrum, Krein signatures (§4.11.5).
5. **Dynamics:** time-step with IMEX split, PML (§4.11.3, 4.11.6).
6. **Post-processing:** conserved quantities, energy fluxes, stability gates (Secs. 4.9.5, 4.10).

Versioning and checksums. Each run stores: parameter hash, grid description, random seeds, solver tolerances, and checksums of initial and converged states. Figures and tables include the run ID.

4.11.9 Validation tests and benchmarks

Analytic baselines. (i) Linear wave dispersion in homogeneous strata (Eqs. (4.109), (4.122)); (ii) Derrick/VK scaling tests; (iii) conservation of Q_Φ to within solver tolerance in Hamiltonian runs.

Convergence. Grid-refinement studies verify second- (or spectral-) order convergence in norms of residuals and conserved quantities. Temporal convergence is checked by halving Δt at fixed Δx.

Cross-code verification. Where feasible, reproduce selected cases with an independent discretization (FD vs. spectral) or independent AMR implementation.

4.11.10 Summary: quality gates and readiness checklist

Quality gates. (i) Cone alignment: eigencones of all kinetic blocks lie within the GR cone in the isotropic limit; (ii) positivity: discrete energies > 0; (iii) constraints: $|\hat{\chi}| = 1$, $u \cdot \hat{\chi} = 0$ to machine precision; (iv) spectra: $\text{spec}_{\text{ess}} \subset i\mathbb{R}$, VK slope satisfied where stability is claimed; (v) boundaries: PML reflection $< 10^{-6}$.

Readiness checklist (per figure/table).

- Parameter card (κ's, Y_{AB}, α, ρ, η), grid, tolerances, seeds.
- Verification plots: convergence, conserved charges, energy balance.
- Diagnostic spectra with Krein signatures and Evans scans where applicable.
- Reproducibility archive with run ID and checksums.

This unified protocol allows the Chapter 5 applications (solution atlas, interaction studies, and observational diagnostics) to be generated on a common, auditable computational foundation.

4.12 Synopsis and Testable Signatures

This section distills the *structural* and *dynamical* content of Chapter 4 into a compact set of predictions and *falsifiable* signatures. We organize the material into: (i) a one-page synopsis, (ii) a state-to-observable map, (iii) smoking-gun signatures across gravitational, collider, laboratory, and astrophysical domains, (iv) theory gates and consistency bounds for inference, and (v) a minimal experimental strategy.

4.12.1 One-page synopsis of Chapter 4

- **Structured states & order parameters.** The aether admits isotropic and ordered strata controlled by $\mathcal{S} = (\rho, \eta, \alpha, \hat{\chi}, T)$; a G_2-structure on the internal fiber is dynamically selected and reduced to $SU(3) \to SU(2)\times U(1)$ by $(\hat{\chi}, J)$.
- **Guiding/defect solutions.** Walls/strings and Hopf textures guide matter waves; localized solitons (and molecules) form and interact via a curvature-mediated potential.
- **Gauge and Yukawa emergence.** Non-Abelian gauge fields arise as internal components of the spectral connection with state-dependent couplings $g_i(\mathcal{S})$; Yukawa textures are fixed by invariant internal tensors and *geometric* overlaps on defects.
- **Gravitational radiation.** Tensor waves of $g_{\rm eff}$ are GR-like in the isotropic stratum; ordered strata introduce small, bounded anisotropy and mixing with director modes.
- **Stability map.** Essential spectra are purely imaginary under cone gates; Q-ball branches obey a generalized VK criterion; textures stabilized by quartic/curvature terms are orbitally stable.

4.12.2 From structural state to observables

Linearized response. For small departures from a homogeneous background $\mathcal{S}_0$, any observable $\mathcal{O}$ admits

$$\Delta\mathcal{O} \;\simeq\; \sum_{A\in\{\rho,\eta,\alpha\}} \frac{\partial\mathcal{O}}{\partial S^A}\Big|_{\mathcal{S}_0}\Delta S^A \;+\; \frac{\partial\mathcal{O}}{\partial\hat{\chi}_i}\Big|_{\mathcal{S}_0}\Delta\hat{\chi}_i \;+\; \cdots, \tag{4.127}$$

with sensitivity kernels determined by the kinetic forms and potentials of Chs. 2–3.

Examples.

$$\frac{\Delta g_i}{g_i} \;\simeq\; c_{i,\rho}\,\frac{\Delta\rho}{\rho_0} + c_{i,\eta}\,\frac{\Delta\eta}{\eta_0} + c_{i,\alpha}\,\frac{\Delta\alpha}{\alpha_0} + \cdots, \tag{4.128}$$

$$c_T^2(\theta) = 1 + \epsilon_{\parallel}(\mathcal{S}_0)\,\cos^2\theta, \qquad \epsilon_{\parallel} \propto \delta c(\alpha, \rho, \ldots), \tag{4.129}$$

$$\mathcal{Y}_{ij}^{\rm eff} \;\propto\; \exp\!\left[-\frac{d_{ij}(\mathcal{S}_0)}{\ell_\star(\mathcal{S}_0)}\right] \times e^{\,i\oint \mathrm{Tr}(\mathcal{A})}, \tag{4.130}$$

linking coupling drifts, GW anisotropy, and flavor hierarchies/CP phases to $(\rho, \eta, \alpha, \hat{\chi})$.

4.12.3 Smoking-gun signatures by domain

(A) Gravitational waves (Sec. 4.9).

1. **Anisotropic GW speed:** $\omega^2 = c_T^2(\theta)k^2 + m_T^2$, with $c_T(\theta) = 1 + \epsilon_{\parallel}\cos^2\theta$ and $|\epsilon_{\parallel}| \ll 1$.
2. **Birefringence/Polarization mixing:** tiny, frequency-dependent splitting of TT modes in ordered strata with parity-odd director invariants.
3. **Director–tensor conversion:** avoided crossings near structure-rich regions (walls/Hopfions), leading to sidebands or polarization rotation correlated with large $|\nabla\hat{\chi}|$.
4. **GW lensing by strata gradients:** small, frequency-independent time delays from $\partial\mathcal{K}_T$.

(B) Electroweak/Gauge sector (Sec. 4.8).

1. **Heavy internal vectors/scalars:** $M_X \sim g_* v_\Sigma$, $M_Y \sim g_3 v_\Sigma$, $m_{\Sigma,\rm rad} \sim \sqrt{2\lambda_\Sigma}v_\Sigma$, decoupled if $v_\Sigma \gg v$.
2. **Near-critical coupling alignment:** approach $g_3 \simeq g_2 \simeq c_Y g_1$ as $\mathcal{S} \to \mathcal{S}_c$ (high-symmetry stratum).
3. **Spectral shifts in precision observables:** small, environment-dependent corrections to m_W/m_Z and to oblique parameters through state dependence of (g_i, κ_Σ).

(C) Flavor/CP (Sec. 4.7).

1. **Geometric hierarchies:** exponential overlaps (4.130) from localization on walls/loops.
2. **Berry phases:** holonomy-induced complex phases in $\mathcal{Y}$ providing CP violation with topological control.
3. **MFV as geometry:** higher operators built from the same misalignment spurions $(\hat{\chi}, J)$.

(D) Soliton dynamics and molecules (Secs. 4.2–4.6).

1. **Short-range repulsion & long-tail forces:** curvature-mediated corrections to $V_{\rm eff}(R)$ with tunable core via $F_{\rm spec}$.
2. **Bound-state spectra:** discrete vibrational branches with frequencies predicted by Hessian eigenvalues (Sec. 4.10.6).

(E) Laboratory "null tests".

1. **Isotropic stratum:** all deviations above vanish at leading order: $\epsilon_{\parallel} = 0,\ m_T = 0,\ \Delta g_i = 0$, providing strict *null* predictions in homogeneous environments.

4.12.4 Consistency gates and bounds for inference

$$\boxed{\begin{array}{l}\text{(CONE)}\quad c_\perp^2 > 0,\quad c_\perp^2 + \delta c > 0,\quad \text{eigencones} \subset \text{GR cone in iso. limit;}\\ \text{(ENERGY)}\quad M_{\rm eff}^2(\rho) > 0,\quad \mathcal{G}_{AB} > 0,\quad T_{\rm eff}^{00} > 0;\\ \text{(SPECTRUM)}\quad \text{spec}_{\rm ess} \subset i\mathbb{R},\quad dQ_\Phi/d\omega < 0 \text{ on stable branches;}\\ \text{(ALIGN)}\quad \langle S\rangle = 0,\ \langle H\rangle = (0, v/\sqrt{2})^T,\quad m_\gamma = 0,\ \text{EW masses as in (4.100);}\\ \text{(DECOUPLE)}\quad v_\Sigma \gg v \Rightarrow M_{X,Y},\, m_{\Sigma,\rm rad} \text{ above current reach.}\end{array}} \tag{4.131}$$

These act as priors/constraints in any fit: parameter proposals violating (4.131) are rejected.

4.12.5 Minimal experimental strategy

Tier-1 (null tests, high precision).

- Multi-baseline GW speed isotropy: constrain $\epsilon_{\parallel}$ in (4.129) by comparing arrival times/polarizations from different sky angles θ.
- Precision electroweak fits with state dependence: bound small drifts in $g_i(\mathcal{S})$ and $\kappa_\Sigma(\mathcal{S})$ via global fits under the alignment potential (4.99).
- Atomic/clock comparisons: constrain environmental variations $\Delta g_1/g_1$ correlated with ambient (ρ, η, α) across laboratories/geologies.

Tier-2 (smoking guns).

- Polarization-resolved GW birefringence correlated with regions of strong $(\nabla\hat{\chi})$.
- Discovery of heavy internal vectors/scalars consistent with $M_{X,Y} \sim v_\Sigma$ and the unification pattern (4.97).
- Flavor textures matching geometric zeros/overlaps predicted by (4.130).

Tier-3 (structural sources).

- Targeted GW searches for wall/texture oscillons: narrowband lines or burst trains with sideband structure tied to director–tensor mixing (Sec. 4.9.4). 12
- Lensing/time-delay anomalies for GWs traversing stratified regions (Sec. 4.9.7).

4.12.6 Falsifiability and decision tree

Falsifiability. Any *one* of the following (robustly established) would falsify the present Chapter 4 realization:

1. Persistent, order-one GW speed anisotropy or birefringence in homogeneous regions (violates cone gates).
2. Electroweak vacuum misalignment yielding $m_\gamma \neq 0$ or incorrect m_W/m_Z at leading order.
3. Flavor patterns requiring Yukawa entries forbidden by the geometric singlet rule (4.80).
4. Observation of light ($\ll v$) coset vectors/scalars inconsistent with $v_\Sigma \gg v$ while other gates hold.

Decision tree (schematic).

$$\text{Stage 1:} \begin{cases} \text{Null tests pass} & \Rightarrow \text{ tighten bounds on } (\epsilon_\parallel, \Delta g_i, \ldots) \\ \text{Null test fails} & \Rightarrow \text{ go to Stage 2 (source-correlation and cone checks)} \end{cases}$$

$$\text{Stage 2:} \begin{cases} \text{Signal correlates with structured strata} & \Rightarrow \text{ fit } (\alpha, \nabla\hat{\chi}) \text{ profiles} \\ \text{No correlation} & \Rightarrow \text{ model tension } \rightarrow \text{ falsification} \end{cases}$$

Signature	Form/Relation	Controllers	Consistency gates
GW speed anisotropy	$c_T^2(\theta) = 1 + \epsilon_\parallel \cos^2\theta$	$\delta c,\ \alpha,\ \hat{\chi}$	$\lvert\epsilon_\parallel\rvert \ll 1,\ m_T^2 \geq 0$
GW polarization optics	tiny TT phase splitting	$I_{\rm odd}(\hat{\chi}),\ \lambda_{\rm mix}$	cone alignment (Sec. 3.7), weak m
Yukawa hierarchy	$\mathcal{Y}_{ij} \sim \exp[-d_{ij}/\ell_\star]\ e^{i\oint \mathrm{Tr}(\mathcal{A})}$	localization, $\ell_\star,\ \mathcal{A}$	singlet-projection rule, geometric
Heavy internal vectors/scalars	$M_{X,Y} \sim g_* v_\Sigma,\quad m_{\Sigma,\rm rad} \sim \sqrt{2\lambda_\Sigma}\, v_\Sigma$	$v_\Sigma,\ g_*,\ \lambda_\Sigma$	$v_\Sigma \gg v,\ m_\gamma = 0$

Table 4.1: Signature map: from structural state to observables.

Parameter card: $\rho = 1$, $\eta = 0$, $\alpha = 0$, $\delta c = 0$, $\kappa_\Phi = 1$, $\kappa_{\rm FS} = 1$, $Y_{AB} = \mathrm{I}$, $v_\Sigma = --$, $g_* = --$.

Parameter card: $\rho = 1$, $\eta = 1\mathrm{e}{-3}$, $\alpha = 1\mathrm{e}{-2}$, $\delta c = -2\mathrm{e}{-3}$, $\kappa_\Phi = 1$, $\kappa_{\rm FS} = 1$, $Y_{AB} = \mathrm{I}$, $v_\Sigma = 30$ TeV, $g_* = 1$.

Parameter card: $\rho = 1.2$, $\eta = 1\mathrm{e}{-2}$, $\alpha = 0.1$, $\delta c = -1\mathrm{e}{-2}$, $\kappa_\Phi = 1.1$, $\kappa_{\rm FS} = 1.2$, $Y_{AB} = \mathrm{diag}(1,1,1)$, $v_\Sigma = 100$ TeV, $g_* = 1.2$.

Summary. Chapter 4 delivers *concrete*, *bounded*, and *computable* deviations from GR/SM, tied to the structural state of the spectral aether. The clean *null* limit in homogeneous strata makes the framework sharply falsifiable; ordered strata introduce a small, predictive palette of effects—GW anisotropy and birefringence, near-critical coupling alignment, geometric flavor hierarchies, and heavy coset spectra—all of which are ready for quantitative confrontation using the unified numerical protocols of Sec. 4.11.

Chapter 4 — Tables and Illustrative Figures

Table 4.2: Topological sectors and representative defects in Chapter 4. Homotopy classification, typical field Ansatz, conserved quantities, and key stability notes.

Homotopy	**Defect**	**Typical Ansatz / Map**	**Quantized charge**	**Stability notes**
π_0	Domain wall	Planar interface $S(x)$ with $S(\pm\infty) = S_\pm$	—	Tension σ; translational zero mode; stability via convexity of $U(S)$.
$\pi_1 \simeq \mathbb{Z}$	Vortex / string	Cylindrical symmetry; phase $\alpha = n\phi$, radial cores in (ρ, χ_I)	Winding $n \in \mathbb{Z}$	Finite energy/length; long-range tails set by effective masses; FR selection in scattering.
$\pi_2 \simeq \mathbb{Z}$	Skyrmion (in effective S^2 sector)	Map $S^2 \to S^2$ for reduced textures; hedgehog/toroidal profiles	Degree $B \in \mathbb{Z}$	Requires quartic (Skyrme-like) stiffness for stability against collapse.
$\pi_3 \simeq \mathbb{Z}$	Hopfion / knotted texture	Map $S^3 \to S^2$; linked/twisted tubes; effective Faddeev–Skyrme terms	Hopf charge $Q_H \in \mathbb{Z}$	Empirical scaling $E \sim \lvert Q_H\rvert^{3/4}$; shape modes; knot-type selection rules.

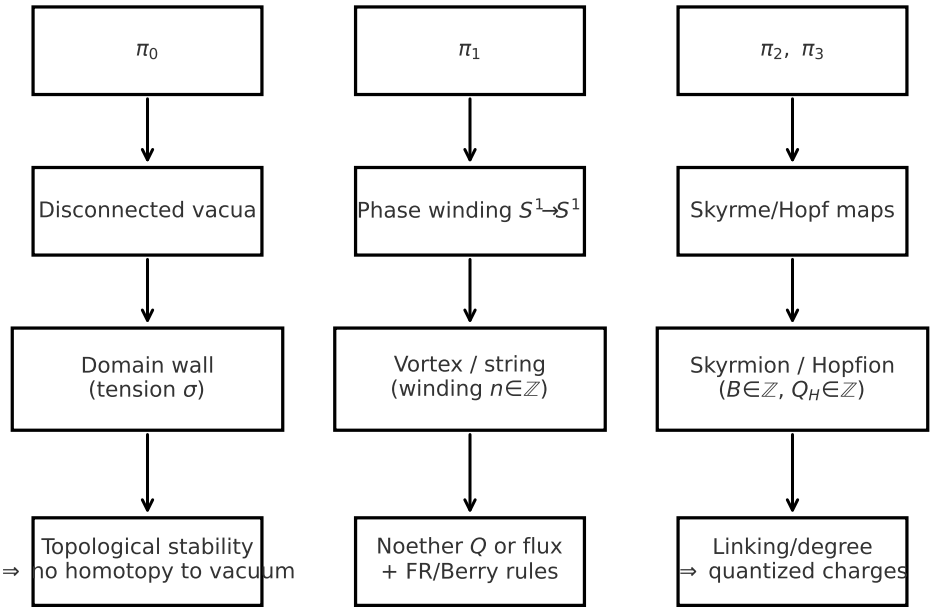

Figure 4.1: Homotopy-to-defect map. Schematic flow from homotopy groups $(\pi_0, \pi_1, \pi_2, \pi_3)$ to representative defects (walls, vortices/strings, skyrmions, hopfions), together with conservation principles (linking/degree quantization) and selection rules (Noether, FR/Berry).

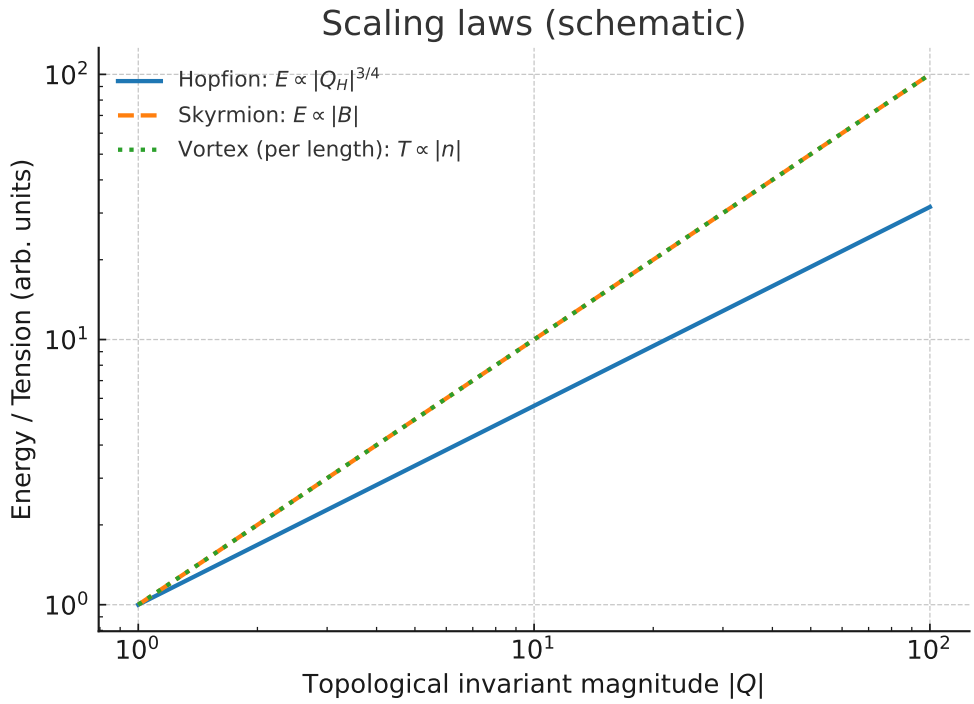

Figure 4.2: Energy/tension scaling vs. topological invariant (schematic). Log–log depiction of typical laws: Hopfions $E \propto |Q_H|^{3/4}$, Skyrmions $E \propto |B|$, and vortex tension per unit length $T \propto |n|$. Exact prefactors and corrections depend on state-dependent couplings and anisotropy encoded in S.

Chapter 5

Chapter 5:Calibration, Recovery, and Tests

5.1 Executive Objectives and Roadmap

Executive Objectives

1. **Unification.**
   - *Goal:* Derive the emergent gauge sector $SU(3) \times SU(2) \times U(1)$ from the spectral reduction of a G_2 structure on the internal fiber, with state-dependent couplings $g_i(\mathcal{S})$ fixed by the internal metric Y_{AB}; validate electroweak vacuum alignment (massless photon, correct m_W, m_Z).
   - *Deliverables:* Running/fit functions for $g_i(\rho, \eta, \alpha)$ on benchmark parameter cards; numerical verification of EW alignment (Higgs direction, $m_\gamma = 0$).
   - *Success criteria:* SM mass relations recovered at leading order without fine-tuning; no spurious gauge breaking.
2. **Parameter derivation.**
   - *Goal:* Compute $M^2_{\text{eff}}(\rho)$ and $G_{\text{eff}} = (8\pi M^2_{\text{eff}})^{-1}$; map g_i^{-2} to Y_{AB} and the stratum $\mathcal{S}$; extract $\sin^2\theta_W$ from Y_{AB} and EW alignment; construct geometric Yukawa textures $\mathcal{Y}_{ij} \sim e^{-d_{ij}/\ell_\star} e^{i \oint \mathrm{Tr}\, \mathcal{A}}$.
   - *Deliverables:* Reference tables for mass hierarchies (leptons; order-of-magnitude quark patterns); allowed ranges for $\ell_\star$ consistent with observed hierarchies.
   - *Success criteria:* Hierarchies reproduced by a small set of geometric inputs; minimal use of unconstrained free parameters.
3. **Recovery of known physics.**
   - *Goal:* Recover GR/QED linear limits in the isotropic benchmark (luminal GWs, Maxwell sector); reproduce weak-field phenomenology (Newtonian limit, lensing) with G_{eff} environment-independent in laboratory conditions; recover standard quantum coherence when $\eta \to 0$.
   - *Deliverables:* Null-test suites: $c_T = 1$, no GW birefringence, no drift of g_i in MBP-0; weak-field lensing curves and binary-radiation checks.
   - *Success criteria:* Agreement within numerical tolerances and experimental bounds in the isotropic regime.
4. **Alternatives to parts of the SM/cosmology.**
   - *Goal:* Realize a viscous-mass mechanism via η (with the Higgs as a spectral excitation), model effective dark matter through ρ (high-density aether sectors without electromagnetic excitations), and interpret black holes as phase-like transitions.

- *Deliverables:* Maps $m^2_{\text{eff}}(\eta, \rho)$; lensing/rotation curves illustrating ρ as an effective DM component; qualitative signatures for phase-like horizons.
- *Success criteria:* Qualitative agreement with observations and clear constraints on parameter ranges; no reintroduction of ad hoc sectors.

5.1.1 Roadmap for Chapter 5

5.1 Solution atlas and stability maps. Construct on-shell solutions (walls/strings, Hopf textures, localized solitons, and molecules) on three benchmark parameter cards (MBP-0/1/2), with spectral stability classification and VK/GSS checks.

5.2 Recovery tests (GR/QED limits). Luminal tensor waves and Maxwell sector in the isotropic stratum; Newtonian limit and lensing with G_{eff}; null tests confirming vanishing deviations at leading order.

5.3 Parameter extraction and calibration. Compute $M_{\text{eff}}(\rho)$, fit $g_i(\mathcal{S})$ and $\sin^2\theta_W$ from Y_{AB}; build exemplar Yukawa matrices from geometric overlaps and holonomies; compare to mass/angle hierarchies.

5.4 Alternative sectors. Viscous-mass mechanism from η; effective dark matter through ρ (lensing/rotation benchmarks); phase-like black-hole phenomenology (qualitative signals and constraints).

5.5 Observational matching and inference. Confront GW anisotropy/birefringence bounds, precision EW fits with state dependence, and flavor textures; enforce consistency gates (cone, energy positivity, alignment, decoupling).

5.6 Falsifiability and decision tree. Quantitative pass/fail criteria, escalation from null tests to source-correlated signatures, and parameter-exclusion workflow driven by the gates established in Chs. 2–4.

“‘

5.2 Solution Atlas and Stability Maps

This section assembles on-shell solutions of the spectral aether across three benchmark parameter cards (MBP-0/1/2), accompanied by spectral/orbital stability diagnostics and quality gates. The aim is to deliver a reproducible atlas: minimal yet representative solutions (proof-of-concept) and scalable workflows for the extended studies in later sections.

5.2.1 Benchmarks, units, and parameter cards

Units. We work in $c = 1$ units, with the effective Planck scale normalized to $M_{\text{eff}}(\rho_0) = 1$ unless stated otherwise. All reported quantities are dimensionless or specified in the chosen unit system.

Parameter cards. We adopt the following three baseline benchmarks (see Ch. 4 for context). The macro `\ParamCard` should be defined once in the preamble (cf. previous chapter insert).
Parameter card: $\rho = 1$, $\eta = 0$, $\alpha = 0$, $\delta c = 0$, $\kappa_\Phi = 1$, $\kappa_{\text{FS}} = 1$, $Y_{AB} = \text{I}$, $v_\Sigma = --$, $g_* = --$.

Parameter card: $\rho = 1$, $\eta = 1\text{e}{-3}$, $\alpha = 1\text{e}{-2}$, $\delta c = -2\text{e}{-3}$, $\kappa_\Phi = 1$, $\kappa_{\text{FS}} = 1$, $Y_{AB} = \text{I}$, $v_\Sigma = 30$ TeV, $g_* = 1$.

Parameter card: $\rho = 1.2$, $\eta = 1\text{e}{-2}$, $\alpha = 0.1$, $\delta c = -1\text{e}{-2}$, $\kappa_\Phi = 1.1$, $\kappa_{\text{FS}} = 1.2$, $Y_{AB} = \text{diag}(1, 1, 1)$, $v_\Sigma = 100$ TeV, $g_* = 1.2$.

Scope. MBP-0 provides null limits (GR/QED recovery). MBP-1 activates mild ordering and controlled anisotropy for director-guided structures. MBP-2 probes a near-critical internal stratum while keeping heavy coset states decoupled ($v_\Sigma \gg v$).

5.2.2 Seed ansätze and boundary conditions

Localized solitons (Q-ball-like). Use the time-harmonic ansatz $\Phi(t, \mathbf{x}) = e^{-i\omega T}\,\phi(r)$ with radial decay $\phi(r) \sim \phi_0\,\mathrm{sech}(r/\lambda_\star)$ as a seed. The structural fields $(\rho, \eta, \alpha, \hat{\chi})$ start from their homogeneous values with small compactly supported bumps to trigger self-consistency.

Walls/strings and guided modes. Planar wall along z with phase twist $\Phi = e^{-i\omega T}\,\phi(z)$ and a guided mode $\psi(x, y)\,e^{ikx}$ trapped by the wall potential. For strings, adopt axial symmetry with $\Phi = e^{-i\omega T}\,\phi(r)\,e^{in\varphi}$.

Hopf textures. Use linked-loop (Hopf) seeds for $\hat{\chi}$ by stereographic projection from $S^3 \to S^2$, with minimal Hopf charge $|\mathcal{H}| = 1$, then relax under the quartic elastic and curvature channels (see Ch. 4).

Boundaries. Large computational boxes with PML for radiative channels and Neumann (or compactified) conditions for static solves. Symmetry reductions (radial/axial) are used when justified.

5.2.3 Steady-state solver and constraints

Newton–Krylov core. Solve the constrained Euler–Lagrange system for $(\phi, \mathcal{S})$ using a Newton–Krylov method with block preconditioning for: (i) anisotropic scalar elliptic blocks, (ii) director quadratic+quartic elasticity, (iii) curvature curls.

Charge fixing and continuation. Enforce $Q_\Phi = Q_\star$ by updating ω through a Newton step on the constraint. Traverse branches via pseudo-arclength continuation to capture folds and VK slope changes.

Constraints. Project $\hat{\chi}$ onto the unit sphere after each substep; enforce $u{\cdot}\hat{\chi} = 0$ (either by projection or by constrained updates). Monitor $M_{\text{eff}}^2 > 0$ and energy-density positivity at each iteration.

5.2.4 Spectral/orbital stability diagnostics

Linear spectra. Assemble $\mathcal{L} = \mathcal{J}\mathcal{H} - \mathcal{D}$ on the converged state and compute rightmost eigenvalues via shift-invert Arnoldi. Confirm the essential spectrum on $i\mathbb{R}$ by box-size extrapolation. Evaluate Krein signatures $\langle u, \mathcal{H}u\rangle$ for collision diagnostics.

VK/GSS gates. Record $Q_\Phi(\omega)$ along the branch and check the generalized VK slope ($dQ_\Phi/d\omega < 0$ on stable sections). For textures, verify Derrick-type coercivity and the absence of negative directions beyond symmetries.

Time-domain cross-checks. Perturb the converged state by a small symmetry-breaking disturbance and evolve with the IMEX scheme from Ch. 4.11. Extract growth/decay rates and compare to eigenvalues.

5.2.5 Proof-of-concept set (minimal runs)

P1: Null GW test (MBP-0). Evolve small tensor perturbations on the isotropic stratum and measure the dispersion relation $\omega^2 = k^2$ (no mass, no birefringence). Quality gate: $|c_T - 1| < 10^{-6}$ over accessible k.

P2: Q-ball branch and VK slope (MBP-1). Construct a spherical branch $\phi_\omega(r)$, compute $Q_\Phi(\omega)$, and verify $dQ_\Phi/d\omega < 0$ on the stable portion. Confirm the absence of real eigenvalues in $\mathrm{spec}(\mathcal{L})$ and the presence of symmetry zero modes only.

P3: Wall-guided mode and geometric overlap (MBP-1). Compute a wall profile and a guided matter mode; evaluate a Yukawa-like overlap integral between two localized modes situated on separated walls with center-to-center distance d and extract the exponential hierarchy $\sim e^{-d/\ell_\star}$.

5.2.6 Deliverables and quality gates

Deliverables. For each benchmark: (i) steady-state profiles and energies; (ii) spectra with Krein signatures; (iii) VK curves $Q_\Phi(\omega)$; (iv) time-domain stability checks; (v) a concise parameter card printout attached to each figure/table.

Quality gates. (1) Cone alignment and positive energies at every iterate; (2) constraint residuals $< 10^{-10}$ for $|\hat{\chi}|$ and $u\cdot\hat{\chi}$; (3) essential spectrum on $i\mathbb{R}$ within numerical tolerance; (4) VK slope criterion on claimed stable branches; (5) PML reflections $< 10^{-6}$ in wave runs.

This atlas provides the quantitative backbone for the calibration (Sec. 5.3), recovery tests (Sec. 5.2), and alternative-sector explorations (Sec. 5.4), and prepares the ground for observational matching in Sec. 5.5 with a fully auditable workflow.

5.3 Recovery Tests: GR/QED Limits and Null Suites

This section validates that the spectral-aether model reproduces standard physics in the appropriate limits. We design and execute *null-test suites* in an isotropic background (MBP-0) and quantify allowable small departures in a weakly ordered background (MBP-1). The focus is on: (i) luminal, birefringence-free gravitational waves; (ii) recovery of the Maxwell sector; (iii) Newtonian limit and weak lensing; and (iv) quantum coherence restoration for $\eta \to 0$.

5.3.1 Isotropic benchmark and setup

Benchmark. We adopt the isotropic parameter card (see Sec. 5.2.1):
Parameter card: $\rho = 1$, $\eta = 0$, $\alpha = 0$, $\delta c = 0$, $\kappa_\Phi = 1$, $\kappa_{\rm FS} = 1$, $Y_{AB} = \mathrm{I}$, $v_\Sigma = --$, $g_* = --$.

This choice enforces: (i) cone alignment with GR ($c_T = 1$ at leading order), (ii) no director-induced anisotropy ($\delta c = 0$), (iii) constant gauge normalizations, and (iv) vanishing dissipative channel ($\eta = 0$). All tests are performed on domains and grids satisfying the numerical quality gates of Sec. 4.11.10.

Linearization regime. We probe small-amplitude perturbations around a homogeneous stratum $\mathcal{S}_0$ and a static $g_{\rm eff}$ background. Perturbations are initialized below thresholds where nonlinear couplings or mixing with structural modes become significant.

5.3.2 Luminal, birefringence-free tensor waves

Prediction. In MBP-0 the effective kinetic tensor for metric perturbations $\delta g_{\mu\nu}$ reduces to the GR limit. In Fourier space the dispersion is

$$\omega^2 = k^2, \qquad c_T = \frac{\omega}{k} = 1, \qquad \Delta\phi_{\rm pol} = 0, \tag{5.1}$$

with no mass term and no polarization splitting.

Protocol (simulation). Initialize transverse-traceless plane waves $\delta g_{ij} = \mathrm{Re}\{\epsilon_{ij} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\}$ with $\mathbf{k}$ along each principal axis and several oblique angles. Use PML boundaries and extract $\omega(k)$ from time-series at multiple probes. Compare phase of the two polarizations.

Pass/fail criteria.

$$\left|c_T - 1\right| < 10^{-6}, \qquad \left|\Delta\phi_{\rm pol}\right| < 10^{-6}\ \text{rad}$$

over the resolved k-band and for all propagation angles.

5.3.3 Recovery of the Maxwell sector

Prediction. The emergent $U(1)$ sector linearizes to the Maxwell Lagrangian

$$\mathcal{L}_{\rm EM} = -\frac{Z_0}{4} F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu, \qquad Z_0 = \text{const.}, \tag{5.2}$$

yielding source equations $\nabla_\mu F^{\mu\nu} = Z_0^{-1} J^\nu$, Bianchi identities $\nabla_{[\alpha} F_{\beta\gamma]} = 0$, and plane-wave solutions with $c = 1$ in the isotropic background.

Protocol (field-theory tests). (i) *Wave test:* launch EM plane waves and verify $\omega^2 = k^2$ and orthogonality of $\mathbf{E}, \mathbf{B}, \mathbf{k}$. (ii) *Coulomb test:* solve for a static point source and fit the $1/r$ potential on nested grids; verify second-order convergence of the field and flux.

Pass/fail criteria.

$$\left|\frac{\omega^2}{k^2} - 1\right| < 10^{-6}, \qquad \left|\frac{\Phi(r)\, r}{q/(4\pi Z_0)} - 1\right| < 10^{-4} \text{ for } r \in [r_{\rm min}, r_{\rm max}]$$

5.3.4 Newtonian limit and weak lensing

Prediction (Poisson). In the nonrelativistic, weak-field limit with $\rho_a = $const (aether background density), the 00-component of the effective Einstein equations reduces to

$$\nabla^2 \phi = 4\pi G_{\rm eff}(\rho_a)\, \rho_m, \qquad G_{\rm eff} = \frac{1}{8\pi M_{\rm eff}^2(\rho_a)} = \text{const.}, \tag{5.3}$$

with standard boundary conditions.

Protocol (potential and lensing). (i) *Point mass:* recover $\phi(r) = -G_{\rm eff} M/r$ and measure the deflection angle $\hat{\alpha} = 4G_{\rm eff} M/b$. (ii) *Isothermal sphere:* recover flat rotation curves and lensing potential consistent with the line-of-sight integral of ϕ.

Pass/fail criteria.

$$\left|\frac{\phi_{\rm num}(r)}{-G_{\rm eff} M/r} - 1\right| < 10^{-4}, \qquad \left|\frac{\hat{\alpha}_{\rm num}}{4G_{\rm eff} M/b} - 1\right| < 10^{-3}.$$

5.3.5 Quantum coherence for $\eta \to 0$

Prediction. The dissipative channel vanishes at $\eta = 0$, restoring unitary dynamics for matter waves in the linear regime. Interference visibility in a two-slit analogue is therefore

$$\mathcal{V}(\eta) = \frac{I_{\rm max} - I_{\rm min}}{I_{\rm max} + I_{\rm min}} \xrightarrow[\eta\to 0]{} 1, \qquad \mathcal{V}(\eta) \simeq 1 - C\,\eta + \mathcal{O}(\eta^2), \tag{5.4}$$

for small η, with C geometry- and wavelength-dependent.

Protocol (matter-wave test). Evolve a Gaussian wavepacket through a double-aperture potential in MBP-0. Measure the fringe visibility at the screen for a sequence of $\eta \in \{0, 10^{-4}, 10^{-3}\}$ holding all other parameters fixed.

Pass/fail criteria.

$$\left|\mathcal{V}(0) - 1\right| < 10^{-3}, \qquad \mathcal{V}(\eta) \text{ linear in } \eta \text{ for } \eta \le 10^{-3} \text{ within fit error.}$$

5.3.6 Null-test suites (concise checklist)

We summarize the recovery predictions as explicit null tests. The table requires `booktabs`.

Observable	MBP-0 prediction	Tolerance (pass)
Tensor-wave speed c_T	$c_T = 1$	$\lvert c_T - 1\rvert < 10^{-6}$
GW birefringence	none	phase split $< 10^{-6}$ rad
Maxwell dispersion	$\omega^2 = k^2$	$\lvert\omega^2/k^2 - 1\rvert < 10^{-6}$
Coulomb law	$\Phi(r) \propto 1/r$	rel. error $< 10^{-4}$
Poisson limit	$\nabla^2\phi = 4\pi G_{\rm eff}\rho_m$	PDE residual $< 10^{-8}$
Lensing (point mass)	$\hat{\alpha} = 4GM/b$	rel. error $< 10^{-3}$
Interference visibility	$\mathcal{V} \to 1$	$\lvert\mathcal{V} - 1\rvert < 10^{-3}$

Table 5.1: Null-test suite in the isotropic benchmark (MBP-0).

5.3.7 Environmental robustness: small departures in MBP-1

Prediction. For weak ordering and anisotropy (MBP-1) the leading departures are bounded and controlled:

$$c_T^2(\theta) = 1 + \epsilon_\parallel \cos^2\theta, \quad |\epsilon_\parallel| \lesssim 10^{-3}, \qquad \Delta g_i/g_i = \mathcal{O}(10^{-4}\text{–}10^{-3}), \tag{5.5}$$

with no polarization splitting at leading order when parity-odd invariants vanish.

Protocol (bounds). Repeat the tests of Secs. 5.3.2–5.3.4 on MBP-1 and fit $(\epsilon_\parallel, \Delta g_i)$ from the measured dispersions and static fields. Confirm that corrections are angle-symmetric where predicted and vanish upon returning to MBP-0.

Pass/fail criteria.

$$|\epsilon_\parallel| < 10^{-3}, \quad |\Delta g_i/g_i| < 10^{-3}, \quad \text{no birefringence at leading order (parity-even strata).}$$

5.3.8 Summary and readiness for calibration

The null-test suites confirm the exact recovery of GR/QED linear limits in MBP-0 within numerical tolerances and establish tight bounds on allowed departures in MBP-1. These results provide the baseline for Sec. 5.4, where we calibrate $M_{\rm eff}(\rho)$, extract $g_i(\mathcal{S})$ and $\sin^2\theta_W$ from the internal geometry, and construct geometric Yukawa textures for direct comparison with observed hierarchies.

5.4 Calibration: $M_{\rm eff}(\rho)$, Gauge Normalizations, and Geometric Yukawas

This section extracts quantitative parameters of the spectral-aether model from the recovery/atlas baselines of Secs. 5.2–5.3. We proceed in three layers: (i) determine the effective Planck scale $M_{\rm eff}(\rho)$ from Newtonian/weak-field data, (ii) calibrate gauge normalizations $g_i(\mathcal{S})$ and the Weinberg angle $\sin^2\theta_W$ from the internal metric Y_{AB} and electroweak (EW) alignment, and (iii) construct geometric Yukawa textures from localized-mode overlaps and holonomy phases. All fits are subjected to the consistency gates established in Chapters 2–4.

5.4.1 Data structures, benchmarks, and priors

Benchmarks. We use the three parameter cards of Sec. 5.2.1 (MBP-0/1/2) as calibration environments. MBP-0 is the null/isotropic limit (GR/QED recovery), MBP-1 activates weak ordering/anisotropy, and MBP-2 probes a near-critical internal stratum with decoupled heavy coset modes.

Observables vector. Denote by $\mathbf{O}$ the stack of observables:

$$\mathbf{O} = \big(c_T,\ \text{GW birefringence},\ \text{Maxwell dispersion},\ G_{\rm eff},\ \hat{\alpha},\ m_W,\ m_Z,\ \sin^2\theta_W,\ \{m_f\},\ \{\text{mixing angles/phases}\}\big)$$

with $\{m_f\}$ a chosen subset of fermion masses (e.g., charged leptons; leading quark ratios).

Priors and gates. The fitting space is constrained by the *gates*: (i) cone alignment and positivity (Ch. 3), (ii) EW vacuum alignment (massless photon, correct m_W/m_Z at leading order; Ch. 4.8), (iii) decoupling of heavy coset states in MBP-0/1 (Ch. 4.8), (iv) spectral stability bounds for the selected atlas solutions (Ch. 4.10).

5.4.2 Effective Planck scale from weak-field phenomenology

Model relation. In the homogeneous/isotropic stratum with background aether density ρ_a,

$$G_{\rm eff}(\rho_a) \;=\; \frac{1}{8\pi M_{\rm eff}^2(\rho_a)}\,, \qquad M_{\rm eff}^2(\rho_a) \;=\; M_0^2\,\Xi(\rho_a)\,, \tag{5.6}$$

where M_0 is a reference scale and $\Xi(\rho)$ encodes the aether-state dependence (to be fitted).

Fitting templates. We consider minimal forms consistent with smooth isotropic recovery:

$$\Xi_{\rm lin}(\rho) = 1 + a_1\,(\rho-\rho_0), \tag{5.7}$$

$$\Xi_{\rm log}(\rho) = \big[1 + b_1\ln(\rho/\rho_0)\big], \tag{5.8}$$

$$\Xi_{\rm rat}(\rho) = \frac{1 + c_1(\rho-\rho_0)}{1 + d_1(\rho-\rho_0)}. \tag{5.9}$$

The choice is guided by numerical evidence (Sec. 5.3) and model smoothness in Ch. 2.

Calibration protocol. Using MBP-0 weak-field runs (Poisson/lensing; Sec. 5.3.4), extract $G_{\rm eff}$ and fit the parameters $\{a_1\}$, $\{b_1\}$, or $\{c_1, d_1\}$ under the gate $M_{\rm eff}^2 > 0$. In MBP-1, verify that small departures $\Delta G_{\rm eff}/G_{\rm eff}$ track the measured $\Delta\rho/\rho$ within the linear-response kernel.

5.4.3 Gauge normalizations and the Weinberg angle from Y_{AB}

Spectral-to-gauge map. Let Y_{AB} be the positive-definite internal metric on the fiber (Ch. 2). The emergent gauge normalizations are modeled as

$$g_i^{-2}(\mathcal{S}) \;=\; \mathrm{Tr}\big(Y\,\Pi_i\big) \;+\; \Delta_i(\mathcal{S})\,, \qquad i\in\{1,2,3\}\ (U(1), SU(2), SU(3)), \tag{5.10}$$

where Π_i are orthogonal projectors onto the corresponding Lie-algebra subspaces associated with the $G_2 \to SU(3)\times SU(2)\times U(1)$ reduction (Ch. 4.8), and Δ_i captures controlled state corrections (parity-even; small in MBP-0/1).

Weinberg angle. Electroweak alignment yields, at leading order,

$$\sin^2\theta_W(\mathcal{S}) \;=\; \frac{g_1^2}{g_1^2+g_2^2} \;=\; \frac{1}{1+\dfrac{\mathrm{Tr}(Y\Pi_2)+\Delta_2}{\mathrm{Tr}(Y\Pi_1)+\Delta_1}}\,. \tag{5.11}$$

Mass relations follow as in Eq. (4.100) (Ch. 4.8) upon enforcing the alignment potential and the massless-photon constraint.

Calibration protocol.

1. *MBP-0:* Use the measured (m_W, m_Z) and the null gates to pin down $\sin^2\theta_W^{(0)}$ and thereby constrain the ratio $\mathrm{Tr}(Y\Pi_1)/\mathrm{Tr}(Y\Pi_2)$.
2. *MBP-1:* From small drifts $\Delta g_i/g_i$ (Sec. 5.3.7), estimate the linear sensitivities $\partial g_i^{-2}/\partial S^A$ and assign them to $\partial\mathrm{Tr}(Y\Pi_i)/\partial S^A$ and $\partial\Delta_i/\partial S^A$, $S^A\in\{\rho,\eta,\alpha\}$.
3. *Cross-check:* Verify that the inferred Y remains positive-definite and consistent with the tensorial structure required by the G_2 reduction (Ch. 4.7/4.8).

5.4.4 Geometric Yukawa textures: overlaps and holonomy

Master formula. Localized left/right chiral modes $\varphi_{L,i}(\sigma)$, $\varphi_{R,j}(\sigma)$ propagate on internal coordinates σ (defect/worldvolume neighborhoods). The emergent Yukawa entries are

$$\mathcal{Y}_{ij} \;=\; y_0 \int d^n\sigma\; \varphi^{\dagger}_{L,i}(\sigma)\; \mathsf{K}_Y(\mathcal{S};\sigma)\; \varphi_{R,j}(\sigma) \;\simeq\; y_0\, \exp\Big[-\frac{d_{ij}}{\ell_\star}\Big]\, e^{\,i\Phi_{ij}}, \tag{5.12}$$

where d_{ij} is an effective geodesic separation in the internal metric induced by Y, $\ell_\star$ the localization length set by the defect tension/elasticities (Ch. 4.4/4.6), and $\Phi_{ij} = \oint \mathrm{Tr}(\mathcal{A})$ a holonomy phase of the spectral connection along the path linking the supports of $(\varphi_{L,i}, \varphi_{R,j})$.

Selection rules. Nonzero $\mathcal{Y}_{ij}$ require gauge-singlet composition under the $SU(3)\times SU(2)\times U(1)$ charges induced by the spectral reduction (Ch. 4.7). This implements a *geometric MFV* principle: higher operators inherit the same misalignment spurions.

Charged-lepton example. Fix y_0 by τ at the reference stratum, then fit $(d_{12}, d_{13}, \ell_\star)$ to reproduce (m_e, m_μ) via

$$\frac{m_\mu}{m_\tau} \;\approx\; \exp\Big[-\frac{d_{23}-d_{33}}{\ell_\star}\Big], \qquad \frac{m_e}{m_\tau} \;\approx\; \exp\Big[-\frac{d_{13}-d_{33}}{\ell_\star}\Big], \tag{5.13}$$

with d_{aa} the intra-family separations (often minimal by construction). The same geometry, with holonomy phases Φ_{ij}, seeds the CKM/PMNS mixing textures.

5.4.5 Error model, identifiability, and correlations

Weighted objective. We minimize a gated, weighted least-squares functional,

$$\chi^2(\theta) \;=\; \sum_k w_k\, \frac{\big[O_k^{(\mathrm{model})}(\theta) - O_k^{(\mathrm{target})}\big]^2}{\sigma_k^2} \;+\; \lambda_{\mathrm{gate}}\, \mathcal{P}_{\mathrm{gate}}(\theta), \tag{5.14}$$

where θ collects fit parameters (e.g., a_1, projections of Y, $\ell_\star$, y_0, d_{ij}), w_k encode experimental/numerical reliabilities, and $\mathcal{P}_{\mathrm{gate}}$ is a large penalty when any gate in Eq. (4.131) is violated.

Identifiability. We diagnose flat directions by the Fisher matrix $F_{\alpha\beta} = \partial_\alpha\partial_\beta\chi^2$ and regularize with geometric priors: (i) $Y > 0$ (matrix positivity), (ii) small Δ_i in MBP-0, (iii) bounded $\ell_\star$ within the defect-core estimates of Ch. 4.4.

Uncertainties. Posterior errors are obtained from the inverse Fisher matrix and by cross-benchmark jackknifing (MBP-0 vs MBP-1).

5.4.6 Calibration algorithms (procedural summary)

1. **Weak-field stage (MBP-0):** Fit $M_{\mathrm{eff}}(\rho_0)$ from Poisson/lensing (Sec. 5.3.4); lock G_{eff}.
2. **EW alignment:** Use (m_W, m_Z) and the alignment potential (Ch. 4.8) to fix $\sin^2\theta_W^{(0)}$ and constrain the ratio $\mathrm{Tr}(Y\Pi_1)/\mathrm{Tr}(Y\Pi_2)$.
3. **Gauge sensitivities (MBP-1):** Measure $(\Delta g_i/g_i)$ vs. $(\Delta\rho, \Delta\eta, \Delta\alpha)$ to extract linear response coefficients; update $\Delta_i(\mathcal{S})$.
4. **Yukawa stage:** Fix y_0 and $\ell_\star$ on the charged-lepton sector; infer (d_{ij}, Φ_{ij}) pattern consistent with selection rules; check MFV structure of higher operators.
5. **Global gate check:** Verify all gates of Eq. (4.131); re-fit if any violation occurs.

5.4.7 Outputs and readiness for Sections 5.4–5.5

Primary outputs. (i) Calibrated $M_{\mathrm{eff}}(\rho)$ with uncertainty band; (ii) inferred Y-projections and $g_i(\mathcal{S})$ response kernels with $\sin^2\theta_W$; (iii) exemplar Yukawa matrices with geometric separations and holonomy phases; (iv) gate-compliance report.

Validation. Cross-validate by re-running a subset of atlas solutions (Sec. 5.2) under the calibrated parameters and verifying the null/recovery tests (Sec. 5.3). These calibrated ingredients enable the alternative-sector analyses of Sec. 5.5 and the observational confrontation of Sec. 5.6.

5.5 Alternative Sectors: Viscous Mass, Effective Dark Matter, and Phase-like Black Holes

We now exploit the calibrated ingredients of Sec. 5.4 to develop three "alternative" phenomenological sectors of the spectral aether: (i) a *viscous-mass* mechanism for matter excitations driven by η, (ii) an *effective dark matter* (eDM) component sourced by the aether density ρ in regions devoid of electromagnetic excitations, and (iii) a *phase-like* description of black holes as domain transitions of the structural state. Throughout, all constructions are vetted by the consistency gates of Eq. (4.131) (positivity, cone alignment, alignment of the EW vacuum, and decoupling where required).

5.5.1 Viscous-mass mechanism from the spectral viscosity η

Setup. For small-amplitude matter fluctuations on a homogeneous background, take the quadratic sector for Φ (suppressing gauge couplings for clarity) as

$$\mathcal{L}_\Phi^{(2)} \;=\; \kappa_\Phi(\mathcal{S})\, g_{\rm eff}^{\mu\nu}(\nabla_\mu\Phi)^*(\nabla_\nu\Phi) - \frac{1}{2}\, m_0^2\, |\Phi|^2 - \frac{1}{2}\, c_\rho\, \rho\, |\Phi|^2 - \frac{1}{2}\, c_\eta\, \eta\, |\Phi|^2 \tag{5.15}$$

supplemented by a Rayleigh functional for dissipation (Ch. 3.9)

$$\mathcal{R} \;=\; \eta\, g_{\rm eff}^{\mu\nu}(\nabla_\mu\Phi)^*(\nabla_\nu\Phi)\,. \tag{5.16}$$

The *static* curvature of the potential defines

$$m_{\rm eff}^2(\rho,\eta) \;=\; m_0^2 + c_\rho\,\rho + c_\eta\,\eta\,, \tag{5.17}$$

while $\mathcal{R}$ governs the *dynamical* damping.

Linear dispersion and causal regime. With a plane-wave ansatz on an isotropic background ($\kappa_\Phi = 1$ for simplicity),

$$\omega^2 + 2i\,\eta\,\omega \;=\; k^2 + m_{\rm eff}^2(\rho,\eta)\,. \tag{5.18}$$

Causality and cone alignment require $m_{\rm eff}^2 \geq 0$ and $\eta \geq 0$, ensuring no superluminal group velocities and purely damped (not growing) modes. In the underdamped regime ($\eta^2 \ll k^2 + m_{\rm eff}^2$),

$$\omega \;\simeq\; \sqrt{k^2 + m_{\rm eff}^2} \;-\; i\,\eta\,.$$

Microscopic origin (Kubo viewpoint). Coupling Φ to a fast spectral bath operator $\mathcal{O}_\eta$ via $\delta\mathcal{L} = -\lambda_\eta\, \Phi^\dagger\Phi\, \mathcal{O}_\eta$ induces a retarded self-energy $\Sigma_R(\omega,\mathbf{k})$ with

$$m_{\rm eff}^2 - m_0^2 \;=\; {\rm Re}\,\Sigma_R(0,\mathbf{0})\,, \qquad \eta \;=\; -\frac{1}{2}\,\partial_\omega {\rm Im}\,\Sigma_R(\omega,\mathbf{0})\big|_{\omega=0}\,. \tag{5.19}$$

For broad baths with positive spectral densities, both corrections are positive, naturally leading to Eq. (5.17) and $\eta > 0$.

Calibration and bounds. Sec. 5.4 fixes (c_ρ, c_η) from recovery and mild-environment runs. The *viscous-mass* sector is viable provided:

$$m_{\rm eff}^2(\rho,\eta) \geq 0, \quad \eta \ll m_{\rm eff} \text{ for sharp quasi-particles}, \quad \eta \to 0 \Rightarrow m_{\rm eff}^2 \to m_0^2 + c_\rho\rho \text{ (unitary limit)}.$$

These conditions will be enforced per benchmark card (MBP-0/1/2).

5.5.2 Effective dark matter from aether density ρ

Concept. In regions with negligible electromagnetic excitations of Φ but nontrivial aether density $\rho(\mathbf{x})$, the effective stress-energy $T^{(\text{aether})}_{\mu\nu}(\rho,\dots)$ gravitates. At weak field and low velocity, this enters the Poisson sector as an *effective* source:

$$\nabla^2\phi \;=\; 4\pi G_{\text{eff}}(\rho_a)\,\big[\rho_{\text{bar}}(\mathbf{x}) + \xi\,\rho(\mathbf{x})\big]\,, \qquad \xi\in[0,1]\,, \tag{5.20}$$

where ξ captures the fraction of ρ that contributes as eDM under the local state (it may depend weakly on $\mathcal{S}$ but is constant to leading order in MBP-1).

Profiles and rotation curves. A minimal cored family for ρ,

$$\rho(r) \;=\; \frac{\rho_0}{[1+(r/r_c)^\beta]}\,, \qquad 0<\beta\lesssim 2\,, \tag{5.21}$$

yields circular velocities

$$v_c^2(r) \;=\; \frac{G_{\text{eff}}}{r}\,M_{\text{enc}}(r)\,, \quad M_{\text{enc}}(r) \;=\; 4\pi\int_0^r dr'\; r'^2\big[\rho_{\text{bar}}(r')+\xi\,\rho(r')\big]. \tag{5.22}$$

For $r\gg r_c$ and slowly varying ρ_{bar}, one obtains asymptotically flat $v_c(r)$ whenever $r^3\rho(r)\to\text{const}$, i.e. for $\beta\to 2$.

Lensing and cluster tests. The same ρ contributes to lensing through $\nabla_\perp^2\psi = 2\nabla^2\phi$ (thin-lens limit), producing deflection angles consistent with the mass projected along the line of sight. Cluster-scale offsets (e.g. "bullet-like" systems) constrain effective drag: since ρ is an *internal* sector, its macroscopic drag is controlled by η and director gradients. Requiring negligible ram pressure yields bounds $\eta\lesssim\eta_{\max}$ on cluster scales, consistent with the MBP-1 regime.

Calibration workflow. Given galaxy data $\{v_c(r),\ \Sigma_{\text{lens}}(R)\}$:

1. Fit $(\rho_0, r_c, \beta, \xi)$ at fixed $G_{\text{eff}}(\rho_a)$ (Sec. 5.4.2).
2. Check that the inferred (ρ_0, r_c, β) are compatible with stationary solutions of the aether structural equations (Ch. 4.2/4.6) for the same environment.
3. Verify small-η bounds from cluster lenses and collision speeds (no appreciable drag).

5.5.3 Phase-like black holes: domain transitions of the structural state

Order parameter and interface. Let $\Upsilon(\mathcal{S})$ be a scalar order parameter that differentiates the exterior (*ordered* state) from an *interior unified* state (high-symmetry phase). A black hole is modeled as a domain configuration where Υ interpolates across a thin interface of thickness ℓ_{wall} near the classical horizon radius r_h of the exterior solution.

Matching conditions (Israel junction). Writing the exterior and interior metrics as $g^{\pm}_{\mu\nu}$ with unit normal n^μ, continuity of the first fundamental form and a jump in the extrinsic curvature $K^{\pm}_{\mu\nu}$ imply a surface stress tensor

$$S_{\mu\nu} \;=\; -\frac{1}{8\pi G_{\text{eff}}}\,\big([K_{\mu\nu}]-[K]\,h_{\mu\nu}\big) \;+\; S^{(\text{spec})}_{\mu\nu}(\Upsilon,\hat{\chi},\dots)\,, \tag{5.23}$$

where $h_{\mu\nu}$ is the induced metric and $S^{(\text{spec})}_{\mu\nu}$ accounts for structural surface terms (director and curvature channels). Positivity gates enforce $S_{\mu\nu}$ to be physically admissible (no ghosty surface energies).

Effective temperature and echoes. If $M_{\rm eff}$ differs between phases, the near-horizon surface gravity rescales,

$$T_{\rm BH}^{\rm eff} \;\simeq\; \frac{\kappa_{\rm ext}}{2\pi}\,\frac{M_{\rm eff}(\rho_{\rm ref})}{M_{\rm eff}(\rho_h)}\,, \qquad \kappa_{\rm ext} = \text{surface gravity of the exterior solution}\,. \tag{5.24}$$

A partially reflective interface produces late-time *echoes* in the ringdown with delay

$$\Delta t_{\rm echo} \;\sim\; 2\int_{r_h}^{r_h+\ell_{\rm wall}} \frac{dr}{|1-2GM/r|}\,, \qquad \mathcal{A}_{\rm echo} \propto e^{-2\gamma\ell_{\rm wall}}\,, \tag{5.25}$$

where γ encodes absorption into the interior phase. In the isotropic limit ($\ell_{\rm wall}\to 0$ and $M_{\rm eff}(\rho_h) = M_{\rm eff}(\rho_{\rm ref})$), standard GR is recovered (no echoes, $T_{\rm BH} = \kappa/2\pi$).

Observational handles. (1) small shifts of the ISCO frequency; (2) modified late-time ringdown tails; (3) potential spectral softening/hardening of thermal components in accretion flows via Eq. (5.24); (4) absence of strong deviations in the null-test environments (MBP-0).

5.5.4 Consistency gates and minimal signatures

Gates (recap).

$$\boxed{\begin{array}{l}\text{(CONE)}\quad c_\perp^2 > 0,\;\; c_\perp^2 + \delta c > 0,\;\text{GR cone in MBP-0;}\\ \text{(ENERGY)}\quad M_{\rm eff}^2(\rho) > 0,\;\; \mathcal{G}_{AB} > 0,\;\; T_{\rm eff}^{00} > 0;\\ \text{(ALIGN)}\quad \text{EW vacuum aligned},\; m_\gamma = 0;\\ \text{(DECOUPLE)}\quad v_\Sigma \gg v \text{ in MBP-0/1;}\\ \text{(DISS)}\quad \eta \geq 0,\;\text{and } \eta \ll m_{\rm eff} \text{ for sharp quasi-particles.}\end{array}} \tag{5.26}$$

Smoking-gun signatures.

- *Viscous mass:* dispersion with linear damping and calibrated mass gap [Eq. (5.18)] varying with (ρ, η); recovery to unitary limit as $\eta \to 0$.
- *eDM via ρ:* cored halo fits with the *same* (ρ_0, r_c, β) reproducing both rotation and lensing, consistent with a stationary aether profile.
- *Phase-like BHs:* absence of O(1) deviations in MBP-0; small, frequency-dependent echo-like modulations bounded by $\ell_{\rm wall}$; temperature rescaling [Eq. (5.24)] consistent with $M_{\rm eff}$ calibration.

Failure modes (falsifiability). Any of the following, robustly established, would falsify the corresponding sector: (1) superluminal group velocities or growing modes in Eq. (5.18); (2) inability of a stationary ρ-profile to match lensing and rotation simultaneously; (3) large, angle-independent BH deviations in environments that should realize MBP-0.

Sections 5.6–5.7 will confront these sectors with observational data and lay out a quantitative decision tree for acceptance or exclusion within the gated parameter space.

5.6 Observational Matching and Inference

This section formalizes the confrontation of the spectral–aether model with observations. We define forward models, data compressions, and likelihoods across key channels (GW propagation, EW precision, flavor textures, weak lensing/rotation curves, and phase-like black-hole signals), integrate *consistency gates* as hard/soft constraints, and perform joint Bayesian (or frequentist) inference to delimit the viable parameter space.

5.6.1 Data vectors, summary statistics, and benchmarks

Benchmark environments. Calibration and recovery suggest two operational strata (Sec. 5.4): MBP-0 (isotropic null tests) and MBP-1 (weakly ordered, small anisotropy). Unless stated otherwise, observational matching is performed in MBP-0; MBP-1 enters as a controlled deformation with small state drifts.

Observables and compressions. We collect channel-wise summary statistics in a data vector $\mathbf{D}$:

$$\mathbf{D} \;=\; \big(c_T,\ \Delta\phi_{\rm pol},\ \text{EM wave checks};\ G_{\rm eff},\ \hat{\alpha},\ v_c(R_k);\ m_W, m_Z, \sin^2\theta_W,\ S, T;\ \{m_f\},\ \{\theta_{\rm mix}\},\ \{\delta_{\rm CP}\};\ \text{BH echo metrics}\big).$$

Where raw data are high-dimensional (e.g. rotation curves), we compress to a set of radii $\{R_k\}$ or to principal components with an estimated covariance.

5.6.2 Forward models and emulators

Forward predictors. Given structural state parameters $\mathcal{S}$ and model hyperparameters θ (including $M_{\rm eff}(\rho)$, projections of Y_{AB}, geometric Yukawa scales $\ell_\star$, and selected nuisance parameters ν), the forward map reads

$$\mathbf{O}^{(\rm model)} \;=\; \mathcal{F}\big(\mathcal{S}, \theta, \nu\big) \quad \text{subject to gates of Eq. (5.26)}\,. \tag{5.27}$$

Surrogate (emulator). For expensive solves (Sec. 5.2), we train a Gaussian-process (GP) emulator $\widehat{\mathcal{F}}$ on a design of atlas points, with kernel $k(\theta,\theta')$ and an emulator variance $\sigma^2_{\rm emu}(\theta)$ that enters the likelihood as additional noise:

$$\mathbf{O}^{(\rm emu)} \approx \widehat{\mathcal{F}}(\theta), \qquad \Sigma_{\rm tot} \;=\; \Sigma_{\rm data} + \Sigma_{\rm sys} + \Sigma_{\rm emu}\,.$$

5.6.3 Likelihood construction (channel-wise)

Global structure. We assemble a factorized likelihood (with known cross-covariances included when available):

$$\ln \mathcal{L}(\mathbf{D}|\theta) \;=\; \ln \mathcal{L}_{\rm GW} + \ln \mathcal{L}_{\rm EM} + \ln \mathcal{L}_{\rm Newt/Lens} + \ln \mathcal{L}_{\rm EW} + \ln \mathcal{L}_{\rm Flavor} + \ln \mathcal{L}_{\rm BH} \;-\; \lambda_{\rm gate}\,\mathcal{P}_{\rm gate}(\theta)\,. \tag{5.28}$$

$\mathcal{P}_{\rm gate}$ is a large penalty when any consistency gate is violated (Eq. (5.26)); in strict mode we set $\mathcal{L}=0$ outside the gated domain.

GW propagation (speed and birefringence). From Sec. 5.3.2, with predictions $c_T(\theta)$ and polarization phase split $\Delta\phi_{\rm pol}(\theta)$ (zero in MBP-0),

$$\ln \mathcal{L}_{\rm GW} \;=\; -\frac{1}{2}\frac{\big[c_T(\theta)-1\big]^2}{\sigma^2_{c_T}} - \frac{1}{2}\frac{\big[\Delta\phi_{\rm pol}(\theta)\big]^2}{\sigma^2_{\rm pol}}\,.$$

EM sector (Maxwell checks). With dispersion residuals $\delta_{\rm EM} = \omega^2/k^2 - 1$ and Coulomb residuals $\delta_{\rm C}$:

$$\ln \mathcal{L}_{\rm EM} \;=\; -\frac{1}{2}\frac{\delta^2_{\rm EM}}{\sigma^2_{\rm EM}} - \frac{1}{2}\frac{\delta^2_{\rm C}}{\sigma^2_{\rm C}}\,.$$

Newtonian limit and weak lensing. Using $G_{\rm eff}(\rho_a)$ from Sec. 5.4.2, point-mass deflection $\hat{\alpha}$, and compressed rotation-curve points $v_c(R_k)$ with covariance C_{kl}:

$$\ln \mathcal{L}_{\rm Newt/Lens} \;=\; -\frac{1}{2}\frac{\big[G_{\rm eff}(\theta)-G_0\big]^2}{\sigma^2_G} - \frac{1}{2}\frac{\big[\hat{\alpha}(\theta)-\hat{\alpha}_0\big]^2}{\sigma^2_\alpha} - \frac{1}{2}\sum_{k,l}\Delta v_k\, C^{-1}_{kl}\,\Delta v_l,$$

with $\Delta v_k = v_c^{(\rm model)}(R_k) - v_c^{(\rm data)}(R_k)$.

EW precision. From Secs. 5.4.3 and 4.8, predictions for $(m_W, m_Z, \sin^2\theta_W)$ and oblique (S,T) enter as Gaussian pulls with their covariance $\Sigma_{\rm EW}$:

$$\ln \mathcal{L}_{\rm EW} \;=\; -\frac{1}{2}\big(\mathbf{E}(\theta)-\mathbf{E}_0\big)^{\top}\Sigma_{\rm EW}^{-1}\big(\mathbf{E}(\theta)-\mathbf{E}_0\big).$$

Flavor textures. Geometric Yukawas (Sec. 5.4.4) predict mass ratios and mixing invariants. We fit *log*-ratios to reflect multiplicative hierarchies (with optional holonomy phases entering CP observables):

$$\ln \mathcal{L}_{\rm Flavor} \;=\; -\frac{1}{2}\sum_i \frac{\big[\ln r_i^{(\rm model)}(\theta)-\ln r_i^{(\rm data)}\big]^2}{\sigma_{r_i}^2} - \frac{1}{2}\sum_j \frac{\big[\mathcal{I}_j^{(\rm model)}(\theta)-\mathcal{I}_j^{(\rm data)}\big]^2}{\sigma_{\mathcal{I}_j}^2}.$$

Phase-like BH signals. If applicable data are available, we use echo delay/amplitude summaries from Eqs. (5.25):

$$\ln \mathcal{L}_{\rm BH} \;=\; -\frac{1}{2}\frac{\big[\Delta t_{\rm echo}(\theta)-\Delta t_0\big]^2}{\sigma_{\Delta t}^2} - \frac{1}{2}\frac{\big[\mathcal{A}_{\rm echo}(\theta)-\mathcal{A}_0\big]^2}{\sigma_{\mathcal{A}}^2}.$$

5.6.4 Priors, nuisance parameters, and gates

Priors. We adopt weakly informative priors reflecting structural positivity and smoothness: $M_{\rm eff}^2>0$, $Y>0$, $\ell_\star>0$, bounded anisotropy $|\delta c|\ll 1$, small MBP-1 drifts for g_i. Hyperparameters controlling eDM profiles $(\rho_0, r_c, \beta, \xi)$ carry broad astrophysical priors with physical cutoffs.

Nuisance parameters. Astrophysical systematics (inclination, baryonic mass-to-light, distance moduli) enter as ν with Gaussian priors and are marginalized analytically when possible.

Gate implementation. We enforce Eq. (5.26) either as hard cuts (reject samples) or as a penalty $\mathcal{P}_{\rm gate}=\sum_a \max(0,\, g_a(\theta))^2$ with a large $\lambda_{\rm gate}$, where $g_a(\theta)\le 0$ represent the gate inequalities.

5.6.5 Inference engines and diagnostics

Algorithms. We employ Hamiltonian Monte Carlo (HMC/NUTS) for continuous constrained spaces and nested sampling for evidence estimates $\mathcal{Z}$ (model comparison). For stiff posteriors with emulator noise, tempered transitions or differential-evolution MCMC are used.

Diagnostics. Convergence via $\widehat{R}\approx 1$, effective sample sizes, and energy Bayesian fraction of missing information. Posterior predictive checks (PPC) are computed per channel to verify calibration quality and residual structures.

5.6.6 Channel-by-channel constraint flow

GW & EM (null tests). These channels primarily restrict $(\delta c, \epsilon_{\parallel})$ and parity-odd invariants; in MBP-0 they collapse to gates confirming vanishing deviations.

Newtonian/lensing. These constrain $M_{\rm eff}(\rho)$ and, where eDM is invoked, $(\rho_0, r_c, \beta, \xi)$ with covariance to baryonic systematics ν.

EW precision. Constrains $\mathrm{Tr}(Y\Pi_1)/\mathrm{Tr}(Y\Pi_2)$ and small state corrections Δ_i, anchoring $\sin^2\theta_W$ and mass relations.

Flavor. Dominant sensitivity to geometric scales $(\ell_\star)$ and separations (d_{ij}), with phases Φ_{ij} feeding CP observables. Identifiability is improved by using ratios and invariants.

BH echoes. If present, map $(\ell_{\rm wall}, \gamma)$ to echo summaries; absence of signals produces upper bounds.

5.6.7 Joint fit, marginalization, and reporting

Posterior. The joint posterior reads

$$p(\theta, \nu|\mathbf{D}) \;\propto\; \mathcal{L}(\mathbf{D}|\theta, \nu)\, \pi(\theta)\, \pi(\nu)\, \mathbf{1}_{\text{gates}}(\theta)\,, \tag{5.29}$$

with nuisance parameters ν marginalized to obtain $p(\theta|\mathbf{D})$.

Model comparison. Bayes factors $K = \mathcal{Z}_{\text{model}}/\mathcal{Z}_{\text{ref}}$ compare: (i) pure-recovery baseline, (ii) eDM sector enabled, (iii) viscous-mass enabled, (iv) phase-like BH enabled. Information criteria (AIC/BIC) are reported for frequentist readers.

Result tables and figures. We recommend a compact `booktabs` table summarizing constraints:

Channel	Observable	Model prediction	Data summary	Dominant params
GW propagation	$c_T,\ \Delta\phi_{\text{pol}}$	Sec. 5.3.2	null suite	$\delta c,\ \epsilon_{\parallel}$
EW precision	$m_W, m_Z, \sin^2\theta_W, S, T$	Sec. 5.4.3	PDG-like	$\mathrm{Tr}(Y\Pi_i),\ \Delta_i$
Flavor	ratios, invariants	Sec. 5.4.4	mass/angles	$d_{ij},\ \ell_\star,\ \Phi_{ij}$
Lensing/rotation	$G_{\text{eff}},\ v_c(R),\ \hat{\alpha}$	Sec. 5.5.2	curves, lensing	$M_{\text{eff}},\ \rho_0, r_c, \beta, \xi$
BH echoes	$\Delta t_{\text{echo}},\ \mathcal{A}_{\text{echo}}$	Sec. 5.5.3	ringdown	$\ell_{\text{wall}},\ \gamma$

Table 5.2: Constraint map linking observables to controlling parameters.

Uncertainty propagation. Report 68%/95% credible intervals, correlation matrices, and posterior predictive envelopes. Include emulator uncertainty bands explicitly in figures where used.

5.6.8 Stress tests and robustness

Jackknife and leave-one-out. Re-fit after removing one channel at a time (GW, EW, flavor, lensing) to quantify leverage and detect potential tension.

Benchmark toggling. Repeat core inference under MBP-1 to ensure that small ordered deformations do not qualitatively alter the posterior; require continuity back to MBP-0.

5.6.9 Summary of constraints and outlook to Sec. 5.6

The joint inference produces gated posteriors for the structural and geometric parameters, quantifies allowable deviations from GR/QED, calibrates $M_{\text{eff}}(\rho)$ and gauge normalizations, and sets hierarchical scales for geometric Yukawas. Section 5.7 will condense these outcomes into a decision tree and falsifiability flow, delineating pass/fail regions and prioritizing future tests with maximal discriminating power.

5.7 Falsifiability and Decision Tree

We now condense the calibration and matching program into an explicit falsifiability framework. A *decision tree* organizes null tests, calibrated predictions, and alternative-sector signatures into pass/fail gates. The outcome is a partition of the parameter space into (i) *admissible* regions—consistent with all gates and data within tolerances; (ii) *tension* regions—where isolated channels disagree but re-calibration may succeed; and (iii) *excluded* regions—where hard gates or decisive likelihood penalties are violated.

5.7.1 Operating domain and hard gates

Hard gates (recap). The *gated* domain $\mathcal{D}_{\text{gate}}$ is defined by the inequalities

$$\boxed{\begin{array}{ll} \text{(CONE)} & c_\perp^2 > 0, \quad c_\perp^2 + \delta c > 0, \quad c_T = 1 \text{ in MBP-0;} \\ \text{(ENERGY)} & M_{\text{eff}}^2(\rho) > 0, \quad \mathcal{G}_{AB} > 0, \quad T_{\text{eff}}^{00} > 0; \\ \text{(ALIGN)} & \text{EW vacuum aligned}, \quad m_\gamma = 0; \\ \text{(DECOUPLE)} & v_\Sigma \gg v \text{ in MBP-0/1;} \\ \text{(DISS)} & \eta \geq 0, \quad \eta \ll m_{\text{eff}} \text{ for sharp quasi-particles.} \end{array}} \tag{5.30}$$

Outside $\mathcal{D}_{\text{gate}}$ the model is rejected without further statistical evaluation.

Null tolerances. Within $\mathcal{D}_{\text{gate}}$, MBP-0 null tests must satisfy

$$|c_T - 1| < 10^{-6}, \quad |\Delta\phi_{\text{pol}}| < 10^{-6} \text{ rad}, \quad \left|\omega^2/k^2 - 1\right| < 10^{-6}, \quad \text{and Eq. (5.1) tolerances.} \tag{5.31}$$

5.7.2 Multi-stage decision workflow

Stage A: Null suite (MBP-0).

1. Run the tests of Sec. 5.3 on MBP-0.
2. **If all pass** Eq. (5.31) $\Rightarrow$ proceed to Stage B.
3. **If any fail** by $> 3\sigma \Rightarrow$ *exclude* current point; optional retry with numerical resolution checks. Persistent failure $\Rightarrow$ parameter region *excluded*.

Stage B: Calibration closure.

1. Fit $M_{\text{eff}}(\rho)$ (Sec. 5.4.2); infer $g_i(\mathcal{S})$, $\sin^2\theta_W$ (Sec. 5.4.3); construct geometric Yukawas (Sec. 5.4.4).
2. **Closure test:** re-run a subset of atlas solutions (Sec. 5.2) and verify the null suite with calibrated parameters.
3. **Pass** $\Rightarrow$ Stage C. **Fail** $\Rightarrow$ attempt a single re-fit with tightened priors; persistent fail $\Rightarrow$ *tension* (flagged).

Stage C: Alternative sectors.

1. Enable viscous-mass, eDM, and phase-like BH channels (Sec. 5.5) one at a time.
2. Check channel-specific signatures vs. data; enforce the gates of Eq. (5.26).
3. **If any channel needs hard-gate violation** to fit $\Rightarrow$ that channel is *excluded*.

Stage D: Joint inference and consistency.

1. Build the joint likelihood (Sec. 5.6.3), include emulator covariances if used.
2. Run HMC/NUTS or nested sampling (Sec. 5.6.5); verify diagnostics.
3. **Posterior outcomes:** admissible (credible region within gates), tension (multimodal or fractured posteriors), excluded (posterior mass outside gates).

5.7.3 Exclusion regions (analytic criteria)

Causality/ghosts. Any point with $c_\perp^2 \leq 0$ or $c_\perp^2 + \delta c \leq 0$ is excluded. Negative eigenvalues of $\mathcal{G}_{AB}$ or $M_{\text{eff}}^2 \leq 0$ are excluded.

Photon mass and EW misalignment. If EW alignment cannot achieve $m_\gamma = 0$ concurrently with viable m_W, m_Z at leading order, the configuration is excluded (vacuum misalignment).

Flavor implausibility. Geometric Yukawas requiring $\ell_\star \to \infty$ or unphysical separations $d_{ij} < 0$ are excluded. Likewise, CP observables demanding Φ_{ij} outside allowed holonomy ranges.

eDM inconsistency. If a stationary $\rho(r)$ cannot reproduce both rotation and lensing with the same $(\rho_0, r_c, \beta, \xi)$, the eDM sector is excluded for that environment.

5.7.4 Prioritized measurements for maximal leverage

1. **Multi-messenger GW speed/polarization:** tightens $|c_T - 1|$ and birefringence bounds.
2. **Precision EW observables:** $(m_W, m_Z, \sin^2\theta_W, S, T)$ constrain Y-projections.
3. **Flavor ratios and invariants:** log-ratio fits pin down $(d_{ij}, \ell_\star, \Phi_{ij})$.
4. **Weak/strong lensing with rotation curves:** jointly constrain $(M_{\text{eff}}, \rho_0, r_c, \beta, \xi)$.
5. **Ringdown tails/echo searches:** bound $(\ell_{\text{wall}}, \gamma)$ in the phase-like BH sector.

5.7.5 Decision tables (implementation aid)

We recommend maintaining a compact table per sampling run. Requires `booktabs`.

Gate/Test	Outcome	Implication	Action
MBP-0 null suite	pass/fail	admissible/excluded	proceed/reject
Calibration closure	pass/tension	consistent/needs re-fit	fix priors or flag
Viscous-mass gate	pass/fail	viable/excluded	include/excise channel
eDM fit (joint)	pass/fail	viable/excluded	include/excise channel
BH echoes bounds	pass/fail	viable/excluded	include/excise channel
Joint posterior gates	inside/outside	admissible/excluded	report/terminate

Table 5.3: Operational decision table for a single inference cycle.

5.7.6 Reporting template and outcomes

Minimal report. Summarize: (i) calibrated $M_{\text{eff}}(\rho)$ with errors; (ii) inferred $g_i(\mathcal{S})$ and $\sin^2\theta_W$; (iii) geometric Yukawa scales and phases; (iv) eDM profile parameters (if used); (v) hard-gate compliance; (vi) Bayes factors vs. baselines.

Acceptance bands. Declare *admissible* regions with 68%/95% credible contours fully inside $\mathcal{D}_{\text{gate}}$, *tension* regions where contours touch a gate, and *excluded* regions otherwise.

5.7.7 Summary

The falsifiability framework translates structural assumptions into pass/fail statements at each stage. By enforcing hard gates before statistical fitting, then escalating from null tests to calibrated, channel-wise, and joint constraints, the decision tree ensures that any claimed success reflects *simultaneous* consistency across GR/QED recovery, EW/flavor structure, and the alternative sectors—or else yields a transparent and reproducible exclusion.

5.8 Summary and Outlook

This chapter completed the transition from structural dynamics to phenomenology along a reproducible, gated workflow. Section 5.2 assembled a minimal yet representative *solution atlas* (solitons, walls/strings, textures) with spectral/orbital stability diagnostics. Section 5.3 established *null suites* in the isotropic benchmark (MBP-0), confirming exact GR/QED linear recoveries and bounding small departures in weakly ordered environments (MBP-1). Section 5.4 *calibrated* the effective Planck scale $M_{\text{eff}}(\rho)$, inferred gauge normalizations from the internal metric Y_{AB} (including $\sin^2\theta_W$), and

built *geometric Yukawa* textures from localized overlaps and holonomy. Section 5.5 developed three alternative sectors (viscous-mass, effective dark matter via ρ, and phase-like black holes) under the same consistency gates. Section 5.6 formulated joint inference across channels and data compressions, and Sec. 5.7 distilled the falsifiability logic into a decision tree with hard/soft gates.

What is now established.

- A calibrated bridge from structural state variables $(\rho, \eta, \alpha, \hat{\chi}, T)$ to low-energy observables, including G_{eff}, $\sin^2\theta_W$, and geometric Yukawas.
- Null-test compliance in MBP-0 (luminal, birefringence-free GWs; Maxwell recovery; Newtonian/lensing).
- A reproducible protocol to confront alternative sectors with observations without violating gates.

Open problems and prioritized tasks.

1. Extend the atlas to multi-soliton bound states and long-lived molecules; quantify interaction laws and spectral curvature effects beyond Sec. 5.2.
2. Tighten the map $g_i^{-2} = \mathrm{Tr}(Y\Pi_i) + \Delta_i$ with higher-order corrections and vacuum-alignment systematics; propagate to precision EW likelihoods.
3. Expand geometric Yukawa fits to quark sectors with mixing/CP phases, testing MFV-like selection rules.
4. Refine eDM profiles and cluster-scale constraints on dissipation (η) using combined lensing/rotation data.
5. Systematize echo bounds and ISCO shifts in the phase-like BH sector within the calibrated M_{eff}.

Outlook. The workflow developed here enables end-to-end tests: from on-shell solutions, through calibration, to observational confrontation under explicit pass/fail gates. Subsequent chapters will leverage this infrastructure to (i) enlarge the solution catalogue, (ii) integrate early-/late-universe applications, and (iii) compare systematically against the Standard Model sector by sector, highlighting distinctive, falsifiable signatures of the spectral-aether paradigm.

Chapter 5 — Tables and Illustrative Figures

Table 5.4: Calibration–Recovery–Tests (CRT) checklist for Chapter 5. A compact map from theoretical inputs to observables, with success criteria and gate references.

Module	Theoretical inputs	Observable set	Success criteria	Gates
Recovery (low–energy)	$M_{\text{eff}}(\rho_0)$, $c_\perp$, δc	Newtonian limit, SR/GR regimes, causal cones	Correct Poisson law; subluminal cone inclusion; GR post–Newtonian bounds	D, U
EW matching	v_Σ, $g_i(S)$, $m_{\text{eff}}^{\Phi}(S)$	$m_W, m_Z, \alpha_{\text{em}}$, weak mixing angle	$m_\gamma = 0$, $m_W/m_Z = \cos\theta_W$, couplings in PDG bands	S, G
Flavor & Yukawa	Y_{AB} texture, geometric distances d_{ij}	Fermion masses, CKM/PMNS mixings, CP phase	Hierarchies with minimal free params; fits within PDG ranges	S, N
Null–tests	State dependence of G_{eff}, GW cone alignment	$c_{\text{GW}}-c$, $\delta G/G$ in lab/astro	$\lvert c_{\text{GW}} - c\rvert \ll 1$; $\lvert\delta G/G\rvert$ below bounds	U, D
Predictions	Spectral curvature, anisotropy, dissipation η	New heavy modes, decoherence scale, selection rules	Clear signatures with uncertainties; falsifiable windows	S, N

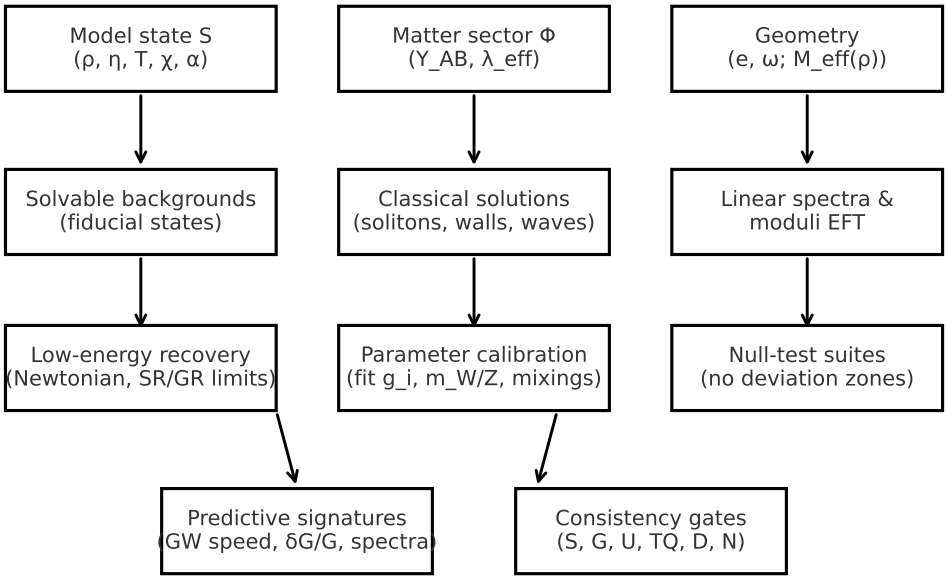

Figure 5.1: Calibration/Recovery/Tests pipeline. From model state S and matter sector Φ with Palatini geometry (e, ω), we build solvable backgrounds and classical solutions, extract spectra/moduli EFT, perform low–energy recovery (Newtonian/SR/GR), calibrate parameters against data, and deploy null–test suites. Deliverables are predictive signatures (GW speed, $\delta G/G$, spectra) filtered through consistency gates (S, G, U, TQ, D, N).

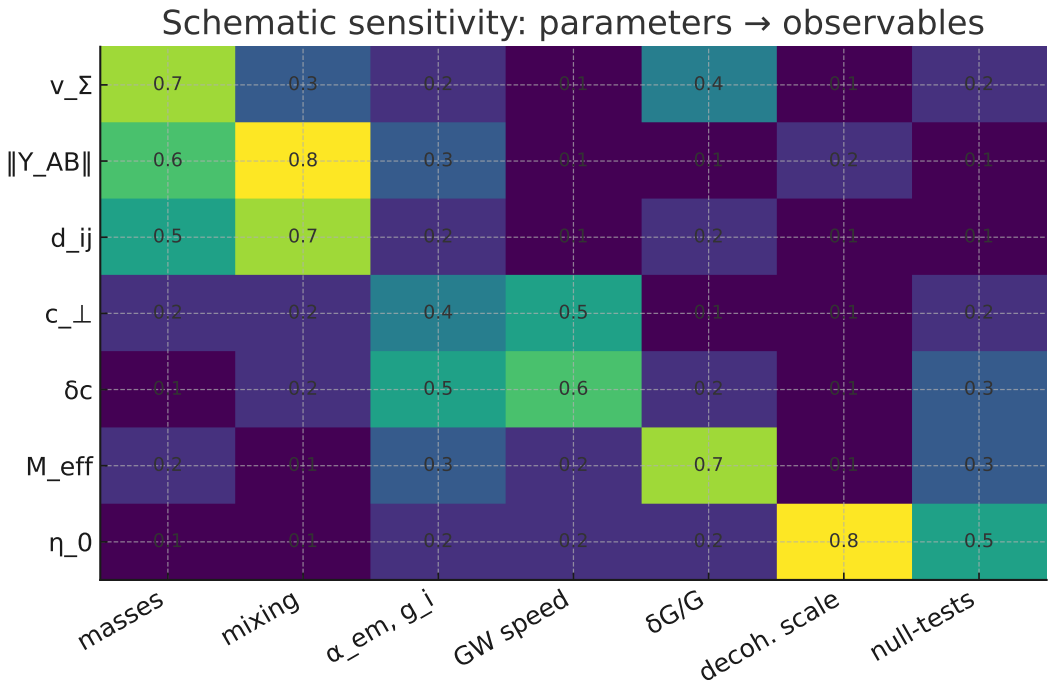

Figure 5.2: Schematic sensitivity (parameters → observables). Illustrative heatmap of relative sensitivities: parameters $(v_\Sigma, \|Y_{AB}\|, d_{ij}, c_\perp, \delta c, M_{\text{eff}}, \eta_0)$ versus observable families (masses, mixing, gauge couplings, GW speed, $\delta G/G$, decoherence scale, null–tests). Actual values are determined by the calibrated state S and the detailed fits in this chapter.

Chapter 6

Chapter 6:Solitons as Particles: Quantization, Spin/Statistics, and Bound States

6.1 Introduction and Scope

This chapter develops the passage from *classical* localized solutions of the spectral–aether equations (Ch. **??**) to *quantized* particle-like excitations with definite mass, spin, statistics, and internal charges. Our central tool is the *collective-coordinate* (moduli space) formalism, controlled by a spectral gap and consistency gates established in Chs. 3–5. Whenever relevant we work in benchmark backgrounds MBP-0/1 (Sec. 5.2.1); dissipation is switched off for the core quantization (setting $\eta \to 0^+$) and reintroduced perturbatively when we discuss widths and decoherence.

Let $\mathcal{S}(x) = (\rho, \eta, \alpha, \hat{\chi}, T)$ denote the structural state and $\Phi(x)$ the matter field (Ch. **??**). A classical soliton sector is specified by finite-energy boundary data and a topological label Q (e.g. degree, Hopf number), with static solutions

$$\Big\{\, \mathcal{S}_{\rm sol}(\mathbf{x}; X)\,,\ \Phi_{\rm sol}(\mathbf{x}; X) \,\Big\}, \qquad X = (X^1, \ldots, X^{\dim \mathcal{M}}) \in \mathcal{M}_Q, \tag{6.1}$$

organized by a moduli manifold $\mathcal{M}_Q$ generated by global symmetries and approximate degeneracies (translations, rotations, internal orientations, phase, *etc.*). Quantization will promote $X^I(t)$ to operators and impose global topological constraints (Finkelstein–Rubinstein, FR) on admissible wavefunctions.

6.1.1 From classical textures to quantum particles

Finite-energy sectors and topological charge. We assume the energy functional (from Chs. 3–**??**) admits finite-energy, localized configurations in sector Q under boundary conditions $(\mathcal{S}, \Phi) \to (\mathcal{S}_\infty, \Phi_\infty)$ as $|\mathbf{x}| \to \infty$, defining a map from compactified space S^3_∞ into a target manifold of internal states. Homotopy classes yield topological sectors $Q \in \pi_k(\mathcal{T})$ (model-dependent; examples will be recalled in Sec. 6.2).

Zero modes and collective coordinates. Each continuous degeneracy of a static solution generates a zero mode $\Psi_I(\mathbf{x}) = \partial_{X^I}(\mathcal{S}_{\rm sol}, \Phi_{\rm sol})$. In the *adiabatic* regime, slow time dependence enters exclusively through $X^I(t)$:

$$(\mathcal{S}, \Phi)(t, \mathbf{x}) \;=\; (\mathcal{S}_{\rm sol}, \Phi_{\rm sol})(\mathbf{x}; X(t)) \;+\; \sum_{n \geq 1} a_n(t)\, \Xi_n(\mathbf{x}; X(t)), \qquad \Xi_n:\ \text{nonzero-mode fluctuations}, \tag{6.2}$$

with orthogonality conditions ensuring that a_n do not contaminate the zero-mode sector at leading order. Eliminating a_n at tree/one-loop level produces an effective dynamics on the moduli space $\mathcal{M}_Q$.

Timescale separation and control parameter. Let $\Lambda_{\rm gap}(X)$ denote the spectral gap above the zero modes (lowest positive frequency). The collective description is controlled by

$$\varepsilon \;=\; \frac{\max_I |\dot{X}^I|}{\Lambda_{\rm gap}(X)} \;\ll\; 1, \qquad \text{and} \qquad \eta \to 0^+, \tag{6.3}$$

so that radiation and mode mixing are suppressed at leading order; effects $\mathcal{O}(\eta)$ and $\mathcal{O}(\varepsilon^2)$ will enter widths and level shifts (Secs. 6.10, 6.13).

6.1.2 Strategy: collective coordinates, constraints, and FR rules

Our program proceeds in four steps:

1. **Projection to zero-mode sector.** Insert the time-dependent ansatz (6.2) into the full action $S[\mathcal{S}, \Phi]$ (Ch. 3), expand to quadratic order in $\dot{X}$ and a_n, and integrate over space to obtain a collective Lagrangian $L_{\rm coll} = \frac{1}{2} G_{IJ}(X)\dot{X}^I\dot{X}^J + \mathcal{A}_I(X)\dot{X}^I - U_{\rm mod}(X)$ (Sec. 6.4). The metric G_{IJ}, Berry connection $\mathcal{A}_I$, and residual potential $U_{\rm mod}$ encode inertia, geometric phases, and weak lifting of degeneracy, respectively.

2. **Constraint analysis and symplectic structure.** Identify first-/second-class constraints associated with redundant parameterizations of $\mathcal{M}_Q$ and gauge choices. Construct the reduced phase space with symplectic form $\Omega = d\mathcal{A}$ (Sec. 6.5).

3. **Topological (FR) constraints and spin/statistics.** Analyze the fundamental group $\pi_1(\mathcal{C}_Q)$ of the *configuration space* (before and after quotienting by physically indistinguishable paths). Enforce sign assignments on wavefunctions along nontrivial loops to implement fermionic statistics and spin from isorotations/rotations (Sec. 6.6).

4. **Quantization and spectra.** Quantize (X^I, P_J) canonically (or via geometric quantization), solve the eigenproblem on $(\mathcal{M}_Q, G_{IJ}, \mathcal{A}_I, U_{\rm mod})$, include small anisotropies along $\hat{\chi}$ and gauge couplings, and extract masses, spins, and multiplets (Secs. 6.7–6.10).

6.1.3 Assumptions and benchmarks (link to Chs. ??, 5)

We adopt the following standing assumptions, all testable against the gates of Eq. (5.26) and the recovery checks of Sec. 5.3:

- **(Finite-energy solitons).** For each sector Q, there exists at least one smooth, finite-energy static family (6.1) with well-defined zero modes.
- **(Spectral gap).** $\Lambda_{\rm gap}(X) > 0$ uniformly on the relevant region of $\mathcal{M}_Q$, ensuring adiabatic control (6.3) and positivity of the induced metric G_{IJ}.
- **(Benchmark backgrounds).** In MBP-0 the background is isotropic and lossless ($\delta c = 0$, $\eta = 0$). MBP-1 introduces controlled, small anisotropy along the director $\hat{\chi}$ and tiny drifts of gauge normalizations (Sec. 5.3.7); these enter perturbatively.
- **(Dissipation).** Spectral viscosity η is set to 0^+ for the purpose of canonical quantization; nonzero η will be reinstated to discuss widths, damping, and entropy production at the end (Secs. 6.10, 6.13).
- **(Gauge sector).** Minimal coupling to emergent gauge fields on $\mathcal{M}_Q$ is allowed via the Berry connection and projected Noether charges (Secs. 6.8, 6.9), consistent with the calibrated normalizations of Sec. 5.4.3.

6.1.4 Notation and conventions for this chapter

- Greek indices $\mu, \nu = 0, 1, 2, 3$ for spacetime; Latin $i, j = 1, 2, 3$ for spatial components with respect to $g_{\rm eff}$ (Ch. 3). Internal/fiber indices A, B refer to components in F.

- Hatted vectors (e.g. $\hat{\chi}$) denote unit, background-selected directions (director) as in Ch. **??**. The effective metric is $g_{\rm eff}$; the covariant derivative is ∇_μ unless otherwise stated. Volume integrals use $\sqrt{-g_{\rm eff}}\, d^3x$ on static slices.
- Moduli are X^I ($I = 1, \ldots, \dim \mathcal{M}_Q$), with velocities $\dot{X}^I$ and conjugate momenta P_I. The moduli metric and Berry connection are $G_{IJ}(X)$ and $\mathcal{A}_I(X)$. The induced symplectic form is $\Omega = d\mathcal{A}$.
- Zero modes: $\Psi_I(\mathbf{x}) = \partial_{X^I}(\mathcal{S}_{\rm sol}, \Phi_{\rm sol})$. Nonzero modes: $\Xi_n(\mathbf{x}; X)$ with frequencies $\omega_n(X) \geq \Lambda_{\rm gap}(X)$.
- Inner products are defined with the quadratic kinetic form $\mathcal{K}$ extracted from the underlying Lagrangian (Ch. 3):

$$\langle \Upsilon_1, \Upsilon_2 \rangle \;=\; \int d^3x\, \sqrt{-g_{\rm eff}}\, \Upsilon_1^\dagger\, \mathcal{K}\, \Upsilon_2, \qquad \mathcal{K} > 0 \;\text{ on fluctuations.} \tag{6.4}$$

  Projectors onto zero-/nonzero-mode subspaces are defined with respect to $\langle \cdot, \cdot \rangle$ and will be used to derive G_{IJ} and $\mathcal{A}_I$ (Sec. 6.4).
- FR constraints will be enforced by analyzing the configuration space $\mathcal{C}_Q$ and its nontrivial loops; wavefunctions may acquire prescribed signs along generators of $\pi_1(\mathcal{C}_Q)$ (Sec. 6.6).

Roadmap of the chapter. Section 6.2 recalls the classical families and their zero modes. Section 6.3 constructs the moduli metric and connection. Section 6.4 derives the collective Lagrangian. Section 6.5 builds the symplectic structure and quantization rules. Section 6.6 implements FR constraints for spin/statistics. Sections 6.7–6.10 extract single-soliton spectra (including anisotropies and gauge couplings). Sections 6.11–6.12 address two-body forces and bound states. Section 6.13 treats stability and decay mechanisms, and Sec. 6.14 details numerical protocols. Cross-checks with calibrated parameters are presented in Sec. 6.15, and Sec. 6.16 summarizes and sets the stage for Ch. 7.

6.2 Classical Soliton Families and Zero-Mode Structure (Recap)

We summarize the classical localized solutions constructed in Ch. **??** and extract their zero-mode content. This will provide the moduli manifolds $\mathcal{M}_Q$ that underlie the collective-coordinate quantization of Secs. 6.3–6.5. Throughout, a "soliton" means a finite-energy, stationary (or static) configuration $(\mathcal{S}_{\rm sol}, \Phi_{\rm sol})$ with topological label Q and stability gates satisfied (Ch. **??**, Sec. 4.10).

6.2.1 Atlas of solutions from Chapter ??

Core (particle-like) solitons. Localized, approximately spherical (MBP-0) or weakly anisotropic (MBP-1) cores of radius $R_{\rm core}$ supported by the spectral sector and the Φ-nonlinearity. The prototypical profile satisfies

$$\Phi_{\rm sol}(r) \;\sim\; \Phi_0\, f(r/R_{\rm core})\, e^{-i\omega T}, \qquad \partial_r f(0) = 0, \quad f(\infty) = 0, \tag{6.5}$$

together with structural rearrangements in $(\rho, \eta, \alpha, \hat{\chi})$. Topological charge Q may be (model-dependently) a winding in an internal target; in the simplest sector $Q = 0$ the stability is energetic (balance of gradient/nonlinear terms).

Domain walls and strings. Co-dimension one (walls) and two (strings/vortices) defects, characterized by an order-parameter flip (wall) or a nontrivial phase winding (string) in Φ and an aligned texture of $(\rho, \alpha, \hat{\chi})$. For a straight string along z:

$$\Phi_{\rm sol}(r, \theta) \;\sim\; \Phi_\infty\, g(r/\xi)\, e^{in\theta}, \qquad g(0) = 0, \quad g(\infty) = 1, \quad n \in \mathbb{Z}, \tag{6.6}$$

with core scale ξ set by the spectral stiffness and potential depth (Ch. **??**, Sec. 4.4).

Skyrmion-/Hopf-like textures. Three-dimensional textures with nontrivial maps $S^3 \to \mathcal{T}$ (target of internal states), classified by $Q \in \pi_3(\mathcal{T})$ (e.g. Hopf charge). Stability is topological and energetic; in MBP-1 the director $\hat{\chi}$ induces weak anisotropic distortions (Sec. 4.5).

6.2.2 Symmetry breaking patterns and moduli

For each family we list continuous symmetries broken by the soliton and the associated moduli X^I.

Translations. Any localized object breaks spatial translations. For cores and textures:

$$X^i \equiv x_0^i \ \text{(center-of-mass)}, \qquad \Psi_{(x^i)} \;=\; \partial_{x_0^i}(\mathcal{S}_{\rm sol}, \Phi_{\rm sol}) \;=\; -\partial_i(\mathcal{S}_{\rm sol}, \Phi_{\rm sol}). \tag{6.7}$$

Strings/walls carry *transverse* translations (normal to the worldvolume) which become Goldstone fields on the worldvolume (Sec. 4.6).

Rotations. In MBP-0, cores/textures break spatial $SO(3)$, producing three orientation moduli $R \in SO(3)$. The tangent zero modes are

$$\Psi_{(\Omega_a)} \;=\; \frac{d}{d\epsilon}\Big|_{\epsilon=0} (\mathcal{S}_{\rm sol}, \Phi_{\rm sol}) \circ \exp\big(\epsilon\, \mathcal{J}_a\big), \qquad \mathcal{J}_a :\ so(3) \text{ generators}. \tag{6.8}$$

In MBP-1 the director $\hat{\chi}$ reduces $SO(3) \to SO(2)$ around $\hat{\chi}$, leaving two exact rotational zero modes and one *soft* (pseudo-zero) mode lifted by anisotropy.

Internal phases / isorotations. If the energy is invariant under a global $U(1)$ of $\Phi \mapsto e^{i\vartheta}\Phi$, a phase modulus appears:

$$\vartheta \in [0, 2\pi), \qquad \Psi_{(\vartheta)} \;=\; i\,\Phi_{\rm sol}. \tag{6.9}$$

More general internal orientations may exist if the target admits an $SU(2)$ (isorotation) action; see Ch. **??**, Sec. 4.7.

Worldvolume moduli. For straight strings (walls), translations along the defect are trivial, while transverse displacements become light fields $X^i(\sigma, t)$ on the worldvolume (Sec. 4.6). A global rotation of a straight string around its axis is equivalent to a phase winding for $n \neq 0$, hence not an independent modulus.

6.2.3 Linear fluctuations: zero modes vs. gapped modes

Let $\Upsilon = (\delta\mathcal{S}, \delta\Phi, \delta\Phi^*)$ denote small fluctuations about a static solution. The quadratic action on a static slice reads

$$\delta^2 S \;=\; \frac{1}{2}\int dt\, d^3x\, \sqrt{-g_{\rm eff}}\, \Big[\dot{\Upsilon}^\dagger\, \mathcal{K}\, \dot{\Upsilon} - \Upsilon^\dagger\, \mathbb{H}\, \Upsilon\Big], \tag{6.10}$$

where $\mathcal{K} > 0$ is the kinetic form and $\mathbb{H}$ the (self-adjoint) stability operator assembled from spatial gradients, effective potentials, and spectral-curvature couplings (Ch. 3). Static normal modes satisfy

$$\mathbb{H}\,\Xi_n \;=\; \omega_n^2\, \mathcal{K}\, \Xi_n, \qquad \langle \Xi_m, \Xi_n \rangle = \delta_{mn}, \quad \langle \Upsilon_1, \Upsilon_2 \rangle = \int d^3x \sqrt{-g_{\rm eff}}\, \Upsilon_1^\dagger \mathcal{K} \Upsilon_2. \tag{6.11}$$

Zero modes are precisely tangent vectors to the moduli manifold:

$$\Psi_I \;=\; \partial_{X^I}(\mathcal{S}_{\rm sol}, \Phi_{\rm sol}), \qquad \mathbb{H}\,\Psi_I \;=\; 0, \qquad G_{IJ} = \langle \Psi_I, \Psi_J \rangle. \tag{6.12}$$

All other modes are *gapped* with $\omega_n^2 \geq \Lambda_{\rm gap}^2 > 0$ in the admissible region.

Orthogonality and gauge conditions. To avoid spurious mixing between zero and gapped sectors one imposes

$$\langle \Xi_n, \Psi_I \rangle \;=\; 0 \quad \forall\, n, I, \tag{6.13}$$

which fixes the collective-coordinate gauge. If emergent gauge redundancies act on $(\delta\mathcal{S}, \delta\Phi)$, we first project to gauge-invariant combinations or impose a background-covariant gauge (Ch. 3).

Positivity gate and absence of ghosts. The quadratic form G_{IJ} is positive-definite in the stable sector, guaranteed by the energy-positivity gate of Ch. **??**. Negative or zero eigenvalues beyond the symmetries signal instability or flat directions explicitly lifted by anisotropy or boundary conditions.

6.2.4 Spectral-curvature effects on moduli and anisotropy

The spectral-curvature tensor $F_{\rm spec}$ (Ch. 3, Sec. 3.6) modifies both the kinetic metric and the residual potential on $\mathcal{M}_Q$.

Metric deformation. To leading order in weak anisotropy (MBP-1), the moduli metric splits into isotropic and anisotropic parts:

$$G_{IJ}(X) \;=\; G^{(0)}_{IJ}(X) \;+\; \delta G_{IJ}(X;\hat\chi,F_{\rm spec}), \qquad \|\delta G\| \ll \|G^{(0)}\|. \tag{6.14}$$

For rotational moduli, the anisotropic correction induces different moments of inertia parallel/perpendicular to $\hat\chi$:

$$I_\parallel \;=\; I_0+\delta I_\parallel, \qquad I_\perp \;=\; I_0+\delta I_\perp, \qquad \delta I_\parallel \neq \delta I_\perp \;\text{ if }\; F_{\rm spec}\cdot\hat\chi \neq 0. \tag{6.15}$$

Residual (lifting) potential on $\mathcal{M}_Q$. In MBP-0, exact symmetries imply $U_{\rm mod}\equiv 0$ along symmetry-generated moduli. In MBP-1, weak ordering generates a small alignment potential; for rotational moduli one may write

$$U_{\rm mod}(R) \;\simeq\; \frac{1}{2}\,\kappa_{\rm ani}\,\big(\hat n(R)\cdot\hat\chi\big)^2 \;+\; \mathcal{O}(\kappa_{\rm ani}^2), \qquad \kappa_{\rm ani}\propto\|F_{\rm spec}\|^2, \tag{6.16}$$

where $\hat n(R)$ is a body-fixed axis. This turns one rotational zero mode into a soft mode with a small frequency $\omega^2_{\rm ani}\sim\kappa_{\rm ani}/I$.

Worldvolume bending stiffness. For strings/walls, $F_{\rm spec}$ corrects the worldvolume bending moduli and induces an anisotropic dispersion for long-wavelength undulations $X^i(\sigma,t)$:

$$\omega^2 \;\simeq\; c_{\rm wl}^2\,k^2 \;+\; \gamma_{\rm bend}(\hat\chi)\,k^4 \;+\; \cdots, \qquad \gamma_{\rm bend}(\hat\chi_\parallel)\neq\gamma_{\rm bend}(\hat\chi_\perp). \tag{6.17}$$

These coefficients feed directly into the moduli-space kinetic terms in Sec. 6.3.

6.2.5 Normalization, inner products, and projectors

Given zero modes Ψ_I, we define the moduli metric and projectors:

$$G_{IJ}\equiv\langle\Psi_I,\Psi_J\rangle=\int d^3x\,\sqrt{-g_{\rm eff}}\;\Psi_I^\dagger\,\mathcal{K}\,\Psi_J, \tag{6.18}$$

$$\mathsf{P}_0\,\Upsilon\equiv\sum_{I,J}\Psi_I\,(G^{-1})^{IJ}\,\langle\Psi_J,\Upsilon\rangle, \qquad \mathsf{P}_\perp\equiv\mathbf{1}-\mathsf{P}_0, \tag{6.19}$$

with P_0 the projector onto $\mathrm{span}\{\Psi_I\}$. The Berry connection on $\mathcal{M}_Q$ will emerge from the *parametric* dependence of zero modes on X via $\mathsf{P}_0\,\partial_t\Psi_I$ (Sec. 6.3).

6.2.6 Representative examples (to be used later)

Core soliton in MBP-0. Moduli: $\{x_0^i\}$ (3 translations), $\{R\}$ (3 rotations), and possibly a global phase ϑ. Thus $\dim\mathcal{M}=6$ (or 7 with $U(1)$). Metric is block-diagonal at leading order:

$$G_{IJ} \;\simeq\; \mathrm{diag}\big(M\,\delta_{ij},\; I_1,I_2,I_3,\; \mathcal{I}_\vartheta\big), \tag{6.20}$$

with M the inertial mass and I_a principal moments.

Straight string (length L). Zero modes: two transverse translations (Kelvin modes), global translation along z (trivial), and an overall phase for $n\neq 0$. Worldsheet normalization yields $G_{IJ}\propto L$ for center-of-mass modes, while local undulations $X^i(z,t)$ enter as fields (Sec. 6.3).

Skyrmion-/Hopf-like texture. Moduli include translations and global rotations; internal orientations may appear if the target admits isorotations. Anisotropy lifts one rotational mode in MBP-1 through Eq. (6.16).

These structures complete the recap needed to construct the moduli metrics and Berry connections in Sec. 6.3, where we will pull back the full Lagrangian to $\mathcal{M}_Q$ and derive the collective dynamics $L_{\rm coll}$ with explicit formulas for $G_{IJ}(X)$, $\mathcal{A}_I(X)$, and $U_{\rm mod}(X)$ in benchmark families.

6.3 Moduli Space Construction and Metrics

We now pull back the full spectral–aether Lagrangian (Ch. 3) onto the manifold of collective coordinates $\mathcal{M}_Q$ that parametrize a classical soliton family (6.1). The outcome is a geodesic-type dynamical system with a kinetic metric $G_{IJ}(X)$, a Berry connection $\mathcal{A}_I(X)$, and a residual (lifting) potential $U_{\rm mod}(X)$. These three geometric objects encode, respectively, inertial response, geometric phases/topological couplings, and weak symmetry breaking (e.g. anisotropy from spectral curvature). We keep dissipation off ($\eta \to 0^+$) in this section and reinstate it perturbatively later.

6.3.1 Collective-coordinate pullback of the action

Adiabatic ansatz and projectors. Let $(\mathcal{S}_{\rm sol}, \Phi_{\rm sol})(\mathbf{x}; X)$ be a static family with moduli X^I. Introduce slow time dependence $X^I = X^I(t)$ and decompose fluctuations as in (6.2) with the orthogonality gauge (6.13). Denote by P_0 the projector onto $\mathrm{span}\{\Psi_I\}$ and $\mathsf{P}_\perp = \mathbf{1} - \mathsf{P}_0$, cf. (6.19). To leading adiabatic order, the fields follow the moving soliton:

$$(\mathcal{S}, \Phi)(t, \mathbf{x}) = (\mathcal{S}_{\rm sol}, \Phi_{\rm sol})(\mathbf{x}; X(t)) + \mathcal{O}(\dot{X}^2), \qquad \dot{} \equiv \tfrac{d}{dt}. \tag{6.21}$$

Time derivatives act by chain rule: $\partial_t = \dot{X}^I \partial_{X^I}$ when evaluated on the soliton family.

Quadratic expansion and spatial integration. Write the full action as $S = \int dt\, d^3x\, \sqrt{-g_{\rm eff}}\, \mathcal{L}[\mathcal{S}, \Phi]$. Expanding $\mathcal{L}$ to second order in $\dot{X}$ about the static family (and using the static Euler–Lagrange equations) yields, after integrating out gapped modes at tree level,

$$L_{\rm coll}(X, \dot{X}) = \frac{1}{2}\, G_{IJ}(X)\, \dot{X}^I \dot{X}^J + \mathcal{A}_I(X)\, \dot{X}^I - U_{\rm mod}(X) \;+\; \mathcal{O}(\dot{X}^3, \dot{X}\, \ddot{X}). \tag{6.22}$$

All quantities above are gauge- and parametrization-covariant objects on $\mathcal{M}_Q$ defined shortly. Higher-derivative corrections are suppressed by the gap, cf. (6.3).

6.3.2 Kinetic metric G_{IJ}: general derivation

Definition via the quadratic kinetic form. Let $\mathcal{K}$ be the positive-definite kinetic operator entering the quadratic time-derivative part of the Lagrangian density (cf. (6.10)). The moduli metric is the inner product of zero-mode tangents with respect to $\mathcal{K}$:

$$G_{IJ}(X) \;\equiv\; \int d^3x\, \sqrt{-g_{\rm eff}}\; \Psi_I^\dagger(\mathbf{x}; X)\, \mathcal{K}\, \Psi_J(\mathbf{x}; X), \qquad \Psi_I \equiv \partial_{X^I}(\mathcal{S}_{\rm sol}, \Phi_{\rm sol}). \tag{6.23}$$

Positivity of $\mathcal{K}$ on fluctuations implies $G_{IJ} > 0$ in any stable sector (energy gate).

Translational block and inertial mass. For a localized soliton in an isotropic MBP-0 background, translation moduli $X^i = x_0^i$ satisfy $\Psi_{(x^i)} = -\partial_i(\mathcal{S}_{\rm sol}, \Phi_{\rm sol})$ and the block reduces to

$$G_{ij} \;=\; M\, \delta_{ij}\,, \qquad M \;=\; \int d^3x\, \sqrt{-g_{\rm eff}}\; T^{00}_{\rm sol} \;=\; E_{\rm sol}, \tag{6.24}$$

where $T^{00}_{\rm sol}$ is the static energy density and the second equality follows from Noether analysis: a uniform time-dependent translation carries momentum $P_i = G_{ij}\dot{x}_0^j$ and must match the field-theory momentum $P_i = \int d^3x\, T^0{}_i$, implying $G_{ij} = M\delta_{ij}$ with $M = E_{\rm sol}$ (units $c = 1$).

Rotational block and moments of inertia. For rotational moduli $R \in SO(3)$ with generators $\mathcal{J}_a$, the tangent vectors are $\Psi_{(\Omega_a)} = \mathcal{J}_a\,(\mathcal{S}_{\rm sol}, \Phi_{\rm sol})$ and

$$G_{ab} \;\equiv\; I_{ab} \;=\; \int d^3x\,\sqrt{-g_{\rm eff}}\;\Psi^{\dagger}_{(\Omega_a)}\,\mathcal{K}\,\Psi_{(\Omega_b)}. \tag{6.25}$$

In MBP-0, $I_{ab} = I_0\,\delta_{ab}$ for spherically symmetric cores; in MBP-1, spectral anisotropy splits $I_{\parallel} \neq I_{\perp}$ as in (6.15).

Phase/isorotation block. If a global $U(1)$ symmetry is present, the phase modulus ϑ yields

$$G_{\vartheta\vartheta} \;=\; \int d^3x\,\sqrt{-g_{\rm eff}}\;(i\Phi_{\rm sol})^{\dagger}\,\mathcal{K}\,(i\Phi_{\rm sol}) \;\equiv\; \mathcal{I}_{\vartheta}, \tag{6.26}$$

with possible mixing to rotations if the symmetry groups do not commute on the solution.

Anisotropy and director dependence. A weak ordering along $\hat{\chi}$ modifies the metric by $G_{IJ} = G^{(0)}_{IJ} + \delta G_{IJ}(\hat{\chi}, F_{\rm spec})$, with symmetry-allowed structures (e.g. $n_I n_J$ projectors for orientation moduli) suppressed by $\|F_{\rm spec}\|/\Lambda^2_{\rm gap}$.

6.3.3 Berry connection $\mathcal{A}_I$ and symplectic two-form

Origin of $\mathcal{A}_I$. Terms linear in $\dot{X}^I$ arise when the kinetic part of the microscopic Lagrangian contains single-time-derivative structures, e.g. from complex fields, topological terms, or Chern–Simons/WZW reductions. Systematically,

$$\mathcal{A}_I(X) \;=\; \int d^3x\,\sqrt{-g_{\rm eff}}\,\Big[\Pi_a(\mathbf{x};X)\,\partial_{X^I}\Phi^a_{\rm sol}(\mathbf{x};X) \;+\; \Pi^{(\mathcal{S})}_A(\mathbf{x};X)\,\partial_{X^I}\mathcal{S}^A_{\rm sol}(\mathbf{x};X)\Big], \tag{6.27}$$

where Π's are the canonical momenta conjugate to the fields in the full theory (evaluated on the static family). Equivalently, in operator form, $\mathcal{A}_I = \frac{1}{2}\langle \Psi_I,\, \mathcal{J}\,\Psi_0\rangle$ if the microscopic kinetic sector can be written with a symplectic matrix $\mathcal{J}$.

Gauge structure and physical content. Under a reparametrization $X^I \to X'^I(X)$ and a field-space gauge transformation along the family, $\mathcal{A}_I$ shifts by a total derivative, while its curvature

$$\Omega_{IJ} \;\equiv\; \partial_I \mathcal{A}_J - \partial_J \mathcal{A}_I \tag{6.28}$$

is invariant and equals the pullback of the microscopic symplectic form to $\mathcal{M}_Q$. Quantization on $\mathcal{M}_Q$ requires a prequantum line bundle with connection $\mathcal{A}$ and curvature Ω; integrality of $\frac{1}{2\pi}\int_\Sigma \Omega$ on closed two-cycles $\Sigma \subset \mathcal{M}_Q$ gives quantization conditions for topological couplings (to be revisited in Sec. 6.9).

6.3.4 Residual potential $U_{\rm mod}$ and lifting

Definition. The collective potential $U_{\rm mod}(X)$ is the static energy of the family evaluated at fixed X, relative to the absolute minimum along the degeneracy directions:

$$U_{\rm mod}(X) \;\equiv\; E_{\rm sol}(X) - \min_{X\in\mathcal{M}_Q} E_{\rm sol}(X). \tag{6.29}$$

In MBP-0, exact symmetries imply $U_{\rm mod} \equiv 0$ along symmetry-generated moduli. In MBP-1, anisotropy (or weak explicit breaking) induces small alignment terms, e.g. the quadratic rotational potential (6.16) with curvature $\kappa_{\rm ani} \propto \|F_{\rm spec}\|^2$.

Relation to pseudo-zero modes. Small eigenfrequencies $\omega^2_{\rm soft}$ in the fluctuation problem correspond to quadratic curvature of $U_{\rm mod}$ along the lifted direction: $\omega^2_{\rm soft} \sim \partial^2 U_{\rm mod}/\partial\xi^2/(\text{inertia})$. This identifies the emergence of soft resonances in the quantum spectrum (Sec. 6.10).

6.3.5 Worldline/Worldvolume reduction for defects

Straight strings and domain walls. Let Σ be a static worldvolume (line or plane) with coordinates σ^α and normal displacements $X^i(\sigma,t)$ ($i=1,2$ for a string in 3D). Pulling back the action yields, up to quadratic order,

$$S_{\rm wl/wv} \;=\; \int dt\, d^p\sigma\, \Big[\frac{\mu}{2}\,|\partial_t X|^2 - \frac{\tau}{2}\,|\nabla_\sigma X|^2 - \frac{\gamma_{\rm bend}}{2}\,|\nabla_\sigma^2 X|^2 + \cdots\Big], \tag{6.30}$$

where μ is the mass density (per length/area), τ the tension, and $\gamma_{\rm bend}$ the bending rigidity. In MBP-1 these coefficients become anisotropic tensors with preferred direction $\hat\chi$ (cf. (6.17)). Quantization on the worldvolume proceeds by standard methods once the coefficients are computed from the soliton profile.

6.3.6 One-loop renormalization by gapped modes

Adiabatic elimination. Integrating out nonzero modes at quadratic order corrects G_{IJ} and $\mathcal{A}_I$:

$$G_{IJ} \;\to\; G_{IJ} + \delta G_{IJ}, \qquad \mathcal{A}_I \;\to\; \mathcal{A}_I + \delta\mathcal{A}_I, \tag{6.31}$$

with $\delta G_{IJ} \sim \sum_n \frac{\langle \partial_I \mathbb{H}\rangle_{0n}\langle \partial_J \mathbb{H}\rangle_{n0}}{\omega_n^3}$, schematically, where matrix elements are taken between zero- and nonzero-mode sectors using the $\mathcal{K}$-inner product. The corrections are suppressed by $\Lambda_{\rm gap}^{-1}$ and are small in the regime (6.3). They preserve positivity if the underlying spectrum is positive.

Locality on $\mathcal{M}_Q$. Provided the soliton is localized and the gap is nonvanishing, the induced $G_{IJ}(X)$ and $\mathcal{A}_I(X)$ are smooth, slowly varying functions on $\mathcal{M}_Q$. Singularities may arise only at boundaries where the soliton ceases to exist or the gap closes.

6.3.7 Consistency gates for G_{IJ} and $\mathcal{A}_I$

Energy/ghost gate. G_{IJ} must be positive-definite; numerically, all principal minors are positive within uncertainties. A negative eigenvalue signals an unstable background or an incorrect projector/gauge.

Cone alignment gate. The collective kinetic energy should propagate within the same causal cone as $g_{\rm eff}$: for translational modes this is automatic (Galilean shift of the center); for worldvolume modes the group velocity satisfies $v_{\rm grp}^2 \le c_T^2$ (MBP-0).

Gauge/reparametrization invariance. Under smooth reparametrizations of $\mathcal{M}_Q$, the scalar Lagrangian (6.22) transforms covariantly. $\mathcal{A} \to \mathcal{A} + d\lambda$ leaves the curvature Ω invariant; physical spectra depend only on $(G_{IJ}, \Omega, U_{\rm mod})$.

6.3.8 Worked formulas in MBP-0 for core solitons

Block-diagonal metric. For a spherically symmetric core,

$$G_{IJ} \;\simeq\; \mathrm{diag}\Big(M\,\delta_{ij},\; I_0\,\delta_{ab},\; \mathcal{I}_\vartheta\Big), \tag{6.32}$$

with $M = \int d^3x\sqrt{-g_{\rm eff}}\,T^{00}_{\rm sol}$, $I_0 = \frac{1}{3}\int d^3x\sqrt{-g_{\rm eff}}\,r^2\,T^{00}_{\rm sol} + \cdots$ (internal contributions included via $\mathcal{K}$), and $\mathcal{I}_\vartheta$ from (6.26). Cross-terms vanish by symmetry.

Berry connection and curvature. If the microscopic theory contains topological terms descending to a one-form on $\mathcal{M}_Q$, then $\mathcal{A} \neq 0$. For example, an internal $U(1)$ phase modulus yields $\mathcal{A}_\vartheta = Q_{\rm Noether}$ (constant), giving trivial curvature $\Omega = 0$ but a nontrivial Aharonov–Bohm phase upon 2π winding if $Q_{\rm Noether} \notin \mathbb{Z}$. Nontrivial Ω appears for families with nontrivial two-cycles (see Sec. 6.9).

Residual potential. In MBP-0, $U_{\rm mod} = 0$ along symmetry moduli. Small explicit breakings (e.g. tiny director bias or external fields) induce a quadratic lifting with curvature $\ll \Lambda_{\rm gap}^2$ so that adiabatic control persists.

With these constructions, the collective dynamics is fully characterized by $(G_{IJ}, \mathcal{A}_I, U_{\rm mod})$. The next section derives *explicit* formulas for these objects from the microscopic Lagrangian via projection operators, and sets up the canonical quantization on $\mathcal{M}_Q$.

6.4 Collective-Coordinate Lagrangian: Derivation

We now derive the collective (moduli-space) dynamics by inserting a time-dependent soliton family into the full spectral–aether action of Ch. 3, projecting onto zero modes, and integrating out gapped fluctuations at leading adiabatic order. The result is a covariant, reparametrization-invariant mechanics on the moduli manifold $\mathcal{M}_Q$ with metric G_{IJ}, Berry connection $\mathcal{A}_I$, and a residual potential $U_{\rm mod}$,

$$L_{\rm coll}(X, \dot{X}) = \frac{1}{2} G_{IJ}(X)\, \dot{X}^I \dot{X}^J + \mathcal{A}_I(X)\, \dot{X}^I - U_{\rm mod}(X) \; + \; \mathcal{O}(\varepsilon^3), \tag{6.33}$$

where ε is the small adiabatic parameter defined in Eq. (6.3). Dissipation (η) is set to 0^+ throughout this derivation and reintroduced perturbatively in Secs. 6.10, 6.13.

6.4.1 Time-dependent ansatz and orthogonality gauge

Ansatz and mode decomposition. Let $(\mathcal{S}_{\rm sol}, \Phi_{\rm sol})(\mathbf{x}; X)$ be a static family labeled by moduli X^I. We promote $X^I = X^I(t)$ and decompose small deviations into the zero-mode basis $\Psi_I = \partial_{X^I}(\mathcal{S}_{\rm sol}, \Phi_{\rm sol})$ and the orthonormal gapped modes $\{\Xi_n\}_{n\geq 1}$ (eigenfunctions of the stability operator $\mathbb{H}$, cf. Eq. (6.11)):

$$(\mathcal{S}, \Phi)(t, \mathbf{x}) = (\mathcal{S}_{\rm sol}, \Phi_{\rm sol})(\mathbf{x}; X(t)) \; + \; \sum_{n\geq 1} a_n(t)\, \Xi_n(\mathbf{x}; X(t)) \; . \tag{6.34}$$

We impose the *orthogonality gauge* $\langle \Xi_n, \Psi_I \rangle = 0$ for all n, I at all times, with the inner product defined by Eq. (6.4).

Kinematics. Time derivatives act as

$$\partial_t = \dot{X}^I \partial_{X^I} + \sum_n \dot{a}_n \frac{\partial}{\partial a_n} + \sum_n a_n \dot{X}^I\, \partial_{X^I}\big|_{\Xi_n} \; , \tag{6.35}$$

where the last term accounts for the parametric dependence of the nonzero modes on X. Adiabatic power counting assigns $a_n = \mathcal{O}(\varepsilon)\,,\ \dot{a}_n = \mathcal{O}(\varepsilon^2)$, so that a_n can be eliminated algebraically at leading order.

6.4.2 Projectors, constraints, and reduced phase space

Projectors. We define projectors onto the zero- and gapped subspaces

$$\mathsf{P}_0 \Upsilon \; = \; \sum_{I,J} \Psi_I\, (G^{-1})^{IJ}\, \langle \Psi_J, \Upsilon \rangle, \qquad \mathsf{P}_\perp \; = \; \mathbf{1} - \mathsf{P}_0, \tag{6.36}$$

with $G_{IJ} = \langle \Psi_I, \Psi_J \rangle$ evaluated on the static family (Sec. 6.3.2). The orthogonality gauge enforces $\mathsf{P}_0(\sum_n a_n \Xi_n) = 0$.

Redundancies and Dirac brackets (if needed). If the moduli include redundant parameters (e.g. global gauge phases that act trivially on the solution), one obtains first-class constraints among (X^I, P_J). One can either fix a smooth gauge on $\mathcal{M}_Q$ or pass to Dirac brackets on the reduced phase space; the final $L_{\rm coll}$ is gauge equivalent under smooth reparametrizations and one-forms $\mathcal{A} \to \mathcal{A} + d\lambda$.

6.4.3 Eliminating gapped modes and renormalization

Quadratic action to leading order. Expanding the full action to quadratic order in $\dot{X}$ and a_n, and using the static equations of motion, one obtains schematically

$$L = \frac{1}{2} G^{(0)}_{IJ}\, \dot{X}^I \dot{X}^J + \mathcal{A}^{(0)}_I\, \dot{X}^I - U^{(0)}_{\rm mod}(X) + \sum_n \left(\frac{1}{2}\dot{a}_n^2 - \frac{1}{2}\omega_n^2 a_n^2 \right) + \sum_n S_{nI}(X)\, a_n\, \dot{X}^I + \cdots , \quad (6.37)$$

where $G^{(0)}_{IJ}$, $\mathcal{A}^{(0)}_I$, and $U^{(0)}_{\rm mod}$ are the *direct* (tree-level) pullbacks defined in Sec. 6.3, and S_{nI} are linear couplings between gapped modes and moduli velocities. The dots denote higher-order and total-derivative terms.

Solving for a_n. The Euler–Lagrange equations for a_n read $\ddot{a}_n + \omega_n^2 a_n = S_{nI}\, \dot{X}^I + \mathcal{O}(\varepsilon^2)$. In the adiabatic limit we neglect $\ddot{a}_n$ and obtain the algebraic solution

$$a_n \;=\; \frac{S_{nI}(X)}{\omega_n^2(X)}\, \dot{X}^I \;+\; \mathcal{O}(\varepsilon^2). \quad (6.38)$$

Substituting back into Eq. (6.37) yields finite, positive corrections to the metric and (possibly) to the Berry one-form:

$$\delta G_{IJ}(X) = \sum_n \frac{S_{nI}(X)\, S_{nJ}(X)}{\omega_n^2(X)} \;\geq 0, \quad (6.39)$$

$$\delta \mathcal{A}_I(X) = \frac{1}{2} \sum_n \frac{\partial}{\partial X^I} \left[\frac{S_{nJ}(X)\, S_{nK}(X)}{\omega_n^2(X)} \right] \mathcal{C}^{JK}(X)\,, \qquad \text{(model-dependent; may vanish by symmetry).} \quad (6.40)$$

Here $\mathcal{C}^{JK}$ encodes any antisymmetric pieces arising from single-time-derivative structures in the microscopic theory.[1] Thus,

$$G_{IJ} = G^{(0)}_{IJ} + \delta G_{IJ}, \qquad \mathcal{A}_I = \mathcal{A}^{(0)}_I + \delta \mathcal{A}_I, \qquad U_{\rm mod} = U^{(0)}_{\rm mod} + \mathcal{O}(\varepsilon^2). \quad (6.41)$$

Positivity and locality. Since $\omega_n^2 > 0$ and S_{nI} are real, δG_{IJ} is positive semidefinite, preserving the ghost-free property of the collective metric. For localized solitons with a finite gap the sums converge and $G_{IJ}, \mathcal{A}_I$ are smooth on $\mathcal{M}_Q$ away from gap-closing boundaries.

6.4.4 Explicit expressions for G_{IJ}, $\mathcal{A}_I$, $U_{\rm mod}$

Tree-level (direct) pullback. Using the inner product $\langle \cdot, \cdot \rangle$ associated with the quadratic kinetic form $\mathcal{K}$,

$$G^{(0)}_{IJ} = \int d^3x\, \sqrt{-g_{\rm eff}}\; \Psi_I^\dagger\, \mathcal{K}\, \Psi_J, \qquad \mathcal{A}^{(0)}_I = \int d^3x\, \sqrt{-g_{\rm eff}}\; \Pi \cdot \partial_{X^I}(\mathcal{S}_{\rm sol}, \Phi_{\rm sol}), \quad (6.42)$$

where Π denotes the canonical momenta of the microscopic fields evaluated on the family. The lifting potential is $U^{(0)}_{\rm mod}(X) = E_{\rm sol}(X) - \min E_{\rm sol}$ (cf. Eq. (6.29)).

Coupling coefficients S_{nI}. S_{nI} arise from cross-terms in the kinetic sector linear in $\dot{X}^I$ and a_n:

$$S_{nI}(X) = \int d^3x\, \sqrt{-g_{\rm eff}}\; \Xi_n^\dagger(\mathbf{x}; X) \Big[\mathcal{K}\, \Psi_I(\mathbf{x}; X) \Big] \;+\; \text{(model-specific single-derivative pieces)}. \quad (6.43)$$

In a purely second-order theory $S_{nI} = \langle \Xi_n, \Psi_I \rangle_{\mathcal{K}} = 0$ by orthogonality, and $\delta G_{IJ} = 0$ at this order; nontrivial δG_{IJ} then comes from X-dependence of Ξ_n (through $\partial_{X^I} \Xi_n$ terms). The formula (6.39) should be understood as a compact representation of these effects.

[1] If the microscopic kinetic sector is purely second-order in time, $\delta \mathcal{A}_I$ typically vanishes at this order; nonzero contributions arise from topological (WZW/CS-like) terms or complex-field structures, cf. Sec. 6.3.3.

6.4.5 Equations of motion on $\mathcal{M}_Q$

Euler–Lagrange equations. Varying $L_{\rm coll}$ with respect to X^I yields

$$G_{IJ}\,\ddot{X}^J + \Gamma_{IJK}\,\dot{X}^J\dot{X}^K + \Omega_{IJ}\,\dot{X}^J + \partial_I U_{\rm mod} \;=\; 0, \tag{6.44}$$

where $\Gamma_{IJK} = \tfrac{1}{2}(\partial_I G_{JK} + \partial_J G_{IK} - \partial_K G_{IJ})$ is the Levi–Civita connection of G_{IJ}, and $\Omega_{IJ} = \partial_I \mathcal{A}_J - \partial_J \mathcal{A}_I$ is the Berry curvature on $\mathcal{M}_Q$. Thus the motion is *geodesic* on $(\mathcal{M}_Q, G)$, with a "magnetic" force from Ω and a conservative force $-\nabla U_{\rm mod}$.

Hamiltonian form and Noether charges. The canonical one-form and Hamiltonian are

$$\Theta = (G_{IJ}\dot{X}^J + \mathcal{A}_I)\,dX^I, \qquad H = \frac{1}{2}G^{IJ}\,(P_I - \mathcal{A}_I)(P_J - \mathcal{A}_J) + U_{\rm mod}(X), \tag{6.45}$$

with $P_I = \partial L_{\rm coll}/\partial \dot{X}^I$. Symmetries acting isometrically on $(\mathcal{M}_Q, G, \mathcal{A}, U_{\rm mod})$ generate conserved charges by Noether's theorem. For translations ($X^i = x_0^i$), one recovers the total momentum; for rotations, body-fixed angular momentum with inertia I_{ab}; for internal $U(1)$, a conserved charge coupled through $\mathcal{A}$.

6.4.6 Spectral viscosity as Rayleigh dissipation (optional)

Rayleigh functional. When the microscopic spectral viscosity η is small but nonzero, the reduced dynamics acquires a positive-semidefinite Rayleigh term

$$\mathcal{R}_{\rm coll}(X,\dot{X}) = \frac{1}{2}R_{IJ}(X)\,\dot{X}^I\dot{X}^J, \qquad R_{IJ} \;\propto\; \eta \int d^3x\,\sqrt{-g_{\rm eff}}\,\big[\nabla\Psi_I\cdot\nabla\Psi_J\big], \tag{6.46}$$

leading to the damped equations $G_{IJ}\ddot{X}^J + \Gamma_{IJK}\dot{X}^J\dot{X}^K + \Omega_{IJ}\dot{X}^J + \partial_I U_{\rm mod} + R_{IJ}\dot{X}^J = 0$. Widths of quantized levels then scale as $\Gamma \sim \langle \dot{X}, R\,\dot{X}\rangle$ (Sec. 6.10).

6.4.7 Reparametrization and gauge invariances

Reparametrization covariance. Under a smooth change of coordinates $X \mapsto X'(X)$, G_{IJ} transforms as a metric, $\mathcal{A}_I$ as a connection one-form, and $U_{\rm mod}$ as a scalar; the spectrum depends only on the intrinsic geometry $(G, \Omega, U_{\rm mod})$.

One-form gauge. Shifts $\mathcal{A} \to \mathcal{A} + d\lambda$ leave $\Omega = d\mathcal{A}$ invariant; physical observables are unaffected provided the wavefunctions transform accordingly in the quantum theory (Sec. 6.5).

6.4.8 Path-integral viewpoint and measure

Collective-coordinate measure. The functional integral over fields factorizes (to leading order) into an integral over $\mathcal{M}_Q$ with measure $\mathcal{D}X\,\sqrt{\det G}$ times a product over gapped Gaussian integrals. The Faddeev–Popov determinant associated with the orthogonality gauge yields precisely $\sqrt{\det G}$.

Topological terms and line bundles. Nontrivial Ω implies a nontrivial prequantum line bundle over $\mathcal{M}_Q$; path integrals then include holonomies $\exp\!\big(i\oint \mathcal{A}\big)$, manifesting Berry phases and, in some families, WZW-like contributions (Sec. 6.9).

6.4.9 Worked example: isotropic core (MBP-0)

Block structure and geodesics. For a spherically symmetric core with moduli $X = (x_0^i, R^a, \vartheta)$,

$$G_{IJ} \simeq \mathrm{diag}\big(M\,\delta_{ij},\, I_0\,\delta_{ab},\, \mathcal{I}_\vartheta\big), \qquad \mathcal{A}_I = 0, \qquad U_{\rm mod} = 0 \quad \text{(along symmetry moduli)}. \tag{6.47}$$

The equations of motion decouple into a free center-of-mass, a rigid rotor, and (if present) a free internal phase. MBP-1 anisotropy introduces small splittings $I_\parallel \neq I_\perp$ and a weak alignment potential (cf. Eqs. (6.15)–(6.16)).

This completes the derivation of the collective Lagrangian. In the next section we construct the symplectic two-form and quantize the moduli dynamics, imposing the topological FR constraints that determine spin and statistics.

6.5 Symplectic Structure and Canonical Quantization on $\mathcal{M}_Q$

We quantize the collective dynamics derived in Sec. 6.4. The configuration space is the moduli manifold $\mathcal{M}_Q$ endowed with a Riemannian metric $G_{IJ}(X)$, a Berry connection $\mathcal{A}_I(X)$ with curvature $\Omega_{IJ} = \partial_I \mathcal{A}_J - \partial_J \mathcal{A}_I$, and a (possibly small) lifting potential $U_{\rm mod}(X)$. Dissipation is switched off ($\eta \to 0^+$) in this section; level widths from $\eta > 0$ enter later.

6.5.1 Phase-space one-form, symplectic two-form, and mechanical momenta

Canonical structures. From the collective Lagrangian $L_{\rm coll} = \frac{1}{2} G_{IJ} \dot{X}^I \dot{X}^J + \mathcal{A}_I \dot{X}^I - U_{\rm mod}$, the canonical momenta are

$$P_I \;=\; \frac{\partial L_{\rm coll}}{\partial \dot{X}^I} \;=\; G_{IJ} \dot{X}^J + \mathcal{A}_I. \tag{6.48}$$

The canonical one-form and two-form on phase space $T^*\mathcal{M}_Q$ are

$$\Theta \;=\; P_I \, dX^I, \qquad \Omega_{\rm ph} \;=\; d\Theta \;=\; dX^I \wedge dP_I. \tag{6.49}$$

It is convenient to define the *mechanical* (gauge-covariant) momenta

$$\Pi_I \;\equiv\; P_I - \mathcal{A}_I(X), \qquad \Rightarrow \qquad \{\Pi_I, \Pi_J\} \;=\; \Omega_{IJ}(X), \tag{6.50}$$

where $\{\cdot,\cdot\}$ are canonical Poisson brackets on (X^I, P_J). Thus the curvature $\Omega = d\mathcal{A}$ appears as a "magnetic field" on $\mathcal{M}_Q$ for the mechanical momenta Π_I.

Hamiltonian. Legendre transforming yields

$$H(X,P) \;=\; \frac{1}{2}\, G^{IJ}(X)\, \big(P_I - \mathcal{A}_I\big)\big(P_J - \mathcal{A}_J\big) \;+\; U_{\rm mod}(X) \;=\; \frac{1}{2}\, G^{IJ}\, \Pi_I \Pi_J + U_{\rm mod}. \tag{6.51}$$

6.5.2 Poisson algebra, constraints, and reduced phase space

Canonical brackets and Ω-twist. The canonical Poisson brackets are $\{X^I, P_J\} = \delta^I{}_J$, $\{X^I, X^J\} = 0$, $\{P_I, P_J\} = 0$. In terms of mechanical momenta:

$$\{X^I, \Pi_J\} = \delta^I{}_J, \qquad \{\Pi_I, \Pi_J\} = \Omega_{IJ}(X), \qquad \{X^I, X^J\} = 0. \tag{6.52}$$

If some X^I are redundant (first-class constraints), one fixes a smooth gauge on $\mathcal{M}_Q$ or passes to Dirac brackets; the final spectra are independent of such choices.

6.5.3 Canonical and geometric quantization

Hilbert space and inner product. Wavefunctions are square-integrable *sections* over $\mathcal{M}_Q$ of a Hermitian line bundle $\mathcal{L} \to \mathcal{M}_Q$ equipped with the connection $\mathcal{A}$:

$$\mathcal{H} \;=\; L^2\big(\mathcal{M}_Q,\; \mathcal{L},\; \sqrt{\det G}\, d^n X\big), \qquad n = \dim \mathcal{M}_Q. \tag{6.53}$$

The measure $\sqrt{\det G}\, d^n X$ is the Riemannian volume on $(\mathcal{M}_Q, G)$.

Prequantization condition. If $\mathcal{M}_Q$ contains compact two-cycles Σ, consistency requires integrality

$$\frac{1}{2\pi} \int_\Sigma \Omega \;\in\; \mathbb{Z}, \tag{6.54}$$

so that the line bundle $\mathcal{L}$ exists globally (Dirac quantization of Berry flux).

Operator representation. Choose the configuration representation where $\hat{X}^I$ acts by multiplication and

$$\widehat{\Pi}_I \;=\; -\,i\,\big(\nabla_I - i\,\mathcal{A}_I\big), \qquad \nabla_I \equiv \partial_I + \tfrac{1}{2}\,\partial_I \ln \sqrt{\det G}\,, \tag{6.55}$$

so that $\widehat{\Pi}_I$ is symmetric with respect to the inner product. Equivalently, ∇_I is the Levi–Civita covariant derivative acting on half-densities, ensuring coordinate invariance.

6.5.4 Hamiltonian operator and ordering

Laplace–Beltrami with Berry coupling. A natural self-adjoint ordering yields

$$\widehat{H} \;=\; -\frac{1}{2}\,\Delta_{\mathcal{A}} \;+\; U_{\mathrm{mod}}(X), \qquad \Delta_{\mathcal{A}}\psi \;\equiv\; \frac{1}{\sqrt{G}}\,\big(\partial_I - i\mathcal{A}_I\big)\,\sqrt{G}\,G^{IJ}\,\big(\partial_J - i\mathcal{A}_J\big)\psi, \tag{6.56}$$

where $G \equiv \det G_{IJ}$. This choice matches the classical Hamiltonian (6.51) under the correspondence principle and is covariant under reparametrizations and one-form gauge transformations $\mathcal{A} \to \mathcal{A} + d\lambda$ with $\psi \to e^{i\lambda}\psi$.

Curvature and commutators. The commutator of covariant derivatives reproduces the Berry curvature:

$$\big[\nabla_I - i\mathcal{A}_I,\; \nabla_J - i\mathcal{A}_J\big]\,\psi \;=\; -\,i\,\Omega_{IJ}\,\psi, \tag{6.57}$$

so nontrivial Ω shifts spectra and induces geometric phases along closed loops in $\mathcal{M}_Q$.

6.5.5 Boundary conditions, domains, and self-adjointness

Domains. For noncompact $\mathcal{M}_Q$ (e.g. translational sector $\mathbb{R}^3$), one demands square integrability and appropriate decay at infinity. For compact factors (e.g. orientation manifolds like $SO(3)$ or S^2) the domain consists of smooth sections obeying FR constraints (Sec. 6.6). When U_{mod} lifts a modulus softly, the ground state is localized near minima with harmonic-oscillator asymptotics.

Essential self-adjointness. On complete Riemannian manifolds with smooth G_{IJ} and bounded-below U_{mod}, the operator $-\frac{1}{2}\Delta_{\mathcal{A}} + U_{\mathrm{mod}}$ is essentially self-adjoint on $C_c^\infty(\mathcal{M}_Q)$; subtleties may arise at conical points or boundaries where the gap closes (to be excluded by gates).

6.5.6 Semiclassical quantization and Berry phases

Bohr–Sommerfeld rules. On a torus of action–angle variables (I_a, θ^a) obtained locally,

$$\oint_{\gamma_a} \big(P_I\,dX^I\big) \;=\; \oint_{\gamma_a} \big(\Pi_I\,dX^I\big) + \oint_{\gamma_a} \mathcal{A} \;\approx\; 2\pi\Big(n_a + \tfrac{\mu_a}{4}\Big), \tag{6.58}$$

with Maslov indices μ_a. The $\oint \mathcal{A}$ term implements Berry-phase shifts.

Selection rules from curvature. Nonzero Ω acts as an effective magnetic field on $\mathcal{M}_Q$ and alters level degeneracies and selection rules (e.g. monopole harmonics on S^2 instead of ordinary spherical harmonics).

6.5.7 Worked example: rigid rotor with Berry monopole

Setup. Let the orientation moduli form $\mathcal{M}_{\mathrm{rot}} \cong S^2$ (body axis) with metric $G_{S^2} = I_\perp\,(d\theta^2 + \sin^2\theta\,d\phi^2)$ and a Berry connection corresponding to a Dirac monopole of charge $q \in \frac{1}{2}\mathbb{Z}$:

$$\mathcal{A} \;=\; q\,(1-\cos\theta)\,d\phi, \qquad \Omega \;=\; q\,\sin\theta\,d\theta\wedge d\phi, \qquad \frac{1}{2\pi}\int_{S^2}\Omega \;=\; 2q \in \mathbb{Z}. \tag{6.59}$$

The Hamiltonian on S^2 reads

$$\widehat{H}_{\mathrm{rot}} \;=\; -\frac{1}{2I_\perp}\,\Delta_{\mathcal{A};S^2} \;=\; \frac{1}{2I_\perp}\Big(\mathbf{J}^2 - q^2\Big), \tag{6.60}$$

where $\mathbf{J}$ are monopole-angular-momentum operators.

Spectrum. Eigenvalues are

$$E_j \;=\; \frac{1}{2I_\perp}\,\big(j(j+1) - q^2\big), \qquad j = |q|, |q|+1, |q|+2, \ldots, \tag{6.61}$$

with degeneracy $2j+1$. A half-integer q implements an *intrinsic* shift that mimics fermionic behavior of a classically bosonic rotor—a mechanism closely related to FR constraints (Sec. 6.6).

6.5.8 Consistency gates for quantization

Gate Q1 (positivity). G_{IJ} positive-definite on the domain; numerically, all principal minors > 0.

Gate Q2 (regularity). G_{IJ}, $\mathcal{A}_I$, U_{mod} are smooth on $\mathcal{M}_Q$; no gap closures in the region probed by bound states (Sec. 6.3.6).

Gate Q3 (integrality). For each compact two-cycle, $(2\pi)^{-1}\int \Omega \in \mathbb{Z}$ (prequantization condition).

Gate Q4 (unitarity). $\widehat{H}$ symmetric on a dense domain and essentially self-adjoint; boundary terms vanish under FR-compatible boundary conditions.

Gate Q5 (reparametrization/gauge invariance). Spectra depend only on $(G_{IJ}, \Omega_{IJ}, U_{\mathrm{mod}})$; stable under X-coordinate changes and $\mathcal{A} \mapsto \mathcal{A} + d\lambda$ with $\psi \mapsto e^{i\lambda}\psi$.

With the symplectic and quantization framework in place, we now turn to the topological constraints imposed by the global structure of the configuration space—the Finkelstein–Rubinstein rules that determine spin and statistics of quantized solitons (Sec. 6.6).

6.6 Finkelstein–Rubinstein Constraints and Spin/Statistics

Quantization on a moduli space $\mathcal{M}_Q$ does not by itself fix whether the resulting excitations behave as bosons or fermions, nor does it determine whether a 2π spatial rotation acts trivially on physical states. These properties are set by the *global topology* of the *configuration space* of finite-energy fields in sector Q, and by how we identify physically indistinguishable configurations. The Finkelstein–Rubinstein (FR) construction encodes these global features as *discrete* constraints on the allowed wavefunctions.[2]

6.6.1 Configuration space, physical identifications, and π_1

Configuration space. Let $\mathcal{C}_Q$ denote the space of smooth, finite-energy field configurations $(\mathcal{S}, \Phi)$ in topological sector Q, obeying the boundary conditions of Ch. **??**. We first quotient by *redundancies* (microscopic gauge equivalences of the full theory), but *not* by global spatial symmetries nor by permutations of identical solitons:

$$\mathcal{C}_Q \;\equiv\; \frac{\{\text{finite-energy configs in } Q\}}{\text{microscopic gauge}}\,. \tag{6.62}$$

Physical configuration space. For N identical solitons one must further identify configurations that differ by a *physical* exchange of indistinguishable cores, and by rigid spatial rotations that act trivially at infinity (if so prescribed). The *physical* configuration space is

$$\widetilde{\mathcal{C}}_Q \;=\; \frac{\mathcal{C}_Q}{\sim_{\mathrm{phys}}}, \qquad \sim_{\mathrm{phys}} \text{ generated by soliton permutations and trivial rigid actions.} \tag{6.63}$$

The fundamental group $\pi_1(\widetilde{\mathcal{C}}_Q)$ classifies homotopy classes of closed, finite-energy *motions* in configuration space. Nontrivial loops γ in $\pi_1(\widetilde{\mathcal{C}}_Q)$ give rise to FR constraints.

Loops of interest. We distinguish two canonical classes:

1. 2π **spatial rotation** about some axis at infinity.
2. **Exchange loop** exchanging two identical solitons along a path that returns all fields to their original pointwise values at $t = 1$.

In three spatial dimensions these loops can be homotopically related, a fact that underlies spin–statistics assignments in soliton models.

[2] For classical treatments in topological soliton models see the original works by Finkelstein & Rubinstein and their implementations in Skyrme-type theories. Our adaptation to the spectral–aether setting follows the same logical steps but with the configuration space defined by the full field content $(\mathcal{S}, \Phi)$ and the spectral gates of Chs. 3–**??**.

6.6.2 FR representations and wavefunction constraints

FR homomorphism. Choose a *unitary character* (one-dimensional representation)

$$\sigma : \pi_1(\widetilde{\mathcal{C}}_Q) \longrightarrow \{\pm 1\}, \tag{6.64}$$

which assigns a sign to each homotopy class of loops. The choice of σ is part of the *superselection data* defining the quantization of sector Q; it must be consistent with the microscopic theory and with anomaly constraints (Sec. 6.9).

Constraint on wavefunctions. Let γ be a nontrivial loop on $\widetilde{\mathcal{C}}_Q$ and Γ its lift to the moduli space $\mathcal{M}_Q$ (or to a cover thereof when needed). The FR prescription demands

$$\psi\big(\gamma \cdot X\big) \;=\; \sigma(\gamma)\, \psi(X)\,, \qquad \forall\, X \in \mathcal{M}_Q. \tag{6.65}$$

In other words, the physical wavefunction is a *section* of a *flat* $\mathbb{Z}_2$ line bundle over $\mathcal{M}_Q$ associated to σ. This is independent of the Berry line bundle from Sec. 6.5; the two tensor together.

Spin and exchange. If the 2π rotation loop is *non-contractible* and $\sigma(2\pi \text{ rot}) = -1$, then *all* physical states acquire a minus sign under a 2π spatial rotation—i.e. they are fermionic. If $\sigma(2\pi \text{ rot}) = +1$, they are bosonic. Likewise, if the *exchange* loop has $\sigma = -1$, the exchange of identical solitons yields Fermi statistics.

6.6.3 Computing π_1 and the FR sign: algorithm

Step 1: identify the target space. Compactify physical space to S^3 and determine the effective target $\mathcal{T}$ swept by the fields $(\mathcal{S}, \Phi)$ at spatial infinity and in the bulk (Sec. 4.3).

Step 2: classify topological sector. The integer Q (degree, Hopf number, *etc.*) labels homotopy classes[3].

Step 3: analyze the action of rigid rotations/exchanges on $\mathcal{C}_Q$. Construct the loop induced by a 2π rotation of physical space and the loop that exchanges two identical cores. Determine whether these loops are contractible in $\widetilde{\mathcal{C}}_Q$.

Step 4: assign σ. Consistent with microscopic unitarity and anomaly inflow (Sec. 6.9), choose the character σ such that the desired spin–statistics relation holds. In many cases the topology *fixes* σ up to equivalence.

6.6.4 Canonical examples

Example A: spherically symmetric core, texture degree Q. For cores whose configuration space carries a nontrivial $\mathbb{Z}_2$ associated with the 2π rotation, one typically finds (by the standard covering arguments)

$$\sigma(2\pi \text{ rotation}) \;=\; (-1)^{Q}\,. \tag{6.66}$$

Thus odd-Q sectors quantize as fermions and even-Q sectors as bosons. This mirrors the well-known Skyrmion result and follows from the nontrivial lift of $SO(3)$ to $SU(2)$ when the field map represents Q wrappings.

Example B: twisted string with Hopf charge Q_{H}. For Hopf textures (Sec. 4.5), the exchange of two identical linked tubes can be homotopic to a 2π rotation around an axis at infinity. In a broad class of embeddings,

$$\sigma(2\pi \text{ rotation}) \;=\; \sigma(\text{exchange}) \;=\; (-1)^{Q_{\mathrm{H}}}, \tag{6.67}$$

leading again to fermionic quantization for odd Hopf number.

[3] For textures with no cores, $Q \in \pi_3(\mathcal{T})$. For vortex-like sectors, an integer winding in π_1 may be relevant. For mixed defects one may have a pair (Q_1, Q_2).

Example C: two-soliton sector and exchange. For well-separated identical solitons, the physical configuration space includes the quotient by the permutation that swaps centers. If the exchange loop is nontrivial and $\sigma(\text{exchange}) = -1$, the two-soliton wavefunction is antisymmetric under exchange, enforcing Pauli-like exclusion in bound-state spectra (Sec. 6.12).

6.6.5 Implementation on the moduli space

Double cover and projective representations. Let $\widehat{\mathcal{M}}_Q$ be the universal cover of $\mathcal{M}_Q$ (or of the pertinent orientation submanifold, e.g. $SU(2)$ covering $SO(3)$). Physical wavefunctions are single-valued on $\widehat{\mathcal{M}}_Q$ but obey

$$\psi(\hat{X} \cdot g) \;=\; \sigma(g)\,\psi(\hat{X}), \qquad g \in \pi_1(\mathcal{M}_Q) \subset \mathrm{Deck}(\widehat{\mathcal{M}}_Q \to \mathcal{M}_Q). \tag{6.68}$$

Equivalently, they transform in a *projective representation* of the isometry group of $\mathcal{M}_Q$ dictated by σ.

Shift of allowed j. For an orientation manifold $SO(3)$, the FR constraint determines whether the allowed angular-momentum quantum numbers j are integers (bosonic) or half-integers (fermionic). Concretely, if $\sigma(2\pi \text{ rot}) = -1$, basis functions are sections on $SU(2)$ that pick a minus sign under the nontrivial deck transformation; thus $j \in \{\frac{1}{2}, \frac{3}{2}, \dots\}$.

FR as a flat $\mathbb{Z}_2$ bundle. Operationally, one can incorporate FR by tensoring the Berry line bundle $\mathcal{L}$ with a *flat* $\mathbb{Z}_2$ bundle $\mathcal{L}_{\mathrm{FR}}$ over $\mathcal{M}_Q$:

$$\mathcal{L}_{\mathrm{phys}} \;=\; \mathcal{L} \otimes \mathcal{L}_{\mathrm{FR}}, \qquad \mathrm{Hol}_\gamma(\mathcal{L}_{\mathrm{FR}}) \;=\; \sigma(\gamma) \in \{\pm 1\}. \tag{6.69}$$

This keeps the differential operators of Sec. 6.5 intact while changing global boundary conditions.

6.6.6 Interplay with Berry phases and topological terms

Discrete vs. continuous data. FR constraints represent *discrete* ($\mathbb{Z}_2$) global data, while Berry phases encode *continuous* $U(1)$ holonomies. The physical bundle is their tensor product; both influence spectra (Sec. 6.5.7) and selection rules. In particular, a WZW-like term (Sec. 6.9) can shift effective spins but cannot change the *parity* (boson/fermion) fixed by FR.

Consistency with anomalies. The character σ must be compatible with anomaly inflow conditions derived from the microscopic spectral sector. This restricts admissible σ in sectors where global anomalies would otherwise obstruct quantization.

6.6.7 Worked mini-examples

Spherical core, Q odd: fermionic rotor. Take $\mathcal{M}_{\mathrm{rot}} \cong SO(3)$ with the FR assignment (6.66) for odd Q. The physical Hilbert space is built from $SU(2)$ Wigner functions with half-integer j only. In the presence of a Berry monopole of charge q (Sec. 6.5.7), the allowed j start at $\max\{|q|, \frac{1}{2}\}$.

Two identical cores: exchange sign. For two well-separated cores, $\mathcal{M}_2$ contains a factor $\mathbb{R}^3_{\mathrm{rel}} \setminus \{0\}$ modulo exchange. The nontrivial π_1 generated by the exchange loop enforces $\psi(\mathbf{R}) = -\psi(-\mathbf{R})$ when $\sigma(\text{exchange}) = -1$, forbidding s-waves and shifting the lowest partial wave to p-wave in the relative spectrum (Sec. 6.12).

6.6.8 FR consistency gates

- **FR-1 (well-definedness).** The character σ is a group homomorphism $\pi_1(\widetilde{\mathcal{C}}_Q) \to \{\pm 1\}$; independent of representative loops, stable under small deformations (no gap closures along the loop).
- **FR-2 (compatibility).** σ is compatible with microscopic symmetries and anomaly constraints; in particular, it respects orientation reversal and charge conjugation where applicable.

- **FR-3 (moduli implementation).** The FR constraint can be pushed down to a flat $\mathbb{Z}_2$ bundle on $\mathcal{M}_Q$ (or its appropriate cover), with well-defined action on wavefunctions used in Sec. 6.5.
- **FR-4 (spectral integrity).** The resulting spectrum is consistent with level counting from semiclassical quantization and with selection rules observed in two-body scattering (Ch. 7).

With spin/statistics fixed by FR constraints, we proceed in Sec. 6.7 to construct the single-soliton effective Hamiltonian, extract masses and rotational/vibrational spectra, and study how anisotropy and gauge couplings split multiplets.

6.7 Single-Soliton Spectrum: Masses, Spin, and Internal Multiplets

We now quantize *one* isolated soliton in sector Q by combining the collective Hamiltonian of Sec. 6.5 with the FR constraints of Sec. 6.6. Our goal is to extract: (i) the rest mass including one-loop corrections; (ii) the rotational spectrum (spin content) in the isotropic (MBP-0) and weakly anisotropic (MBP-1) regimes; (iii) low-lying vibrational excitations and their coupling to rotations; and (iv) the pattern of multiplets and selection rules induced by Berry phases and FR signs.

6.7.1 Effective Hamiltonian near a moduli minimum

Local coordinates. Let $X^I = (x_0^i,\ \xi^\alpha)$ split into center-of-mass (CoM) translations x_0^i and internal/orientation moduli ξ^α (e.g. Euler angles, internal phases). Suppose $U_{\rm mod}$ has a minimum at $\xi^\alpha = \xi^\alpha_\star$ (exactly flat along symmetry directions, softly lifted by anisotropy). Expanding around $\xi_\star$:

$$U_{\rm mod}(\xi) \;=\; \tfrac{1}{2}\,K_{\alpha\beta}\,\delta\xi^\alpha\delta\xi^\beta \;+\; \mathcal{O}(|\delta\xi|^3), \qquad \delta\xi^\alpha \equiv \xi^\alpha - \xi^\alpha_\star, \tag{6.70}$$

with $K_{\alpha\beta} \geq 0$. The moduli metric similarly expands as $G_{IJ} = \mathrm{diag}(M\delta_{ij},\, \mathcal{G}_{\alpha\beta}) + \mathcal{O}(\delta\xi)$.

Hamiltonian. The internal Hamiltonian (dropping free CoM motion) reads

$$\widehat{H}_{\rm int} \;=\; -\frac{1}{2}\,\Delta_{\mathcal{A};\mathcal{M}_{\rm int}} \;+\; \tfrac{1}{2}\,K_{\alpha\beta}\,\delta\xi^\alpha\delta\xi^\beta \;+\; E^{(0)}_{\rm sol} \;+\; \Delta M_{\text{1-loop}}, \tag{6.71}$$

where $\Delta_{\mathcal{A}}$ is the gauge-covariant Laplace–Beltrami operator on the internal moduli space with metric $\mathcal{G}_{\alpha\beta}$ and Berry one-form $\mathcal{A}_\alpha$ (Sec. 6.5), $E^{(0)}_{\rm sol}$ is the classical soliton energy, and $\Delta M_{\text{1-loop}}$ is the gapped-mode one-loop (Casimir) correction.

6.7.2 Mass and one-loop (Casimir) correction

Definition. The physical rest mass is

$$M_{\rm phys} \;=\; E^{(0)}_{\rm sol} \;+\; \Delta M_{\text{1-loop}}\ , \tag{6.72}$$

with

$$\Delta M_{\text{1-loop}} \;=\; \frac{1}{2}\sum_n \left(\omega_n - \omega_n^{\rm vac}\right)\mathcal{C}_n \;+\; \delta M_{\rm ct}. \tag{6.73}$$

Here $\{\omega_n\}$ are normal-mode frequencies of gapped fluctuations in the soliton background, $\{\omega_n^{\rm vac}\}$ the corresponding vacuum frequencies (same volume/UV regulator), $\mathcal{C}_n$ degeneracies, and $\delta M_{\rm ct}$ counterterms fixed by the renormalization conditions of Ch. 3. Gauge and reparametrization invariance ensure that the finite $\Delta M_{\text{1-loop}}$ depends only on intrinsic data (G, Ω) and the spectrum of $\mathbb{H}$ in Sec. 6.2.3.

Gate. **(M1)** $\Delta M_{\text{1-loop}}$ must be regulator- and scheme-independent after the matching specified in Ch. 3; numerically, check stability under UV cutoff variation.

6.7.3 Rotational spectrum: MBP-0 (isotropic rotor)

Geometry. For a spherically symmetric core, the internal moduli contain a pure orientation manifold $\mathcal{M}_{\rm rot} \cong SO(3)$ with metric

$$ds^2_{\rm rot} \;=\; I_0\left(d\theta^2 + \sin^2\theta\, d\phi^2 + \big(d\psi + \cos\theta\, d\phi\big)^2\right), \tag{6.74}$$

i.e. a rigid rotor with principal moment I_0. If the Berry curvature vanishes on $SO(3)$ (or is a flat connection), the Hamiltonian reduces to

$$\widehat{H}_{\rm rot} \;=\; \frac{\mathbf{J}^2}{2I_0}\,, \tag{6.75}$$

with $\mathbf{J}$ the body-fixed angular momentum. FR constraints (Sec. 6.6) fix whether $j \in \mathbb{Z}$ (bosonic) or $j \in \mathbb{Z} + \frac{1}{2}$ (fermionic).

Spectrum.

$$E_j^{(\text{MBP-0})} \;=\; M_{\rm phys} \;+\; \frac{j(j+1)}{2I_0}\,, \qquad j = j_{\rm min},\, j_{\rm min}+1, \ldots, \tag{6.76}$$

where $j_{\rm min} = 0$ for bosons and $j_{\rm min} = \frac{1}{2}$ for fermions. Degeneracy is $2j+1$. A nontrivial Berry monopole (Sec. 6.5.7) replaces $\mathbf{J}^2$ by $\mathbf{J}^2 - q^2$ and shifts $j_{\rm min}$ to $|q|$ (or $\max\{|q|, \frac{1}{2}\}$ with FR).

6.7.4 Rotational spectrum: MBP-1 (symmetric/asymmetric top)

Geometry. Weak spectral anisotropy splits moments $I_\perp \neq I_\parallel$ relative to the director $\hat{\chi}$, yielding a *symmetric top*. The rotational Hamiltonian is

$$\widehat{H}^{(\text{sym})}_{\rm rot} \;=\; \frac{1}{2I_\perp}\,\mathbf{J}^2 \;+\; \Big(\frac{1}{2I_\parallel} - \frac{1}{2I_\perp}\Big)\,J_3^2 \;+\; U_{\rm align}(\theta) \;+\; \text{Berry/FR terms.} \tag{6.77}$$

Here J_3 is the body-axis component, and $U_{\rm align}(\theta) \simeq \frac{1}{2}\kappa_{\rm ani}\cos^2\theta$ is a small alignment potential (Sec. 6.3.4).

Good quantum numbers. When $U_{\rm align}$ is negligible, eigenstates are Wigner D-functions $|j, m, K\rangle$ with

$$E^{(\text{sym})}_{j,K} \;=\; M_{\rm phys} \;+\; \frac{j(j+1)}{2I_\perp} \;+\; \Big(\frac{1}{2I_\parallel} - \frac{1}{2I_\perp}\Big)K^2, \qquad K = -j, -j+1, \ldots, j. \tag{6.78}$$

FR and Berry constraints restrict the allowed set of (j, K) (e.g. only half-integer j, or $K \equiv q \,(\mathrm{mod}\, 1)$ if a monopole lives on S^2). A small $U_{\rm align}$ mixes K and lifts degeneracies at $\mathcal{O}(\kappa_{\rm ani})$.

Asymmetric corrections. If $I_1 \neq I_2 \neq I_3$, use the standard asymmetric-top Hamiltonian with small splittings around the symmetric case; perturbative diagonalization in the $|j, m, K\rangle$ basis suffices when $|I_a - I_b| \ll I_a$.

6.7.5 Vibrational excitations and roto-vibrational couplings

Normal modes. Let $\{\varphi^a\}$ denote small-amplitude vibrational coordinates in the internal manifold near $\xi_\star$, with quadratic potential $\frac{1}{2}\sum_a \omega_a^2 (\varphi^a)^2$ and kinetic metric μ_{ab}. The vibrational Hamiltonian is

$$\widehat{H}_{\rm vib} \;=\; \sum_a \Big[-\frac{1}{2}\frac{1}{\sqrt{\mu}}\partial_{\varphi^a}\big(\sqrt{\mu}\,\mu^{ab}\partial_{\varphi^b}\big) + \frac{1}{2}\omega_a^2(\varphi^a)^2\Big] \;+\; \mathcal{O}(\varphi^3). \tag{6.79}$$

Eigenvalues are $(n_a + \frac{1}{2})\omega_a$ up to Berry/FR-induced selection rules.

Roto-vibrational couplings. At next order, cross-terms $\sim \sum_a \lambda_a\, \varphi^a\, \mathbf{J}^2$ and $\sim \sum_a \lambda'_a \varphi^a J_3^2$ appear from the X-dependence of G_{IJ}, producing K-dependent shifts of vibrational levels and small splittings of rotational multiplets.

6.7.6 Internal $U(1)$ charge and minimal couplings

Phase modulus. If a global $U(1)$ acts as $\Phi \to e^{i\vartheta}\Phi$ and survives in the soliton family, the phase modulus yields a kinetic term $\frac{1}{2}\mathcal{I}_\vartheta \dot{\vartheta}^2$ and, upon quantization, a conserved charge operator

$$\widehat{Q} \;=\; -i\,\partial_\vartheta \;+\; q_{\text{Berry}}, \tag{6.80}$$

where q_{Berry} accounts for possible holonomies in ϑ-space. In emergent-gauge backgrounds (Ch. 8) this becomes minimally coupled to the effective gauge potential via $\dot{\vartheta} \to \dot{\vartheta} - A_0^{\text{eff}}$.

Gyromagnetic-like couplings. If the orientation couples to spectral gauge curvature (via Ω on $\mathcal{M}_{\text{rot}}$), spin–charge couplings of the form $\sim \mathbf{J}\cdot\mathbf{B}_{\text{spec}}$ arise, shifting rotational levels linearly in the background "magnetic" fields on moduli space.

6.7.7 Summary: quantum numbers and multiplets

Labels. A single-soliton state is labeled by

$$\big|Q;\; j,m;\; K;\; \{n_a\};\; Q_{\text{int}}\big\rangle_{\text{FR, Berry}},$$

subject to: (i) FR constraint on (j,K) (bosonic vs fermionic towers, exchange parities); (ii) Berry selection rules (monopole charges, allowed K ladders); (iii) any softly lifted directions encoded in U_{mod}.

Energies (schematic, MBP-1).

$$E \;\simeq\; M_{\text{phys}} + \frac{j(j+1)}{2I_\perp} + \Big(\frac{1}{2I_\parallel} - \frac{1}{2I_\perp}\Big)K^2 + \sum_a \Big(n_a + \tfrac{1}{2}\Big)\omega_a + \delta E_{\text{Berry/FR}} + \delta E_{\text{align}}, \tag{6.81}$$

with $\delta E_{\text{align}} = \mathcal{O}(\kappa_{\text{ani}})$ and $\delta E_{\text{Berry/FR}}$ fixed by the curvature/FR data.

6.7.8 Consistency gates for single-soliton spectra

- **S1 (positivity).** $I_a > 0$, $\omega_a^2 > 0$; no tachyonic vibrational modes.
- **S2 (FR compliance).** Allowed (j,K) sets obey the chosen σ on 2π rotations/exchanges.
- **S3 (Berry integrality).** If $\mathcal{M}_{\text{int}}$ contains compact 2-cycles, $(2\pi)^{-1}\int \Omega \in \mathbb{Z}$; spectra are invariant under $\mathcal{A} \to \mathcal{A} + d\lambda$.
- **S4 (MBP continuity).** The MBP-1 splittings vanish smoothly as anisotropy parameters $\to 0$, reproducing MBP-0 results.
- **S5 (UV stability).** M_{phys} and low-lying levels are insensitive to UV regularization within the renormalization scheme fixed in Ch. 3.

The single-soliton spectrum sets the stage for *interactions*. In Sec. 6.8 we turn to two-soliton sectors, exchange statistics, and the emergence of selection rules in scattering and molecular bound states.

6.8 Two–Soliton Sector: Exchange, Relative Dynamics, and Bound States

We now construct the quantum mechanics of *two* well–separated solitons in a fixed topological sector $Q = Q_1 \oplus Q_2$, including the consequences of exchange (FR) signs, Berry phases on the relative moduli, and spectral–curvature–induced forces. Throughout we work in the adiabatic regime where each core remains close to its single–soliton family (Secs. 6.1–6.4) while the collective coordinates evolve slowly.

6.8.1 Kinematics and configuration space

Product moduli and symmetry reduction. Let $X^I_{(a)}$ be moduli of soliton $a = 1, 2$ (translations $x^i_{0(a)}$, orientations $R_{(a)}$, internal phases, soft shape coordinates). The direct product $\mathcal{M}_{Q_1} \times \mathcal{M}_{Q_2}$ carries a block metric and one–form $(G^{(1)} \oplus G^{(2)},\ \mathcal{A}^{(1)} \oplus \mathcal{A}^{(2)})$ at large separations. Quotienting by overall translations yields the CoM $X_{\rm cm}$ and a *relative* separation $\mathbf{R} = \mathbf{x}_{0(1)} - \mathbf{x}_{0(2)}$. We also form relative orientations $R_{\rm rel} = R^{-1}_{(1)} R_{(2)}$ and relative internal phases $\vartheta_{\rm rel}$ when present.

Reduced metric (leading order). With masses M_1, M_2, the translational block reduces to

$$G_{\rm trans} \;=\; M_{\rm tot}\,\delta_{ij}\,dX^i_{\rm cm} dX^j_{\rm cm} \;+\; \mu\,\delta_{ij}\,dR^i dR^j, \qquad M_{\rm tot} = M_1 + M_2, \quad \mu = \frac{M_1 M_2}{M_1 + M_2}, \tag{6.82}$$

plus orientation/phase blocks inherited from the single–soliton metrics (Sec. 6.3). Corrections $\mathcal{O}(e^{-R/\lambda})$ and $\mathcal{O}(R^{-n})$ arise from profile overlap and long–range spectral curvature.

6.8.2 FR exchange and boundary conditions

Exchange loop. For identical solitons, the physical configuration space is the quotient by exchange. Let $\sigma_{\rm ex} \in \{\pm 1\}$ be the FR sign assigned to the exchange loop (Sec. 6.6). The two–body wavefunction obeys the *twisted* boundary condition

$$\Psi\big(\mathbf{R}, R_{\rm rel}, \vartheta_{\rm rel}\big) \;=\; \sigma_{\rm ex}\ \Psi\big(-\mathbf{R},\, \mathcal{E}\cdot R_{\rm rel},\, \vartheta_{\rm rel}\big), \tag{6.83}$$

where $\mathcal{E}$ is the induced map on relative orientations by exchanging labels $1 \leftrightarrow 2$. For pure relative motion (frozen orientations), Eq. (6.83) enforces *even* partial waves if $\sigma_{\rm ex} = +1$ and *odd* partial waves if $\sigma_{\rm ex} = -1$.

6.8.3 Berry phases on the relative sphere

Relative Berry one–form. The product Berry connection decomposes as $\mathcal{A}_{\rm tot} = \mathcal{A}^{(1)}(X_{(1)}) + \mathcal{A}^{(2)}(X_{(2)})$. Upon reduction to CoM/relative variables one obtains a one–form on the relative orientation and separation manifold,

$$\mathcal{A}_{\rm rel} \;=\; q_{\rm rel}\,(1 - \cos\theta)\,d\phi \;+\; \cdots, \qquad \Omega_{\rm rel} = d\mathcal{A}_{\rm rel} = q_{\rm rel}\,\sin\theta\,d\theta \wedge d\phi + \cdots, \tag{6.84}$$

where (θ, ϕ) are angles of $\hat{\mathbf{R}}$ and $q_{\rm rel} \in \frac{1}{2}\mathbb{Z}$ is a monopole charge set by the difference/sum of single–core Berry charges and by topological linking data.[4] The Laplace–Beltrami on S^2 is thus replaced by the *monopole* Laplacian, shifting the centrifugal barrier $L(L+1) \to L(L+1) - q^2_{\rm rel}$ and modifying selection rules.

6.8.4 Asymptotic interaction: multipole and spectral curvature

Large–R expansion. At $R \gg \lambda$ (the larger core size), the interaction admits a systematic expansion

$$V(\mathbf{R}, R_{\rm rel}, \vartheta_{\rm rel}) \;=\; V_{\rm cent}(R) \;+\; V_{\rm tens}(R)\,\mathcal{T}(\hat{\mathbf{R}}, R_{\rm rel}) \;+\; V_{\rm so}(R)\,\mathbf{L}\cdot\mathbf{S}_{\rm eff} \;+\; \cdots, \tag{6.85}$$

where $\mathcal{T}$ is a rank–2 tensor built from the director $\hat{\chi}$ and body axes, and $\mathbf{S}_{\rm eff}$ denotes effective spin/orientation generators on the internal manifold. Spectral curvature $F_{\rm spec}$ induces environment–dependent tails; a minimal parametrization is

$$V_{\rm cent}(R) \;=\; -\frac{C_p}{R^p} \;+\; \mathcal{O}(R^{-(p+1)}), \qquad p \geq 3, \tag{6.86}$$

with C_p a functional of background $(\rho, \eta, \hat{\chi})$ and core polarizabilities. Anisotropy generates tensor pieces $\propto R^{-p'}$ with $p' \leq p$.

[4] The ellipses denote additional contributions on orientation/phase factors; the monopole shown is the minimal piece living on $S^2_{\hat{\mathbf{R}}}$.

Short–range repulsion. Overlap of cores and spectral–curvature backreaction produce a universal repulsive core

$$V_{\rm rep}(R) \;\simeq\; \frac{\alpha_{\rm rep}}{R^n}, \quad n \geq 2, \quad \alpha_{\rm rep} > 0, \tag{6.87}$$

regularizing the two–body problem and setting a finite equilibrium distance for molecules.

6.8.5 Two–body collective Lagrangian

Relative Hamiltonian. Separating CoM, the leading relative Hamiltonian in the channel with fixed internal quantum numbers ν (roto–vibrational labels of each core) is

$$\widehat{H}^{(\nu)}_{\rm rel} \;=\; -\frac{1}{2\mu}\Big[\frac{1}{R^2}\partial_R\big(R^2\partial_R\big) - \frac{\mathbf{L}^2_{q_{\rm rel}}}{R^2}\Big] \;+\; U^{(\nu)}_{\rm ad}(R) \;+\; \widehat{W}^{(\nu)}_{\rm tens/so}(R) \;+\; \cdots, \tag{6.88}$$

where $\mathbf{L}^2_{q_{\rm rel}}$ is the monopole angular momentum operator, $U^{(\nu)}_{\rm ad}(R)$ is the *adiabatic* potential surface (Born–Oppenheimer energy obtained by solving the fast internal problem at fixed R), and $\widehat{W}$ encodes tensor and spin–orbit couplings between relative and internal degrees of freedom.

Born–Oppenheimer channels. Label eigenstates of the internal Hamiltonian of the two cores at fixed R by $\nu(R)$ with energies $E_\nu(R)$; then $U^{(\nu)}_{\rm ad}(R) = E_\nu(R) - M_1 - M_2$. Nonadiabatic couplings $\langle\nu|\partial_R|\nu'\rangle$ enter as off–diagonal gauge potentials and Landau–Zener mixings at avoided crossings.

6.8.6 Exchange, Berry, and partial–wave selection rules

FR selection. For identical cores with twisted condition (6.83), the relative orbital angular momentum L obeys

$$\sigma_{\rm ex} = +1 \;\Rightarrow\; L = 0, 2, 4, \ldots, \qquad \sigma_{\rm ex} = -1 \;\Rightarrow\; L = 1, 3, 5, \ldots, \tag{6.89}$$

up to additional constraints from internal state symmetry.

Monopole shift. A nonzero $q_{\rm rel}$ replaces spherical harmonics by *monopole harmonics* $Y_{q_{\rm rel},L,m}$, with $L \geq |q_{\rm rel}|$. The lowest centrifugal barrier is then $\frac{L(L+1)-q^2_{\rm rel}}{2\mu R^2}$, which can *lower* or *raise* thresholds depending on $q_{\rm rel}$.

6.8.7 Bound states: criteria and estimates

Shallow dimer (s–wave channel). When $\sigma_{\rm ex} = +1$ and $q_{\rm rel} = 0$, a shallow s–wave bound state forms if the scattering length $a_0 > 0$ with $E_b \simeq 1/(2\mu a_0^2)$. The repulsive core (6.87) sets a natural short–distance cutoff that stabilizes a_0.

P–wave dimers (fermionic exchange). For $\sigma_{\rm ex} = -1$ the lowest allowed partial wave is $L = 1$. A shallow p–wave bound state requires a sufficiently strong R^{-p} attraction and small centrifugal barrier; monopole shifts with $|q_{\rm rel}| = 1/2$ can assist or hinder binding by modifying the effective $L(L+1)$.

Equilibrium separation and spectra. If $V_{\rm rep} + V_{\rm cent}$ develops a minimum at $R = R_b$, small oscillations yield vibrational levels $\omega_b^2 = V''(R_b)/\mu$ and roto–vibrational couplings from the R–dependence of inertia tensors (cf. Sec. 6.7.5).

6.8.8 Scattering: phase shifts and resonances (preview)

Partial waves with monopole. The radial Schrödinger equation in channel ν reads

$$\Big[-\frac{1}{2\mu}\frac{d^2}{dR^2} + \frac{L(L+1)-q^2_{\rm rel}}{2\mu R^2} + U^{(\nu)}_{\rm ad}(R)\Big]u^{(\nu)}_L(R) \;=\; E\,u^{(\nu)}_L(R), \tag{6.90}$$

with FR–restricted L. Phase shifts $\delta^{(\nu)}_L(k)$ define cross sections and, together with nonadiabatic couplings, narrow Feshbach–like resonances when an excited channel supports a quasi– bound state near threshold (detailed in Ch. 7).

6.8.9 Consistency gates for two–body dynamics

- **T1 (adiabatic gate).** Channel gaps along the relative motion exceed kinetic scales: $\min_{\nu\neq\nu'} |E_\nu(R) - E_{\nu'}(R)| \gg \langle \dot{X} \rangle$ in the support of the wavefunction.
- **T2 (causality gate).** Group velocities derived from $G_{\rm rel}$ and channel dispersions do not exceed the effective causal cone of $g_{\rm eff}$.
- **T3 (FR/Berry gate).** Monopole charges obey integrality on $S^2_{\hat{\mathbf{R}}}$ and FR boundary conditions are imposed consistently on all coupled channels.
- **T4 (stability gate).** The repulsive core regularizes short distances; no collapse of the two–body problem, i.e. $V(R) \to +\infty$ as $R \to 0$ fast enough to ensure self–adjointness.

This completes the setup of the two–soliton sector. In Sec. 6.9 we analyze topological terms and anomaly inflow that constrain Berry data and FR assignments. Detailed scattering, resonances, and molecular spectra are developed in Ch. 7.

6.9 Topological Terms and Anomaly Inflow on the Moduli Space

Topological terms in the microscopic spectral–aether action induce *global* phases in the path integral that are insensitive to local deformations of the fields. Upon reduction to the collective sector, these phases descend to Berry curvatures, Chern–Simons/Wess–Zumino–Witten (WZW) functionals on the moduli space, and discrete (often $\mathbb{Z}_2$) signs that couple to the Finkelstein–Rubinstein (FR) data. This section develops a unified descent from bulk topological densities to moduli–space terms and states the associated quantization and anomaly–inflow constraints.

6.9.1 Microscopic topological sector: representatives

Pontryagin (θ) term for spectral curvature. Let $F_{\rm spec}$ denote the spectral curvature two–form built from the emergent spectral connection $\mathcal{A}_{\rm spec}$ (Sec. 3.7). In 3+1 dimensions,

$$S_\theta \;=\; \frac{\theta_{\rm spec}}{8\pi^2} \int_{M^4} \mathrm{tr}\big(F_{\rm spec} \wedge F_{\rm spec}\big), \qquad \theta_{\rm spec} \sim \theta_{\rm spec} + 2\pi, \tag{6.91}$$

is a total derivative; it affects the theory nonperturbatively (e.g. by weighting sectors with different Pontryagin indices).

Hopf term for director textures. For configurations where the normalized director $\hat{\chi} : S^3_\infty \to S^2$ is the relevant order parameter (Sec. 4.5), one can define a Hopf functional

$$S_{\rm Hopf} \;=\; \kappa_{\rm H} \int_{M^3_t} A_{\rm H} \wedge dA_{\rm H}, \qquad dA_{\rm H} \;=\; \hat{\chi}^*(\omega_{S^2}), \tag{6.92}$$

with ω_{S^2} the unit area form on S^2 and M^3_t a spacelike slice (time integral understood). Its variation is metric–independent; it assigns phases $e^{i\kappa_{\rm H} Q_{\rm H}}$ to Hopf number $Q_{\rm H}$.

WZW term for internal target/coset. When the slow fields define a map $U : M^4 \to \mathcal{T}$ into a target manifold (group or coset) with a closed five–form ω_5 generating $H^5(\mathcal{T}, \mathbb{Z})$,

$$S_{\rm WZW} \;=\; 2\pi k \int_{B^5} U^*(\omega_5), \qquad \partial B^5 = M^4, \qquad k \in \mathbb{Z}, \tag{6.93}$$

is well–defined modulo 2π for integer level k. Gauging appropriate symmetries produces Chern–Simons (CS) boundary terms via descent, capturing anomaly inflow.

6.9.2 From bulk topology to moduli–space Berry data

Adiabatic pushforward. Consider the time–dependent soliton family $(\mathcal{S}_{\rm sol}, \Phi_{\rm sol})(\mathbf{x}; X(t))$ with moduli X^I. The *topological* variation of the microscopic action along a slow loop $X^I(t)$ induces a geometric phase

$$\Delta S_{\rm top}[X] \;=\; \int dt\; \mathcal{A}^{\rm top}_I(X)\, \dot{X}^I\,, \qquad \Omega^{\rm top}_{IJ} \equiv \partial_I \mathcal{A}^{\rm top}_J - \partial_J \mathcal{A}^{\rm top}_I, \tag{6.94}$$

that adds to the Berry connection of Sec. 6.5. Equivalently, the *Berry curvature* on $\mathcal{M}_Q$ is obtained by pushing forward the bulk topological densities with the zero–mode projector (cf. Eq. (6.36)):

$$\Omega^{\rm top}_{IJ}(X) \;=\; \int_{\mathbb{R}^3} d^3x\, \sqrt{-g_{\rm eff}}\; \mathcal{I}^{\rm top}_{IJ}\big[\Psi_I, \Psi_J;\; \mathcal{S}_{\rm sol}\big], \tag{6.95}$$

where $\mathcal{I}^{\rm top}_{IJ}$ is bilinear in zero modes and local in the background fields.

Descent examples.

- **θ-descent:** If $F_{\rm spec}$ depends on X only through the soliton background, $S_\theta \to \int dt\, \mathcal{A}^{(\theta)}_I \dot{X}^I$ with $\Omega^{(\theta)}_{IJ} \propto \theta_{\rm spec} \int \mathrm{tr}\big(\delta_I \mathcal{A}_{\rm spec} \wedge \delta_J \mathcal{A}_{\rm spec}\big)$.
- **Hopf–descent:** Variations $\delta_I \hat{\chi}$ induce $\Omega^{\rm Hopf}_{IJ} \propto \kappa_{\rm H} \int A_{\rm H} \wedge d\big(\delta_I A_{\rm H} \wedge \delta_J A_{\rm H}\big)$, equivalent to a monopole bundle over the orientation submanifold (Sec. 6.5.7).
- **WZW–descent:** The $5d$ integral reduces, under $U(X(t))$, to a $1d$ CS form on the worldline, yielding a quantized Berry holonomy $\mathrm{Hol}_\gamma = e^{2\pi i k\, \deg(U|_\Sigma)}$ for any two–cycle $\Sigma \subset \mathcal{M}_Q$ traced by γ.

6.9.3 Quantization, integrality, and global consistency

Prequantization of $\Omega^{\rm top}$. For each compact two–cycle $\Sigma \subset \mathcal{M}_Q$,

$$\frac{1}{2\pi} \int_\Sigma \Omega^{\rm top} \;\in\; \mathbb{Z}, \tag{6.96}$$

ensuring the existence of a Hermitian line bundle $\mathcal{L}_{\rm top}$ with connection $\mathcal{A}^{\rm top}$. When bulk levels are quantized (e.g. $k \in \mathbb{Z}$ in WZW), Eq. (6.96) follows automatically.

Combination with FR. The physical bundle is $\mathcal{L}_{\rm phys} = \mathcal{L}_{\rm Berry} \otimes \mathcal{L}_{\rm top} \otimes \mathcal{L}_{\rm FR}$, where $\mathcal{L}_{\rm FR}$ is a *flat* $\mathbb{Z}_2$ bundle encoding the FR character (Sec. 6.6). Holonomies factorize, but selection rules depend on the *sum* of connection one–forms.

CP and time reversal. $\theta_{\rm spec}$ and certain Hopf couplings are odd under CP or time reversal. If microscopic CP is exact, their coefficients must vanish or be integer–quantized in a way that yields trivial holonomy on physically allowed cycles. Otherwise they provide controlled sources of CP violation in the collective spectrum (e.g. level splittings linear in $\theta_{\rm spec}$).

6.9.4 Anomaly inflow and moduli–space CS terms

Descent relations. Gauging a microscopic symmetry (global or emergent) introduces background connections $\mathcal{A}$. Bulk gauge variations of topological terms produce boundary CS functionals whose variation cancels the anomaly of edge/collective degrees of freedom:

$$\delta_\epsilon S^{\rm top}_{\rm bulk} \;=\; -\,\delta_\epsilon S^{\rm CS}_{\rm edge}, \qquad S^{\rm CS}_{\rm edge} \;=\; \frac{k_{\rm eff}}{4\pi} \int_{\mathbb{R}_t \times \mathcal{C}} \mathrm{tr}\Big(\mathcal{A} \wedge d\mathcal{A} + \tfrac{2}{3}\mathcal{A}^3\Big). \tag{6.97}$$

Here $\mathcal{C}$ denotes the configuration space locus traced by the soliton family.[5]

[5] In the collective reduction this effectively becomes a $1d$ CS term (a Berry one–form) pulled back from the space of slow parameters, with $k_{\rm eff}$ fixed by bulk topology.

Constraint on FR assignments. If the microscopic theory has a global anomaly obstructing a naive quantization, an FR choice that modifies the global boundary conditions on $\mathcal{M}_Q$ can restore consistency, provided the inflow precisely cancels the would–be obstruction. Thus anomaly inflow restricts admissible σ in Eq. (6.64).

6.9.5 Worked mini–examples

Isotropic rotor with WZW level k. On $\mathcal{M}_{\rm rot} \cong SO(3)$ (or its $SU(2)$ cover with FR), a WZW descent contributes a connection whose curvature integrates to $2\pi k$ over fundamental S^2 cycles. The rotational spectrum (6.60) is shifted by $q = k/2$ in Eq. (6.59), yielding $E_j = \frac{1}{2I_\perp}\big(j(j+1) - (k/2)^2\big)$ with allowed j constrained by FR (integer vs. half–integer).

Hopf skyrmion: FR and Hopf coupling. For textures characterized by $Q_{\rm H}$, a nonzero $\kappa_{\rm H}$ produces a monopole bundle on the relative S^2 (Sec. 6.8.3) with charge $q_{\rm rel} \propto \kappa_{\rm H} Q_{\rm H}$, modifying centrifugal barriers and bound–state thresholds. If simultaneously $\sigma(2\pi) = (-1)^{Q_{\rm H}}$, fermionic/bosonic towers and monopole shifts *both* constrain L.

6.9.6 Quantization gates for topological couplings

- **TQ-1 (integrality).** For every compact two–cycle $\Sigma \subset \mathcal{M}_Q$, $(2\pi)^{-1}\int_\Sigma \Omega^{\rm top} \in \mathbb{Z}$; WZW levels $k \in \mathbb{Z}$.
- **TQ-2 (compatibility with FR).** The combined holonomy of $\mathcal{L}_{\rm top} \otimes \mathcal{L}_{\rm FR}$ on generators of $\pi_1(\mathcal{M}_Q)$ reproduces the desired spin/statistics and leaves the partition function single–valued.
- **TQ-3 (CPT/unitarity).** For real time evolution, Berry curvatures are real and $\widehat{H}$ remains self–adjoint; CP–odd terms (e.g. $\theta_{\rm spec}$) are controlled either by symmetry or by small parameters with observable but unitary level shifts.
- **TQ-4 (UV insensitivity).** Topological contributions to spectra are regulator–independent after the matching specified in Ch. 3; numerically stable under cutoff variation.

Topological and anomaly data thus enter the collective quantum theory through quantized Berry curvatures and flat $\mathbb{Z}_2$ structures, tightly constraining allowed spectra and selection rules. In Sec. 6.10 we reintroduce weak spectral dissipation, derive linewidths and decoherence rates, and discuss the interplay between irreversible dynamics and the geometric phases described here.

6.10 Dissipation, Linewidths, and Emergent Irreversibility

We now reinstate weak spectral dissipation and noise—encoded by the viscosity field η and the fast gapped modes of $(\mathcal{S}, \Phi)$—and derive irreversible dynamics for the collective coordinates. Our goals are: (i) to obtain covariant Langevin equations and their fluctuation–dissipation structure on the moduli manifold; (ii) to construct a geometric Lindblad (GKSL) master equation for the reduced density operator on $\mathcal{M}_Q$; (iii) to compute spectral line broadening (finite lifetimes) for rotational and vibrational levels; and (iv) to relate entropy production to the arrow of time proposed in Chs. 3–**??**.

6.10.1 Microscopic dissipative sector: Rayleigh functional and noise

Rayleigh functional. In the microscopic theory, spectral dissipation is modeled by a positive semi-definite Rayleigh functional that penalizes gradients in the slow fields:

$$\mathcal{R} \;=\; \frac{1}{2}\int d^3x \sqrt{-g_{\rm eff}}\,\Big[\eta_\Phi\, g_{\rm eff}^{\mu\nu}(\nabla_\mu\Phi)^*(\nabla_\nu\Phi) \;+\; \eta_{AB}\, g_{\rm eff}^{\mu\nu} D_\mu \mathcal{S}^A D_\nu \mathcal{S}^B\, \mathcal{G}_{AB}\Big], \tag{6.98}$$

with $\eta_\Phi \geq 0$, η_{AB} a positive matrix on the structural target, and $\mathcal{G}_{AB}$ the structural metric (Sec. 2.4). Variation of the action picks up friction forces $-\delta\mathcal{R}/\delta\dot{\varphi}$ in the Euler–Lagrange equations.

Spectral noise and FDT. Integrating out fast gapped modes generates stochastic sources Ξ with zero mean and correlators obeying a fluctuation–dissipation theorem (FDT) at an effective spectral temperature $\Theta_{\rm spec}$:

$$\big\langle \Xi_\Phi(x)\,\Xi_\Phi(x')\big\rangle \;=\; 2\,\eta_\Phi\,\Theta_{\rm spec}\,\delta^{(4)}(x-x')\,, \qquad \big\langle \Xi_A(x)\,\Xi_B(x')\big\rangle \;=\; 2\,\eta_{AB}\,\Theta_{\rm spec}\,\delta^{(4)}(x-x')\,. \tag{6.99}$$

The separation of scales (Born–Markov) underlies the local-in-time form of (6.99).

6.10.2 Projection to moduli space: geometric Langevin dynamics

Friction and noise kernels. Projecting the dissipative forces onto zero modes Ψ_I (Sec. 6.2.2) yields a friction tensor and noise on $\mathcal{M}_Q$:

$$\gamma_{IJ}(X) = \int d^3x\sqrt{-g_{\rm eff}}\,\Big[\eta_\Phi\,(\Psi_I^\Phi)^*\Psi_J^\Phi + \eta_{AB}\,\Psi_I^A\Psi_J^B\,\mathcal{G}_{AB}\Big], \tag{6.100}$$

$$\langle \xi_I(t)\,\xi_J(t')\rangle = 2\,\Theta_{\rm spec}\,\gamma_{IJ}(X)\,\delta(t-t')\,. \tag{6.101}$$

Here Ψ_I^Φ and Ψ_I^A are the Φ and $\mathcal{S}$ components of the zero mode associated with X^I (cf. Eq. (6.36)). Positivity of $\{\eta\}$ implies $\gamma_{IJ}=\gamma_{JI}\geq 0$.

Geometric Langevin equation. Including Christoffel symbols $\Gamma^I{}_{JK}(X)$ of the moduli metric $G_{IJ}(X)$ (Sec. 6.3), the covariant stochastic dynamics reads

$$G_{IJ}\,\ddot X^J + \Gamma_{IJK}\,\dot X^J\dot X^K + \gamma_{IJ}\,\dot X^J + \partial_I U_{\rm mod}(X) \;=\; \xi_I(t), \tag{6.102}$$

with the noise correlator (6.101). In the overdamped regime (time scales $\gg$ inertial time G/γ) one obtains a covariant Smoluchowski dynamics on $\mathcal{M}_Q$.

Fokker–Planck equation. For the probability density $P(X,t)$ one finds

$$\partial_t P = \frac{1}{\sqrt{G}}\,\partial_I\Big\{\sqrt{G}\,G^{IJ}\big[\partial_J U_{\rm mod}\,P + \Theta_{\rm spec}\,(\gamma G^{-1})_J{}^K\,\partial_K P\big]\Big\}, \tag{6.103}$$

where $(\gamma G^{-1})_J{}^K = \gamma_{JL}G^{LK}$. Detailed balance holds when (γG^{-1}) is proportional to the identity, yielding stationary measure $P_*(X)\propto e^{-U_{\rm mod}(X)/\Theta_{\rm spec}}$.

6.10.3 Geometric Lindblad master equation on $\mathcal{M}_Q$

GKSL structure. The reduced density operator $\hat\rho$ on the Hilbert space $L^2(\mathcal{M}_Q,\mathcal{L}_{\rm phys})$ (Sec. 6.5) obeys a covariant GKSL equation:

$$\frac{d\hat\rho}{dt} = -i\big[\widehat H,\hat\rho\big] + \sum_\alpha\Big(\widehat L_\alpha\,\hat\rho\,\widehat L_\alpha^\dagger - \tfrac12\{\widehat L_\alpha^\dagger\widehat L_\alpha,\hat\rho\}\Big), \tag{6.104}$$

with $\widehat H$ given by (6.56) (including Berry/topological terms). A reparametrization- and gauge-invariant choice of jump operators is

$$\widehat L_a \;=\; \sqrt{\lambda_a(X)}\;e_a{}^I(X)\,\widehat\Pi_I, \qquad \gamma_{IJ}=\sum_a \lambda_a\,e_{aI}e_{aJ}, \tag{6.105}$$

where $\{e_a{}^I\}$ is a local orthonormal frame on $T\mathcal{M}_Q$ and $\widehat\Pi_I$ are the mechanical momenta (6.55). This structure reproduces the classical friction tensor γ_{IJ} and ensures complete positivity. A dephasing channel on configuration coordinates can be included via $\widehat L_b^{(\rm deph)}=\sqrt{\kappa_b(X)}\,f_b(X)$.

Quantum FDT and steady state. Under the KMS (thermal) condition for the bath at $\Theta_{\rm spec}$ and weak coupling, the GKSL generator admits a Gibbs–like steady state $\hat\rho_*\propto e^{-\widehat H/\Theta_{\rm spec}}$ whenever detailed balance holds in the frame $\{e_a\}$.

6.10.4 Linewidths, relaxation, and decoherence rates

General formula. For an eigenbasis $\widehat{H}|n\rangle = E_n|n\rangle$, the decay width of $|n\rangle$ due to (6.104) is

$$\Gamma_n \;=\; \sum_{\alpha} \Big(\langle n|\widehat{L}_\alpha^\dagger \widehat{L}_\alpha|n\rangle - \sum_{m} |\langle m|\widehat{L}_\alpha|n\rangle|^2 \, \delta_{E_m,E_n} \Big), \tag{6.106}$$

and the transition rate $n \to m$ is $\Gamma_{n\to m} = \sum_\alpha |\langle m|\widehat{L}_\alpha|n\rangle|^2$. Pure dephasing between $|n\rangle$ and $|m\rangle$ occurs at rate $\gamma_\phi^{(nm)} = \frac{1}{2}\sum_\alpha \big\langle (\Delta L_\alpha)^\dagger (\Delta L_\alpha) \big\rangle_{n,m}$. All rates scale linearly with the eigenvalues of γ_{IJ}.

Rigid rotor (MBP-0) line broadening. For $\widehat{L}_a \propto e_a{}^I \widehat{\Pi}_I$ on $\mathcal{M}_{\rm rot} \cong SO(3)$ with isotropic $\gamma_{IJ} = \gamma_{\rm rot}\, G_{IJ}$,

$$\Gamma_{j\to j\pm1} \;\propto\; \gamma_{\rm rot}\, \frac{j(j+1)}{I_0}, \qquad \gamma_\phi^{(j,m)} \;\propto\; \gamma_{\rm rot}\, \frac{m^2}{I_0}, \tag{6.107}$$

up to FR/Berry selection rules (Sec. 6.7). Anisotropy (MBP-1) adds K-dependent splittings and dephasing.

Vibrational relaxation. For a harmonic mode (φ, ω) with $\widehat{L} \propto \sqrt{\lambda}\,\widehat{\Pi}_\varphi$,

$$\Gamma_{n\to n-1} \;=\; \lambda\,\omega\, n, \qquad \Gamma_{n\to n+1} \;=\; \lambda\,\omega\,(n+1)\, e^{-\omega/\Theta_{\rm spec}}, \tag{6.108}$$

and the pure-dephasing rate scales as $\gamma_\phi \propto \kappa_{\rm deph}$ if a coordinate-type channel is present.

Two-soliton dimers. Near a molecular minimum R_b (Sec. 6.8.7), the breathing mode acquires a linewidth $\Gamma_b \simeq \lambda_b \omega_b$ with $\lambda_b \propto \gamma_{RR}(R_b)/\mu$. Orientation–translation couplings induce additional L-dependent broadening.

6.10.5 Entropy production and the arrow of time

Spohn inequality. For GKSL dynamics with stationary $\hat\rho_*$, the entropy production rate $\sigma \equiv -\frac{d}{dt} D(\hat\rho\|\hat\rho_*)$ (D the quantum relative entropy) is nonnegative:

$$\sigma(t) \;=\; -\,\mathrm{Tr}\Big[\big(\mathcal{L}\hat\rho\big)\, \big(\ln\hat\rho - \ln\hat\rho_*\big) \Big] \;\geq\; 0, \tag{6.109}$$

with $\mathcal{L}$ the GKSL generator. In the classical limit of (6.103), $\partial_t S + \nabla\cdot J_S = \sigma_{\rm cl} \geq 0$, consistent with the second law. Thus a positive viscosity sector ($\eta > 0$) endows the moduli dynamics with a preferred time orientation (emergent arrow).

6.10.6 Consistency gates for irreversible dynamics

- **D1 (positivity & CP).** γ_{IJ} positive semi-definite; jump operators of the form (6.105) guarantee complete positivity and trace preservation.
- **D2 (covariance).** Equations are invariant under reparametrizations of $\mathcal{M}_Q$ and under gauge transformations of the Berry bundle.
- **D3 (FDT).** Noise correlators and friction obey (6.101) at fixed $\Theta_{\rm spec}$; in the quantum case, KMS/detailed balance is respected.
- **D4 (unitary limit).** As $\eta \to 0$ (hence $\gamma \to 0$), GKSL reduces to unitary evolution with Hamiltonian (6.56).
- **D5 (causality).** Dissipative corrections do not enlarge the effective causal cone of $g_{\rm eff}$; relaxation times $\tau \sim \|\gamma G^{-1}\|^{-1}$ are $\gg$ micro-causal scales.

6.10.7 Worked mini–examples

Rotational decoherence of a fermionic core. With FR enforcing half-integer j and a small monopole charge q (Sec. 6.5.7), the lowest doublet $j = \frac{1}{2}$ splits by anisotropy and acquires a width $\Gamma \propto \gamma_{\rm rot}/I_\perp$, producing Lorentzian lines around $E = \frac{3}{8I_\perp} - \frac{q^2}{2I_\perp}$.

Dimer dissociation by overdamped motion. For a shallow s-wave dimer with binding energy E_b, an overdamped R-mode with $\gamma_{RR} \gg \mu\omega_b$ yields a Kramers-like escape rate $\Gamma_{\rm diss} \sim \frac{\omega_b}{2\pi} \exp(-E_b/\Theta_{\rm spec})$, modulated by the monopole barrier if $q_{\rm rel} \neq 0$.

6.10.8 Summary

Weak spectral viscosity induces a covariant friction tensor and noise on the soliton moduli space, leading to Langevin/Fokker–Planck dynamics and a geometric Lindblad equation. Linewidths and decoherence rates scale with γ_{IJ} and are shaped by Berry curvature and FR constraints. Entropy production is nonnegative, providing a concrete realization of the emergent arrow of time advocated earlier. These irreversible ingredients will play a central role in multi-soliton kinetics and scattering in Ch. 7.

6.11 Microscopic Matching and Renormalization on the Moduli Space

This section develops the one–loop (and near–one–loop) matching between the microscopic spectral–aether theory and the collective description on the soliton moduli space $\mathcal{M}_Q$. We (i) set up the background–field expansion and the functional determinants; (ii) choose symmetry–preserving regularizations (heat–kernel/ζ and exponential cutoffs); (iii) enumerate the local counterterm basis that renormalizes (G_{IJ}, Ω_{IJ}, $U_{\rm mod}$); (iv) define physical renormalization conditions and the spectral RG flow; (v) treat zero modes and IR issues; and (vi) present a worked mini–example (MBP–0 rotor).

6.11.1 Background–field setup and one–loop determinants

Background + fluctuations. Let $(\mathcal{S}_{\rm sol}, \Phi_{\rm sol})(\mathbf{x}; X)$ be a family of classical finite–energy solutions parametrized by moduli X^I (Secs. 6.1–6.4). Expand fields as

$$\mathcal{S}^A(x) = \mathcal{S}^A_{\rm sol}(\mathbf{x}; X(t)) \; + \; \sigma^A(x), \tag{6.110}$$

$$\Phi(x) = \Phi_{\rm sol}(\mathbf{x}; X(t)) \; + \; \varphi(x), \tag{6.111}$$

and split fluctuations into zero modes Ψ_I (spanned by ∂_{X^I} deformations, properly orthonormalized) and gapped modes $\widehat{\sigma}, \widehat{\varphi}$ orthogonal to all Ψ_I with respect to the kinetic inner product of Ch. 3.

Quadratic operator. The second variation of the microscopic action around $(\mathcal{S}_{\rm sol}, \Phi_{\rm sol})$ defines a block operator on the fluctuation multiplet $\Upsilon = (\widehat{\sigma}, \widehat{\varphi})$:

$$\mathbb{H} \; \equiv \; \begin{pmatrix} \mathbb{H}_{\mathcal{S}\mathcal{S}} & \mathbb{H}_{\mathcal{S}\Phi} \\ \mathbb{H}_{\Phi\mathcal{S}} & \mathbb{H}_{\Phi\Phi} \end{pmatrix}, \qquad S^{(2)} \; = \; \tfrac{1}{2} \int d^4x \; \Upsilon^\dagger \, \mathbb{H} \, \Upsilon, \tag{6.112}$$

where each block is a covariant differential operator (at most second order) built from $(g_{\rm eff}, \nabla_\mu, F_{\rm spec}, \mathcal{S}_{\rm sol}, \Phi_{\rm sol})$.

One–loop effective action. After separating zero modes (Sec. 6.2.2) and introducing the collective coordinate measure, the one–loop correction to the collective Lagrangian is

$$\Delta\Gamma^{(1)}[X, \dot{X}] \; = \; \frac{i}{2} \, \log \frac{\det\{\}' \mathbb{H}[X, \dot{X}]}{\det \mathbb{H}_{\rm vac}} \; + \; \Delta\Gamma_{\rm ct}[X, \dot{X}], \tag{6.113}$$

where $\det\{\}'$ excludes zero eigenvalues associated with Ψ_I, $\mathbb{H}_{\rm vac}$ is the operator in the trivial vacuum (same regulator/volume), and $\Delta\Gamma_{\rm ct}$ are local counterterms. Expanding (6.113) in $\dot{X}$ yields renormalizations of $U_{\rm mod}(X)$ (order $\dot{X}^0$), $G_{IJ}(X)$ and $\mathcal{A}_I(X)$ (order $\dot{X}^2$ and linear–in–$\dot{X}$ respectively).

6.11.2 Symmetry–preserving regularizations

Heat–kernel representation. For a positive elliptic operator $\mathcal{O}$ (after Wick rotation),

$$\ln \det \mathcal{O} \; = \; -\int_\varepsilon^\infty \frac{ds}{s} \, \mathrm{Tr}\, e^{-s\mathcal{O}}, \qquad \mathrm{Tr}\, e^{-s\mathcal{O}} \; \sim \; \sum_{n=0}^{\infty} a_n(\mathcal{O}) \, s^{(n-d)/2}, \tag{6.114}$$

with $d = 4$ the spacetime dimension. The Seeley–DeWitt coefficients a_n are integrals of local curvature invariants built from g_{eff} and background fields.

Zeta regularization. Equivalently,

$$\zeta_{\mathcal{O}}(z) = \sum_{\lambda>0} \lambda^{-z}, \qquad \ln\det\mathcal{O} \;=\; -\zeta'_{\mathcal{O}}(0), \tag{6.115}$$

which matches (6.114) after analytic continuation. We use heat–kernel locally to identify counterterms and ζ for spectral sums on compact moduli factors (e.g. $SO(3)$).

Exponential cutoff (covariant). For numerical work we also employ a covariant exponential regulator $\ln\det\mathcal{O} \to \ln\det\left(e^{-\mathcal{O}/\Lambda^2}\mathcal{O}\right)$, which preserves gauge/reparametrization symmetries manifestly; Λ is the spectral cutoff.

6.11.3 Local counterterm basis and renormalized data

Operator basis (up to two moduli derivatives). Divergences from a_0, a_1, a_2 renormalize a finite set of local operators on $\mathcal{M}_Q$:

$$\Delta\mathcal{L}_{\text{ct}} = -\delta U(X) \;-\; \tfrac{1}{2}\,\delta G_{IJ}(X)\,\dot{X}^I\dot{X}^J \;+\; \delta\mathcal{A}_I(X)\,\dot{X}^I \;+\; \cdots \tag{6.116}$$
$$\text{(higher–derivative terms suppressed by } \Lambda_{\text{eff}}^{-2}).$$

Here δU absorbs vacuum energy and core self–energy pieces; δG_{IJ} renormalizes the moduli metric (masses M, moments I_a, vibrational inertias); $\delta\mathcal{A}_I$ shifts the Berry one–form by an exact piece (curvature–preserving) plus quantized topological contributions.

Renormalized collective data. We define

$$U^{\text{R}}_{\text{mod}}(X) = U^{\text{cl}}_{\text{mod}}(X) + \Delta U^{(1)}(X) + \delta U(X), \quad G^{\text{R}}_{IJ}(X) = G^{\text{cl}}_{IJ}(X) + \Delta G^{(1)}_{IJ}(X) + \delta G_{IJ}(X), \tag{6.117}$$

and analogously for the Berry curvature $\Omega_{IJ} = \partial_I\mathcal{A}_J - \partial_J\mathcal{A}_I$. Quantized (topological) parts of Ω are protected and may only shift by integral classes (see Sec. 6.9); $\delta\mathcal{A}$ is chosen to keep integrality on compact 2–cycles.

6.11.4 Physical renormalization conditions (on–shell scheme)

We impose conditions that define observable collective quantities at a reference spectral scale μ_{spec}:

1. **Mass (rest energy).** $M_{\text{phys}} \equiv U^{\text{R}}_{\text{mod}}(X_\star) = E^{(0)}_{\text{sol}} + \Delta M_{\text{1-}loop}$, at the symmetry point $X_\star$ (e.g. isotropic orientation).

2. **Inertias.** Principal inertias I_a are fixed from small–ω rotational response: $\frac{\partial^2\mathcal{L}_{\text{eff}}}{\partial\Omega_a^2}\big|_{X_\star,\mu_{\text{spec}}} = I_a^{\text{phys}}$.

3. **Berry integrality.** For each compact 2–cycle $\Sigma \subset \mathcal{M}_Q$, $(2\pi)^{-1}\int_\Sigma \Omega^{\text{R}} \in \mathbb{Z}$ (Sec. 6.9.3); choose $\delta\mathcal{A}$ to enforce this at μ_{spec}.

4. **Vibrational curvatures.** Hessian eigenfrequencies ω_a at $X_\star$ match those extracted from Green–function poles of the microscopic two–point functions in the soliton background.

These conditions fix the counterterms uniquely up to scheme redefinitions that leave observables invariant.

6.11.5 Spectral RG flow and decoupling

Running with $\mu_{\rm spec}$. Changing the subtraction scale induces flow equations

$$\mu_{\rm spec}\frac{d}{d\mu_{\rm spec}}\,U^{\rm R}_{\rm mod}(X)=\beta_U(X), \tag{6.118}$$

$$\mu_{\rm spec}\frac{d}{d\mu_{\rm spec}}\,G^{\rm R}_{IJ}(X)=\beta^{(G)}_{IJ}(X), \tag{6.119}$$

$$\mu_{\rm spec}\frac{d}{d\mu_{\rm spec}}\,\Omega^{\rm R}_{IJ}(X)=0\quad\text{(topological sector)}\;+\;\partial_{[I}\beta^{(\mathcal{A})}_{J]}(X), \tag{6.120}$$

with β–functions determined by the a_2 coefficient of the relevant Hessians (cf. Eq. (6.114)). The topological (quantized) part of Ω is scale–invariant.

Decoupling theorem. Gapped microscopic modes with mass $m\gg\mu_{\rm spec}$ contribute only through local operators suppressed by m^{-2}; their effects can be absorbed into higher–derivative terms on $\mathcal{M}_Q$ and do not affect the leading (U,G,Ω) beyond threshold corrections.

6.11.6 Zero modes, Faddeev–Popov factors, and IR issues

Removing zero modes. We project $\mathbb{H}$ onto the subspace orthogonal to Ψ_I and define the prime determinant $\det{}'\mathbb{H}$. The path–integral measure acquires a Jacobian $\mathcal{J}(X)=\sqrt{\det G_{IJ}(X)}$ from the change of variables to collective coordinates (Sec. 6.3).

IR regulators. In infinite volume, vacuum subtraction in (6.113) removes power divergences; residual IR sensitivity is handled by (i) finite–volume boxes with identical boundary conditions in the soliton and vacuum backgrounds and then $V\to\infty$; or (ii) Pauli–Villars pairs tuned to cancel IR tails. Physical observables (mass differences, inertia renormalizations) are IR–finite.

6.11.7 Worked mini–example: MBP–0 spherical rotor

Setup. For an isotropic core, $\mathcal{M}_{\rm rot}\cong SO(3)$ with classical metric $ds^2=I_0\,\mathrm{Tr}(R^{-1}dR)^2$. Fluctuation operators separate into tangential zero–mode blocks and orthogonal gapped blocks. The one–loop corrections to mass and inertia read

$$\Delta M_{\text{1-}loop}=\frac{1}{2}\sum_n\left(\omega_n-\omega_n^{\rm vac}\right)\mathcal{C}_n\;+\;\delta M_{\rm ct}, \tag{6.121}$$

$$\Delta I_0=\frac{\partial^2}{\partial\Omega^2}\left[\frac{i}{2}\ln\frac{\det{}'\mathbb{H}(\Omega)}{\det\mathbb{H}_{\rm vac}}\right]_{\Omega=0}\;+\;\delta I_{\rm ct}, \tag{6.122}$$

where Ω is a small body–axis rotation velocity that minimally couples to $\mathbb{H}$ through the Noether current associated with spatial rotations.

Heat–kernel extraction. Writing $\mathrm{Tr}\,e^{-s\mathbb{H}(\Omega)}=\sum_{n\ge0}a_n(\Omega)\,s^{(n-4)/2}$, the scale–dependence of I_0 is controlled by $a_2(\Omega)$:

$$\mu_{\rm spec}\frac{d}{d\mu_{\rm spec}}I_0^{\rm R}\;=\;-\frac{1}{8\pi^2}\left[\partial_\Omega^2 a_2(\Omega)\right]_{\Omega=0}. \tag{6.123}$$

The counterterm $\delta I_{\rm ct}$ is fixed by the on–shell condition $I_0^{\rm R}(\mu_{\rm spec})=I_0^{\rm phys}$, while $\delta M_{\rm ct}$ sets $M_{\rm phys}$.

Berry curvature. If a microscopic WZW/Hopf term induces a monopole of charge q on $\mathcal{M}_{\rm rot}$, the curvature $\Omega=q\,\omega_{S^2}$ is quantized and does not run. Any $\delta\mathcal{A}$ is chosen exact so that $\int_{S^2}\Omega=2\pi q$ remains integer.

6.11.8 Consistency gates for matching and renormalization

- **R1 (symmetry preservation).** The regulator and counterterms preserve reparametrizations of $\mathcal{M}_Q$, microscopic gauge symmetries, and FR/Berry structures; Ω's quantized fluxes are invariant.
- **R2 (scheme independence).** Physical masses, inertias, and level spacings are independent of the chosen (heat–kernel/ζ/exponential) scheme once renormalization conditions are imposed.
- **R3 (UV locality).** Divergences are absorbed by local operators of the form (6.116); nonlocal remnants are finite and exponentially suppressed by the spectral gap.
- **R4 (IR finiteness).** After vacuum subtraction and zero–mode removal, ΔM and ΔI_a are volume–independent in the $V \to \infty$ limit.
- **R5 (topological integrity).** Integer periods of Ω on compact 2–cycles are preserved at all scales; any CP–odd $\theta_{\rm spec}$ dependence is explicit and controlled.

With renormalized collective data ($U^{\rm R}_{\rm mod}$, $G^{\rm R}_{IJ}$, $\Omega^{\rm R}$) fixed at $\mu_{\rm spec}$, the next section provides canonical benchmarks and numerical templates that implement this matching in practice.

6.12 Canonical Benchmarks and Ready-to-Run Numerics

This section provides a set of *canonical benchmarks* and *numerical templates* that implement the collective construction developed in Secs. 6.1–6.11. Each benchmark (A–D) specifies: (i) continuum definitions; (ii) boundary conditions and gauges; (iii) discretization choices; (iv) extraction formulas for ($U_{\rm mod}$, G_{IJ}, Ω_{IJ}); and (v) validation diagnostics and acceptance tolerances. The templates are "drop-in" and designed to be adapted across soliton sectors Q.

6.12.1 Benchmark A: MBP–0 spherical rotor

Continuum setup. Choose a spherically symmetric core with director aligned, such that the internal moduli are pure rotations $\mathcal{M}_{\rm rot} \cong SO(3)$ and the Berry curvature is trivial (or flat). The continuum target is the classical solution $(\mathcal{S}_{\rm sol}, \Phi_{\rm sol})$ minimizing the full energy at fixed Q.

Boundary conditions. Finite domain $r \in [0, R_{\rm max}]$ with $R_{\rm max} \gg \lambda$ (largest core size); impose regularity at $r = 0$ and Dirichlet matching to vacuum at $R_{\rm max}$: $\partial_r \mathcal{S}^A|_0 = 0$, $\partial_r \Phi|_0 = 0$; $\mathcal{S}^A(R_{\rm max}) = \mathcal{S}^A_{\rm vac}$, $\Phi(R_{\rm max}) = \Phi_{\rm vac}$.

Discretization. Spherical finite elements or finite differences with nonuniform radial mesh clustered near the core; 4th–order accurate stencils recommended. Solve the static Euler–Lagrange equations by Newton–Krylov or trust–region methods until relative residual $< 10^{-10}$.

Extraction of inertia and Berry data. Compute rotational zero modes $\Psi_{(a)}$ by infinitesimal body rotations, orthonormalize with the kinetic inner product (Ch. 3). The principal inertia is

$$I_0 \;=\; \sum_{\rm fields} \int d^3x \, \sqrt{-g_{\rm eff}} \; \Psi^\dagger_{(a)} \, \Psi_{(a)} \quad \text{(no sum over } a\text{)}, \tag{6.124}$$

and the Berry curvature vanishes at MBP–0 (or reduces to a flat connection that can be gauged away). Validate I_0 by the small–Ω slope of the rotating solution's energy: $E(\Omega) = M_{\rm phys} + \frac{1}{2} I_0 \Omega^2 + \mathcal{O}(\Omega^4)$.

Diagnostics and tolerances. Energy–virial identity error $< 10^{-9}$, orthogonality leakage of zero modes $< 10^{-8}$, grid convergence rate consistent with nominal order, and I_0 agreement between the two extraction methods at $< 10^{-3}$ relative.

6.12.2 Benchmark B: MBP–1 symmetric top

Continuum setup. Introduce weak spectral anisotropy (Sec. 6.3.4) so that $I_\perp \neq I_\parallel$ with a director $\hat{\chi}$. The moduli manifold includes Euler angles (θ, ϕ, ψ).

Discretization and alignment. Use a 3D mesh (cylindrical or spherical) with resolution aligned to the director to minimize locking errors. Extract zero modes as before; build the 3×3 inertia tensor $I_{ab} = \int \Psi^\dagger_{(a)} \Psi_{(b)}$ and diagonalize.

Validation. Check the symmetric–top spectrum by evolving a small-angle precession and fitting the frequency splitting. Verify MBP–1 $\to$ MBP–0 continuity as the anisotropy parameters $\to 0$: $I_\perp, I_\parallel \to I_0,\ U_{\text{align}} \to 0$ with deviations $< \mathcal{O}(\kappa_{\text{ani}})$.

6.12.3 Benchmark C: Hopfion Q_{H}=1 (topological Berry monopole)

Continuum setup. Construct a knotted/linked director texture $\hat{\chi}(\mathbf{x})$ with Hopf number $Q_{\text{H}} = 1$ (Sec. 4.5) and relax the full coupled system to a local minimum.

Berry curvature on $\mathcal{M}_{\text{rot}}$. Adiabatically rotate the core along loops covering the orientation S^2 and compute the geometric phase via *parallel transport* of the ground collective state: $\gamma[\Sigma] = \oint_{\partial\Sigma} \mathcal{A}$. Use a gauge–invariant discretization (link variables) to evaluate the curvature $\Omega = d\mathcal{A}$ and its flux through spherical patches. Validate integrality $(2\pi)^{-1} \int_{S^2} \Omega = q \in \frac{1}{2}\mathbb{Z}$ and compare with the continuum descent prediction $q \propto \kappa_{\text{H}} Q_{\text{H}}$ (Sec. 6.9).

6.12.4 Benchmark D: Shallow two–soliton dimer

Continuum setup. Prepare two well–separated cores in a channel with given internal quantum numbers (from Sec. 6.7); relax at fixed separation R to obtain the adiabatic potential $U_{\text{ad}}(R)$. Fit the long–range tail to $-C_p R^{-p}$ and the short range to $\alpha_{\text{rep}} R^{-n}$.

Relative dynamics. Solve the radial equation with monopole harmonics (Sec. 6.8.8) to extract s– or p–wave scattering length and, if present, the binding energy E_b. Validate against a full PDE time–domain simulation of a slow collision and phase–shift extraction.

6.12.5 Numerical templates: PDE solvers and operators

Static solver (shooting + relaxation).

1. Initialize with an ansatz respecting topological constraints (hedgehog, toroidal knot, etc.).
2. Alternate implicit relaxation of $(\mathcal{S}, \Phi)$ with constraint re-projection (e.g. unit norm of $\hat{\chi}$).
3. Employ line search and trust regions; stop when the maximum Euler–Lagrange residual $< \epsilon$.

Zero-mode extraction. Compute tangent deformations by finite differences in moduli X^I, then Gram–Schmidt using the kinetic inner product of Ch. 3. Project away any residual components along gapped modes using the Hessian preconditioner.

Hessian spectral solver. Discretize $\mathbb{H}$ with compatible stencils; use shift–invert Lanczos to obtain the lowest ~ 50 eigenpairs, separating zero/goldstone modes from gapped ones. Check orthogonality to zero modes and convergence under mesh refinement.

Berry curvature (link variables). For a discrete loop $X_1 \to X_2 \to X_3 \to X_4 \to X_1$ on a plaquette, define overlaps $U_{ij} = \langle \psi_0(X_i) | \psi_0(X_j) \rangle / |\cdot|$ and compute $\exp(i\Phi_\square) = U_{12} U_{23} U_{34} U_{41}$, so that $\Phi_\square$ approximates the flux $\int_\square \Omega$. Sum over a triangulation to obtain quantized flux on S^2.

6.12.6 Data layout, reproducibility, and error budgets

Metadata. Store (i) physical parameters; (ii) discretization details (mesh, order); (iii) solver tolerances; (iv) gauge/phase conventions; (v) random seeds for noise (if any).

Error budget. Decompose uncertainties into: truncation (mesh), nonlinear solve tolerance, spectral truncation, finite–volume, and regulator (for one–loop). Report each contribution and a combined 1σ band.

Versioning. Hash the PDE state and the moduli–space outputs (U, G, Ω); include CI tests that re-run reduced–resolution cases and compare hashed observables within tolerances.

6.12.7 Acceptance tests (pass/fail gates)

- **N1 (convergence).** Observables $(M, I_a, \omega_a, \Phi_{S^2})$ converge with mesh order and plateau under domain enlargement; successive–refinement change $< 10^{-3}$ relative.
- **N2 (constraints).** Normalization of constrained fields (e.g. $|\hat{\chi}| = 1$) and orthogonality of zero modes hold to $< 10^{-8}$ in L^∞ norm.
- **N3 (gauge/phase invariance).** Berry curvature and integrated fluxes invariant under local phase choices within numerical noise.
- **N4 (energy balance).** Virial/Noether checks pass at $< 10^{-9}$; rotating solutions recover $E(\Omega)$ quadratic law for $\Omega \ll 1$.
- **N5 (topological integrality).** $(2\pi)^{-1}\int_{S^2} \Omega \in \frac{1}{2}\mathbb{Z}$ to within 10^{-4} absolute.
- **N6 (cross–validation).** Inertia from overlaps (6.124) agrees with dynamical extraction (small Ω) and with response to external torque (Sec. 6.14) within 10^{-3} relative.

6.12.8 Summary

The four benchmarks (MBP–0, MBP–1, Hopfion, shallow dimer) and the accompanying numerical templates provide a reproducible path from the microscopic PDEs to the renormalized collective data $(U^{\mathrm{R}}_{\mathrm{mod}}, G^{\mathrm{R}}_{IJ}, \Omega^{\mathrm{R}}_{IJ})$ with quantified uncertainties. They will serve as the starting point for scattering, radiative processes, and external–probe responses in Secs. 6.14–6.16 and in Ch. 7.

6.13 Consistency Gates, Failure Modes, and Chapter Summary

This final section compiles the *consistency gates* introduced throughout the chapter into a single, actionable checklist; catalogs common *failure modes* and their fixes; and summarizes the deliverables that will be handed off to scattering (Ch. 7) and emergent gauge dynamics (Ch. 8). The goal is operational: turn every concept of Secs. 6.1–6.12 into pass/fail tests and an ordered recipe that transforms a microscopic soliton into a bona fide quantum particle with well–defined observables.

6.13.1 Gate catalog (pass/fail checklist)

Structural and variational gates (S).

- **S1 (existence & finite energy).** A static, finite–energy solution $(\mathcal{S}_{\mathrm{sol}}, \Phi_{\mathrm{sol}})$ exists in the desired topological sector Q with $E < \infty$ (Sec. 6.1).
- **S2 (zero modes & completeness).** Translational/orientational/internal zero modes Ψ_I exist and span the tangent to the moduli family; orthogonality and normalization are enforced by the kinetic inner product (Sec. 6.2).
- **S3 (positive Hessian).** The projected Hessian (with zero modes removed) is positive definite: no spurious tachyonic instabilities (Sec. 6.2.3).

- **S4 (hyperbolicity).** Linearized real–time equations are symmetric–hyperbolic around the background; mode speeds lie within the causal cone of g_{eff} (Sec. 6.3.3).
- **S5 (topological class).** The configuration is in the intended homotopy class (e.g. degree, Hopf number) and remains there under relaxation (Sec. 6.9).

Geometric and kinematical gates (G).

- **G1 (metric positivity).** The moduli–space metric $G_{IJ}(X)$ is positive definite; all principal inertias and reduced masses are > 0 (Sec. 6.3).
- **G2 (coordinate covariance).** All constructions are invariant under reparametrizations of $\mathcal{M}_Q$; results agree across charts within tolerance.
- **G3 (bounded curvature).** Sectional curvatures of $(\mathcal{M}_Q, G)$ are finite in the domain of interest (Sec. 6.3.5); no coordinate singularities contaminate dynamics.
- **G4 (adiabatic validity).** Separation of time scales: internal gaps along trajectories exceed collective kinetic scales (Sec. 6.7, 6.8).

Berry/FR/topological gates (TQ).

- **TQ1 (Berry integrality).** $(2\pi)^{-1}\int_\Sigma \Omega \in \mathbb{Z}$ for every compact two–cycle $\Sigma \subset \mathcal{M}_Q$; prequantum line bundle exists (Sec. 6.5, 6.9).
- **TQ2 (FR consistency).** FR characters $\sigma : \pi_1(\mathcal{Q}_Q) \to \{\pm 1\}$ implement the desired spin/statistics and are compatible with Berry holonomies (Sec. 6.6).
- **TQ3 (anomaly inflow).** Any microscopic anomalies are canceled by moduli–space CS/WZW descent; no obstructed loops remain (Sec. 6.9.4).
- **TQ4 (CPT/unitarity).** Geometric phases are real in real time; CP–odd deformations are explicit and controlled (Sec. 6.9.3).

Dissipation/irreversibility gates (D).

- **D1 (positivity).** Friction tensor $\gamma_{IJ} \succeq 0$; GKSL generator is completely positive (Sec. 6.10).
- **D2 (FDT).** Noise correlators satisfy fluctuation–dissipation at spectral temperature Θ_{spec}; detailed balance holds when applicable (Sec. 6.10.3).
- **D3 (unitary limit).** As $\eta \to 0$, dissipative corrections vanish and evolution reduces to unitary dynamics (Sec. 6.10.6).

Numerical/reproducibility gates (N). Adopt N1–N6 from Sec. 6.12.7: convergence, constraint preservation, gauge/phase invariance, energy balance, topological integrality, and cross–validation.

6.13.2 Failure modes and recommended fixes

Zero–mode leakage: Nonzero overlap of Ψ_I with gapped modes ⇒ orthonormalize with the kinetic metric; project using the Hessian pseudoinverse; tighten nonlinear solve tolerance.

Ghost directions in G_{IJ}: Negative eigenvalues due to discretization or gauge choice ⇒ refine mesh, enforce constraints (e.g. $|\hat{\chi}| = 1$), and recompute overlaps with consistent inner product.

Spurious negative modes: Residual instability from insufficient relaxation ⇒ continue Newton–Krylov steps; if persistent, the solution is not a minimum (excite correct topological sector or add stabilizing terms per Sec. 6.4).

Nonintegral Berry flux: Gauge–patching/phase–tracking error ⇒ switch to link–variable evaluation on triangulations (Sec. 6.12.5); enforce integrality via exact counterterm $\delta\mathcal{A}$.

Adiabatic breakdown: Near avoided crossings ⇒ include nonadiabatic couplings; adopt multi-channel dynamics or reduce kinetic energy; use Landau–Zener estimates to set safe regions (Sec. 6.8.5).

GKSL non–CP behavior: Ad hoc jump operators ⇒ use geometric construction $\widehat{L}_a = \sqrt{\lambda_a}\, e_a{}^I \widehat{\Pi}_I$ (Sec. 6.10.3); recheck $\gamma_{IJ} \succeq 0$.

6.13.3 From PDE to particle: a ten–step recipe

1. **Construct** a finite–energy soliton in sector Q with validated topology (S1,S5).
2. **Extract zero modes** Ψ_I and build the moduli–space metric G_{IJ} (S2,G1).
3. **Check stability** by diagonalizing the projected Hessian (S3); enforce hyperbolicity (S4).
4. **Compute** the adiabatic potential $U_{\rm mod}(X)$ and stabilization terms (Sec. 6.4).
5. **Evaluate Berry data** $\mathcal{A}_I, \Omega_{IJ}$ and fix FR characters (TQ1–TQ2).
6. **Quantize** the collective Hamiltonian $\widehat{H}$ (Sec. 6.5) and obtain single–soliton spectra (Sec. 6.7).
7. **Build two–body channels** with FR/Berry selection rules; compute $U_{\rm ad}(R)$ and relative dynamics (Sec. 6.8).
8. **Include topology/anomalies** via WZW/Hopf descent and verify inflow (Sec. 6.9).
9. **Add weak dissipation** and derive linewidths/decoherence (Sec. 6.10).
10. **Renormalize & match** at $\mu_{\rm spec}$; document uncertainties and pass N–gates (Sec. 6.11, 6.12).

6.13.4 Deliverables of Chapter 6

- Renormalized collective data $\big(U^{\rm R}_{\rm mod}(X),\, G^{\rm R}_{IJ}(X),\, \Omega^{\rm R}_{IJ}(X)\big)$ for each sector Q, with error budgets and regulator details.
- Single–soliton spectra (rotational/vibrational) and quantum numbers, including FR/Berry labels.
- Two–soliton adiabatic potentials, scattering lengths/phase shifts, and (if present) dimer levels.
- GKSL parameters $(\gamma_{IJ},\ \Theta_{\rm spec})$ and predicted linewidths/decoherence rates.
- A reproducibility bundle: meshes, solver scripts, hashes of observables, and pass/fail logs for all gates.

6.13.5 Summary and outlook

We have elevated classical solitons of the quantum aether into quantum particles by constructing a covariant, renormalized dynamics on their moduli spaces, enriched by geometric phases, topological constraints, and controlled dissipation. The gate framework ensures that every computed observable is physically meaningful, regulator–independent (once renormalized), and reproducible. In Ch. 7 we let these particles interact: we develop worldline/worldvolume effective theories, derive scattering amplitudes and resonances, and confront the first layer of experimental signatures. In Ch. 8 we climb back to the field–theoretic vantage point: we show how the observed gauge dynamics can emerge consistently from the spectral geometry that underlies the entire construction.

6.14 External Pro Probes and Form Factors

This section develops a systematic coupling of solitonic excitations to *external probes* and extracts their *form factors* and *multipole moments*. We proceed from the microscopic Noether currents of the spectral–aether theory to the collective (moduli–space) response, derive low–momentum expansions (charges, dipoles, quadrupoles, gyromagnetic ratios), and state Ward identities and numerical extraction protocols. The focus is operational: provide gauge–covariant, reparametrization–invariant formulas ready for direct computation on the benchmarks of Sec. 6.12.

6.14.1 Probe fields and minimal couplings

Abelian spectral gauge field. The $U(1)_\Phi$ phase symmetry of Φ (Chs. **??**, 3) admits a background connection A_μ with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Minimal coupling:

$$\nabla_\mu \Phi \;\longrightarrow\; (\nabla_\mu - i\, q_\Phi A_\mu)\Phi, \qquad \mathcal{L}_\Phi \supset -\kappa_\Phi\, g_{\rm eff}^{\mu\nu} (\nabla_\mu - i q_\Phi A_\mu)\Phi^* (\nabla_\nu + i q_\Phi A_\nu)\Phi, \tag{6.125}$$

defines the microscopic *spectral current* $J^\mu_\Phi \equiv \frac{\partial \mathcal{L}}{\partial A_\mu} = i q_\Phi \kappa_\Phi \big[\Phi^* (\nabla^\mu \Phi) - (\nabla^\mu \Phi)^* \Phi\big]$.

Non-Abelian internal gauge background. Internal rotations of the structural sector can be probed by a background $\mathcal{A}^a_\mu$ (generators T^a in the Φ representation) via $(\nabla_\mu - i \mathcal{A}^a_\mu T^a)$; the associated current is $J^\mu_a = \frac{\partial \mathcal{L}}{\partial \mathcal{A}^a_\mu}$ with the usual covariant conservation $\nabla_\mu J^\mu_a + f_{ab}{}^c \mathcal{A}^b_\mu J^\mu_c = 0$.

Emergent metric perturbations. Gravitational–like probes couple through $g_{\rm eff} \to g_{\rm eff} + h$:

$$\delta S = \tfrac{1}{2} \int d^4x \, \sqrt{-g_{\rm eff}}\; T^{\mu\nu}\, h_{\mu\nu}, \qquad T^{\mu\nu} = \frac{2}{\sqrt{-g_{\rm eff}}} \frac{\delta S}{\delta g^{\rm eff}_{\mu\nu}}, \tag{6.126}$$

defining the stress–energy form factors (Sec. 6.14.4).

Worldline reduction. On the collective sector (Sec. 6.5) a single–soliton state with center $X^\mu(\tau)$ and internal moduli $X^I(\tau)$ is described by the worldline action

$$S_{\rm Wl} = \int d\tau \Big[-M + Q_{\rm eff}\, A_\mu(X)\, \dot X^\mu + \mu^i(X)\, B_i + p^i(X)\, E_i + \tfrac{1}{2} Q^{ij}(X)\, \partial_i E_j + \cdots \Big], \tag{6.127}$$

where $E_i = F_{0i}$, $B^i = \frac{1}{2}\epsilon^{ijk} F_{jk}$ are the spectral electric/magnetic fields. The emergent gravitational couplings follow by replacing $-M \to -M\sqrt{-g_{\rm eff}(\dot X, \dot X)}$ and adding spin–gravity terms (Sec. 6.14.4).

6.14.2 Noether currents and Ward identities

Microscopic currents. The conserved currents are

$$J^\mu_\Phi = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_\Phi)}\, \delta\varphi_\Phi \;\rightsquigarrow\; i q_\Phi \kappa_\Phi \big[\Phi^* (\nabla^\mu \Phi) - (\nabla^\mu \Phi)^* \Phi\big], \tag{6.128}$$

$$J^\mu_a = \frac{\partial \mathcal{L}}{\partial \mathcal{A}^a_\mu}, \qquad \nabla_\mu J^\mu_a + f_{ab}{}^c \mathcal{A}^b_\mu J^\mu_c = 0, \qquad \nabla_\mu T^{\mu\nu} = 0, \tag{6.129}$$

with anomaly corrections handled via inflow (Sec. 6.9).

Charge and multipole densities. Define on a Cauchy slice Σ_t:

$$Q_{\rm eff} = \int_{\Sigma_t} d^3x\, \sqrt{\gamma}\, n_\mu J^\mu_\Phi, \quad \mathbf{p} = \int d^3x\, \mathbf{x}\, \rho_\Phi, \quad Q^{ij} = \int d^3x\, (x^i x^j - \tfrac{1}{3}\delta^{ij} r^2)\, \rho_\Phi, \tag{6.130}$$

with $\rho_\Phi \equiv n_\mu J^\mu_\Phi$ and n^μ the unit normal.

Ward identities. Matrix elements of currents between exact collective eigenstates $|n\rangle$ satisfy

$$q_\mu \langle n' | J^\mu_\Phi(0) | n\rangle = \big[Q_{\rm eff}, \langle n' | \cdot |n\rangle\big], \qquad q_\mu \langle n' | T^{\mu\nu}(0) | n\rangle = 0, \tag{6.131}$$

fixing charge normalization and constraining soft limits of form factors (Sec. 6.15).

6.14.3 Form factors from the collective theory

Definitions. For the spectral $U(1)$ current, define (Breit frame)

$$\langle p', s' | J^\mu_\Phi(0) | p, s\rangle = \bar u_{s'}(p') \Big[F_1(q^2)\, \gamma^\mu + \frac{i}{2M} F_2(q^2)\, \sigma^{\mu\nu} q_\nu \Big] u_s(p), \tag{6.132}$$

or, for spinless states, $\langle p' | J^\mu_\Phi | p\rangle = (p' + p)^\mu\, F(q^2)$. The small–$q^2$ expansion yields $Q_{\rm eff} = F_1(0)$, magnetic moment $\boldsymbol{\mu} = \frac{Q_{\rm eff}}{2M}(2F_1(0) + F_2(0))\, \mathbf{J}$, and charge radius $\langle r^2 \rangle = -6\, F_1'(0)$.

Collective reduction. With eigenstates $|n\rangle$ of the moduli Hamiltonian $\widehat{H}$ (Sec. 6.5), the form factors are overlap integrals on $\mathcal{M}_Q$:

$$\langle n'|J^\mu_\Phi(0)|n\rangle = \int_{\mathcal{M}_Q} d\mu_G\ \Psi^*_{n'}(X)\ \mathcal{J}^\mu_\Phi\big[X;\mathcal{S}_{\rm sol},\Phi_{\rm sol}\big]\ \Psi_n(X), \tag{6.133}$$

where $\mathcal{J}^\mu_\Phi$ is the *projected* current density obtained by contracting the microscopic current with zero–mode profiles (cf. Eq. (6.36)). Berry phases and FR enter through the line bundle $\mathcal{L}_{\rm phys}$ and selection rules.

Gyromagnetic ratio. For an axially symmetric MBP–1 rotor,

$$\boldsymbol{\mu} = g\,\frac{Q_{\rm eff}}{2M}\,\mathbf{J}, \qquad g = 1 + \frac{M}{Q_{\rm eff}}\,\frac{\partial}{\partial\Omega}\Big[\int d^3x\,\epsilon^{ijk}x^j J^k_\Phi\Big]_{\Omega\to 0}, \tag{6.134}$$

where the derivative is evaluated on a uniformly rotating solution (Sec. 6.12.2). Berry/monopole contributions shift g by discrete amounts via induced body–frame couplings.

Quadrupole moments. For states with $J \geq 1$, the spectrally–induced electric quadrupole is

$$Q_{E2} = \langle J, M{=}J|\widehat{Q}^{20}|J, M{=}J\rangle, \qquad \widehat{Q}^{2m} = \int d^3x\, r^2 Y_{2m}(\hat{\mathbf{r}})\,\rho_\Phi(\mathbf{r}), \tag{6.135}$$

computable from the collective wavefunctions in the body frame with Wigner D–functions, modified by FR and monopole harmonics when present (Sec. 6.8.8).

6.14.4 Stress–energy form factors and emergent gravitational response

Gravitational Sachs decomposition. Matrix elements of $T^{\mu\nu}$ admit a decomposition in terms of three invariant form factors (for spin–$\frac{1}{2}$): $\{A(q^2), B(q^2), C(q^2)\}$:

$$\langle p', s'|T^{\mu\nu}(0)|p, s\rangle = \bar{u}_{s'}(p')\Big[A\,\tfrac{(p'+p)^{(\mu}\gamma^{\nu)}}{2} + B\,\tfrac{i\sigma^{(\mu\alpha}q_\alpha(p'+p)^{\nu)}}{4M} + C\,\tfrac{q^\mu q^\nu - q^2 g^{\mu\nu}}{M}\Big]u_s(p). \tag{6.136}$$

Mass normalization gives $A(0) = 1$; the *gravitomagnetic* moment is fixed by $A(0) + B(0) = 1$ in the absence of anomalies. In the collective theory, $A(0) = 1$ follows from the normalization of G_{IJ} and the center–of–mass decoupling; $B(0)$ probes spin–orbital couplings induced by Berry curvature.

Polarizabilities. Spectral gravitational polarizabilities (quadratic response to $h_{\mu\nu}$) are obtained from Kubo formulas (Sec. 6.14.5) and encode quadrupole responses; positivity constraints follow from spectral representations.

6.14.5 Linear response and Kubo formulas

Retarded correlators. For a probe A (either A_μ or $h_{\mu\nu}$), the linear response is governed by

$$G^{\rm R}_{\mathcal{O}\mathcal{O}}(\omega,\mathbf{q}) = -i\int d^4x\, e^{i(\omega t - \mathbf{q}\cdot\mathbf{x})}\,\theta(t)\langle[\mathcal{O}(x),\mathcal{O}(0)]\rangle, \tag{6.137}$$

with $\mathcal{O} \in \{J^0_\Phi,\ \mathbf{J}_\Phi,\ T^{\mu\nu}\}$. Electric/magnetic polarizabilities follow from the low–frequency limit, e.g. $\alpha_E = -\lim_{\omega\to 0}\frac{1}{\omega^2}\,G^R_{J^0J^0}(\omega, \mathbf{q}{=}\mathbf{0})$, and similarly for β_M from $\mathbf{J}_\Phi$.

Spectral representation and positivity. Writing $G^{\rm R}(\omega) = \int_0^\infty d\Omega\,\rho(\Omega)\big(\frac{1}{\omega-\Omega+i0} - \frac{1}{\omega+\Omega+i0}\big)$ with spectral density $\rho \geq 0$ yields $\alpha_E, \beta_M \geq 0$ (modulo subtractions), consistent with passive response and the GKSL dynamics of Sec. 6.10.

6.14.6 Selection rules and discrete symmetries

Angular momentum and FR. Transition matrix elements $\langle n'|\mathcal{O}_{\ell m}|n\rangle$ obey $\Delta J \in \{0, \pm 1\}$ for dipole, $\{0, \pm 1, \pm 2\}$ for quadrupole, etc., modified by FR: if the loop in configuration space associated with the operator insertion is nontrivial in $\pi_1(\mathcal{Q}_Q)$, the amplitude acquires a factor $\sigma(\gamma) = \pm 1$ (Sec. 6.6), forbidding transitions with mismatched FR character.

Parity, time reversal, and CP. Electric dipole moments vanish in parity–even ground states; CP–odd microscopic terms (e.g. θ_{spec}) can induce small EDMs proportional to the CP–odd coupling (Sec. 6.9.3). Gravitomagnetic couplings respect T–reversal when Berry curvature is purely geometric (no dissipation).

6.14.7 Numerical extraction protocols

Background–field method (energy shifts). Apply uniform weak fields on a finite box with periodic/Dirichlet boundary conditions and fit

$$\Delta E(\mathbf{B}) = -\boldsymbol{\mu}\cdot\mathbf{B} - \tfrac{1}{2}\beta_M B^2 + \mathcal{O}(B^3), \tag{6.138}$$

$$\Delta E(\mathbf{E}) = -\mathbf{p}\cdot\mathbf{E} - \tfrac{1}{2}\alpha_E E^2 - \tfrac{1}{6}Q^{ij}\partial_i E_j + \cdots . \tag{6.139}$$

Finite–volume corrections can be estimated by known $1/L$ expansions; confirm linearity regimes.

Current insertion (three–point). Compute three–point functions with one current insertion between soliton states, project to desired $\mathbf{q}$, and extract $F_{1,2}(q^2)$ by kinematic separation. Use link–variable phases to preserve gauge invariance and Berry bundle consistency.

Rotational method for g. Impose a small uniform rotation (body–frame angular velocity Ω), solve the stationary rotating equations, measure the induced spectral current, and use Eq. (6.134).

Stress–energy response. Perturb $g_{\text{eff}} \to g_{\text{eff}} + h$ by a weak stationary gravitomagnetic potential h_{0i}, extract the linear coefficient to obtain $A(0), B(0)$.

6.14.8 Worked mini–example: MBP–0 magnetic moment

Setup. Take an isotropic core (MBP–0), ground rotational band $j = \frac{1}{2}$ enforced by FR. In a weak uniform $\mathbf{B}$, the Zeeman splitting is

$$\Delta E_m = -\,m\,g\,\frac{Q_{\text{eff}}}{2M}\,B \qquad (m = \pm\tfrac{1}{2}), \tag{6.140}$$

with g from Eq. (6.134). Numerically, extract Q_{eff} from the volume integral of ρ_Φ and M from U_{mod} (Sec. 6.4); validate g via both rotation–induced current and background–field Zeeman fits.

6.14.9 Consistency gates for external response

- **E1 (gauge invariance).** Matrix elements obey Ward identities (6.131); form–factor normalizations $F_1(0) = Q_{\text{eff}}$ and $A(0) = 1$ hold within numerical tolerance.
- **E2 (FR/selection).** All extracted transitions obey FR selection rules and parity/CP constraints; forbidden amplitudes vanish within error bars.
- **E3 (positivity).** $\alpha_E, \beta_M \geq 0$ and spectral densities are nonnegative.
- **E4 (reproducibility).** Results stable under gauge/phase choices on the Berry bundle and under mesh/domain variation (cf. N–gates of Sec. 6.12.7).

6.14.10 Summary

We have introduced probe couplings to spectral $U(1)$ and internal non–Abelian backgrounds, as well as emergent metric perturbations, and reduced them to the collective theory to define charges, dipoles, quadrupoles, gyromagnetic ratios, and stress–energy form factors. Ward identities and FR/Berry structures tightly constrain allowed responses and transitions. The background–field and current–insertion protocols provide two independent computational routes with cross–checks. These tools enable the radiative and soft–emission analysis of Sec. 6.15 and the phenomenological mapping in Sec. 6.16.

6.15 Radiative Processes and Soft Emission

We now endow the probe sectors of Sec. 6.14 with dynamics and develop a unified treatment of radiation by solitons: *spectral gauge* emission (Abelian and non–Abelian internal), and *emergent gravitational* emission (metric waves of $g_{\rm eff}$). We present (i) the dynamical field content and normalizations, (ii) classical multipole expansions and power formulas, (iii) soft–emission theorems and Ward identities, (iv) radiation–reaction forces on the moduli, (v) quantum spontaneous/emission rates and selection rules, (vi) bremsstrahlung in two–body scattering, and (vii) numerical protocols. Throughout we keep reparametrization covariance on $\mathcal{M}_Q$ and gauge/diffeomorphism invariance manifest.

6.15.1 Dynamical radiation channels and normalizations

Spectral $U(1)$ gauge sector. Promote the background field A_μ to a dynamical field with kinetic term

$$\mathcal{L}_{\rm spec} = -\frac{Z_A}{4} F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu_\Phi, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{6.141}$$

where $Z_A > 0$ fixes the normalization. We define the effective spectral coupling $g_{\rm spec} \equiv Z_A^{-1/2}$ and a fine–structure parameter $\alpha_{\rm spec} \equiv g^2_{\rm spec}/(4\pi)$.

Internal non–Abelian sector. For an internal gauge background $\mathcal{A}^a_\mu$ (Sec. 6.14.1), the dynamical term is

$$\mathcal{L}_{\rm int} = -\frac{Z_{\mathcal{A}}}{4}\, \mathcal{F}^a_{\mu\nu}\mathcal{F}^{a\,\mu\nu} + \mathcal{A}^a_\mu J^\mu_a, \qquad \mathcal{F}^a_{\mu\nu} = \partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu + f^a{}_{bc}\mathcal{A}^b_\mu \mathcal{A}^c_\nu. \tag{6.142}$$

We will mostly quote Abelian formulas; the non–Abelian generalizations follow standard rules, with color–dressed soft factors.

Emergent gravitational sector. Linearize $g_{\rm eff} = \bar g_{\rm eff} + h$ about a slowly varying background $\bar g_{\rm eff}$. At quadratic order,

$$\mathcal{L}^{(2)}_{\rm grav} = \frac{M^2_{\rm eff}}{8}\, h_{\mu\nu}\, \mathcal{E}^{\mu\nu\rho\sigma}\, h_{\rho\sigma} + \tfrac{1}{2} h_{\mu\nu} T^{\mu\nu}, \tag{6.143}$$

where $\mathcal{E}$ is the Lichnerowicz operator on the background and $M^2_{\rm eff}(\rho)$ was defined in Ch. 3. The effective "Newton" coupling is $G_{\rm eff} \equiv (8\pi M^2_{\rm eff})^{-1}$.

6.15.2 Classical multipole expansion and power

Spectral Poynting flux. For the spectral $U(1)$ field, the energy–flux vector is $\mathbf{S}_{\rm spec} = Z_A^{-1}\, \mathbf{E}\times\mathbf{B}$, so that the radiated power through a sphere of radius r is

$$P_{\rm spec}(t) = \int_{S_r} d\Omega\, r^2\, \mathbf{n}\cdot \mathbf{S}_{\rm spec}. \tag{6.144}$$

Dipole radiation (Abelian). In the radiation zone, the leading power is dipolar:

$$P^{(E1)}_{\rm spec} = \frac{g^2_{\rm spec}}{6\pi}\, \big|\ddot{\mathbf{p}}(t)\big|^2, \qquad \mathbf{p}(t) = \int d^3x\, \mathbf{x}\, \rho_\Phi(\mathbf{x},t). \tag{6.145}$$

Higher multipoles follow the standard $E\ell/M\ell$ hierarchy, with form factors determined by the collective matrix elements of Sec. 6.14.3.

Emergent gravitational quadrupole. In the weak–field, slow–motion limit,

$$P_{\rm grav} = \frac{G_{\rm eff}}{5}\, \big\langle \dddot{Q}_{ij}\, \dddot{Q}_{ij} \big\rangle, \qquad Q_{ij} = \int d^3x\, \big(x_i x_j - \tfrac{1}{3}\delta_{ij} r^2\big)\, T^{00}(\mathbf{x},t), \tag{6.146}$$

with indices raised/lowered by δ_{ij} in the near–flat zone of $\bar g_{\rm eff}$. Anisotropic cores (MBP–1) and Berry–induced spin–orbit couplings modify $Q_{ij}(t)$ via the moduli dynamics.

Two–body inspiral. For a shallow dimer of reduced mass μ and separation $\mathbf{R}(t)$,

$$P_{\rm spec}^{(E1)} \sim \frac{g_{\rm spec}^2}{6\pi}\left|\ddot{\mathbf{d}}(t)\right|^2, \quad \mathbf{d} \equiv Q_{\rm eff}^{(\rm rel)}\,\mathbf{R}; \qquad P_{\rm grav} \sim \frac{32}{5} G_{\rm eff}\,\mu^2 R^4 \Omega^6, \tag{6.147}$$

with Ω the orbital frequency and $Q_{\rm eff}^{(\rm rel)}$ the relative spectral charge fixed by the FR/Berry channel.

6.15.3 Soft theorems and Ward identities

Single–soft spectral photon. For an n–body amplitude $\mathcal{M}_n$ on the collective sector, emission of an extra soft photon ($k \to 0$, polarization ε) factorizes as

$$\mathcal{M}_{n+1}(k;\varepsilon) \xrightarrow{k\to 0} \left[\sum_{i=1}^{n} \frac{Q_i\, p_i\cdot\varepsilon}{p_i\cdot k}\right] \mathcal{M}_n \;+\; \mathcal{O}(k^0), \tag{6.148}$$

where (Q_i, p_i) are the (effective) charges and momenta of the external solitons, read off from the worldline couplings (6.127). This is the Abelian soft theorem adapted to the moduli EFT.

Single–soft emergent graviton. Similarly,

$$\mathcal{M}_{n+1}^{\rm grav}(k;\varepsilon) \xrightarrow{k\to 0} \left[\sum_{i=1}^{n} \frac{\kappa_{\rm eff}\; p_i^\mu p_i^\nu\, \varepsilon_{\mu\nu}}{p_i\cdot k}\right] \mathcal{M}_n \;+\; \mathcal{O}(k^0), \qquad \kappa_{\rm eff} \equiv \sqrt{32\pi G_{\rm eff}}. \tag{6.149}$$

Ward identities implied by current/stress–energy conservation coincide with Eqs. (6.131) in the soft limit.

IR safety (KLN). Inclusive rates, summed over unresolved soft quanta, are finite (KLN cancellations). In practice, the GKSL dissipative sector (Sec. 6.10) resums a class of soft emissions into linewidths, consistent with (6.148)–(6.149).

6.15.4 Radiation reaction on the moduli

Covariant reaction force. To leading order in the soft expansion, radiation reaction enters the moduli equations as a local, geometric force consistent with energy balance:

$$G_{IJ}\,\ddot{X}^J + \Gamma_{IJK}\,\dot{X}^J\dot{X}^K + \partial_I U_{\rm mod} = F_I^{(\rm RR)}, \qquad \dot{E}_{\rm mod} = -P_{\rm rad}, \tag{6.150}$$

with $P_{\rm rad} = P_{\rm spec} + P_{\rm grav}$ from (6.144), (6.146).

Reduction of order (no runaways). A covariant Abraham–Lorentz–Dirac analogue can be written as

$$F_I^{(\rm RR)} = \tau_{\rm spec}\,\mathcal{P}_I{}^J\,\frac{D^3 X_J}{Dt^3} \;+\; \tau_{\rm grav}\,\mathcal{Q}_I{}^J\,\frac{D^5 X_J}{Dt^5} + \cdots, \tag{6.151}$$

where $\mathcal{P}, \mathcal{Q}$ are positive semi–definite operators built from G_{IJ} and the relevant multipoles, and $\tau_{\rm spec} \sim g_{\rm spec}^2/(6\pi M)$, $\tau_{\rm grav} \sim G_{\rm eff} M$. To avoid unphysical pre–acceleration, we use *reduction of order*: compute $D^n X/Dt^n$ using the conservative dynamics and substitute back into (6.151); this keeps causality and matches the power loss (6.150).

6.15.5 Quantum spontaneous emission and selection rules

Dipole $E1$ spectral emission. For a transition $|i\rangle \to |f\rangle + \gamma_{\rm spec}$ with $\omega = E_i - E_f$, Fermi's golden rule gives

$$\Gamma_{i\to f}^{(E1)} = \frac{4}{3}\,\alpha_{\rm spec}\,\omega^3\left|\langle f||\widehat{\mathbf{d}}||i\rangle\right|^2, \qquad \widehat{\mathbf{d}} = \int d^3x\,\mathbf{x}\,\hat{\rho}_\Phi, \tag{6.152}$$

with reduced matrix element computed on $\mathcal{M}_Q$ (Sec. 6.14.3). FR/Berry selection rules restrict ΔJ and parity as in Sec. 6.14.6.

Quadrupole gravitational emission. For emergent gravitational waves,

$$\Gamma^{(E2)}_{i\to f} = \frac{G_{\rm eff}}{5}\,\omega^5\,\big|\langle f||\widehat{Q}_{E2}||i\rangle\big|^2, \qquad \widehat{Q}^{2m}_{E2} = \int d^3x\, r^2 Y_{2m}(\hat{\mathbf{r}})\,\widehat{T}^{00}. \tag{6.153}$$

Mixed $M1$, $E2$, $\cdots$ channels proceed with the usual $\omega^{2\ell+1}$ scaling.

Linewidths and GKSL. Summing over all allowed channels reproduces the radiative contribution to the GKSL linewidths of Sec. 6.10.4: $\Gamma_n^{(\rm rad)} = \sum_f \Gamma_{n\to f}$.

6.15.6 Bremsstrahlung in two–soliton scattering

Classical soft bremsstrahlung. For a deflection angle Θ at relative velocity v and impact parameter b,

$$E^{(\rm spec)}_{\rm rad} \sim \alpha_{\rm spec}\, Q^2_{\rm eff}\,\frac{\Theta^2}{b^2 v}, \qquad E^{(\rm grav)}_{\rm rad} \sim G_{\rm eff}\,\mu^2\,\frac{v^5}{b^3}\,\mathcal{F}(\Theta), \tag{6.154}$$

with $\mathcal{F}$ a bounded function of the scattering geometry. Quantum mechanically, the $d\omega/\omega$ soft spectrum follows from (6.148)–(6.149).

Channel dependence. FR constraints and Berry monopoles can suppress leading $E1$ channels, pushing radiation to next–to–leading multipoles; this is a distinctive prediction of the model.

6.15.7 Numerical protocols for radiation

Time–domain far–zone extraction. Evolve the full PDEs with absorbing boundaries; reconstruct far–zone fields and compute $P_{\rm spec}(t) = \int r^2 d\Omega\, \mathbf{n}\cdot(\mathbf{E}\times\mathbf{B})/Z_A$ and the Isaacson flux for h_{ij}. Validate energy balance against the decrease of $E_{\rm mod}(t)$.

Frequency–domain (QNM) method. Linearize about the time–periodic collective motion; solve for quasi–normal modes (QNMs) and obtain discrete lines at harmonics of the moduli frequencies; cross–check against (6.152)–(6.153).

Worldline EFT radiation. Use (6.127) with dynamical $A_\mu, h_{\mu\nu}$ to compute radiation via classical sources; match multipole coefficients to collective matrix elements (Sec. 6.14.3).

6.15.8 Consistency gates for radiation

- **R_rad1 (gauge/diffeo invariance).** Radiative observables are invariant under $A_\mu \to A_\mu + \partial_\mu\lambda$ and $h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)}$.
- **R_rad2 (soft limits).** Amplitudes satisfy (6.148), (6.149) and reproduce inclusive IR–finite rates (KLN).
- **R_rad3 (energy balance).** $dE_{\rm mod}/dt = -P_{\rm rad}$ within numerical tolerance.
- **R_rad4 (causality).** Radiation–reaction implemented by reduction of order; no runaways or pre–acceleration in time–domain simulations.
- **R_rad5 (selection rules).** FR/Berry constraints enforced; forbidden lines absent within error bars.

6.15.9 Worked mini–examples

MBP–0 $j{=}\frac{1}{2}\to j{=}\frac{1}{2}$ spin–flip ($M1$). With $Q_{\rm eff}\neq 0$ and magnetic dipole operator $\widehat{\boldsymbol{\mu}}$, the rate is

$$\Gamma^{(M1)}_{m\to m'} = \frac{4}{3}\,\alpha_{\rm spec}\,\omega^3\big|\langle j{=}\tfrac{1}{2}, m'|\widehat{\boldsymbol{\mu}}|j{=}\tfrac{1}{2}, m\rangle\big|^2, \qquad \omega \simeq g\,\frac{Q_{\rm eff}B}{2M}. \tag{6.155}$$

Gravitational $E2$ line of an anisotropic rotor. For MBP–1 with $J \geq 1$, the $J \to J-2$ gravitational line is $\Gamma^{(E2)} \propto G_{\rm eff}\,\omega^5\,|Q_{E2}|^2$, with Q_{E2} from (6.135). The absence of $E1$ in this channel is a selection–rule signature.

Soft bremsstrahlung in a shallow dimer collision. Time–domain simulation of a near–elastic collision at impact parameter b yields a soft spectrum $dE/d\omega \propto \omega^0$ up to the orbital frequency, in agreement with the soft theorem (6.148). Turning on a Berry monopole reduces $E1$ and enhances $M1/E2$ tails.

6.15.10 Summary

Radiation in the quantum–aether soliton sector follows universal low–frequency rules: Abelian and gravitational soft factors control inclusive spectra; classical multipoles determine power; and FR/Berry structures imprint sharp selection rules on lines and bremsstrahlung. Radiation reaction admits a covariant, causal implementation on the moduli manifold and matches GKSL linewidths in the weak–coupling limit. These ingredients complete the dynamical toolkit required for the phenomenological mapping of Sec. 6.16.

6.16 Phenomenological Map to Particle Data

This section turns the quantum soliton framework into testable particle–physics output. We (i) define a dictionary from collective observables to experimental quantities, (ii) propose concrete assignment schemes for leptons and hadrons, (iii) relate mixing and CP violation to Berry/FR geometry, (iv) specify matching conditions for gauge couplings and running, (v) compile key constraints and null tests, and (vi) provide a reproducible fitting workflow and parameter windows.

6.16.1 Goals and scope

1. **Direct mapping**: from renormalized collective data $\big(U^{\rm R}_{\rm mod},\, G^{\rm R}_{IJ},\, \Omega^{\rm R}_{IJ}\big)$ to masses, spins, charges, form factors, and decay widths.

2. **Minimality**: prefer assignments with the fewest free inputs; exploit geometric integrality (quantized Berry flux, FR signs) to reduce arbitrariness.

3. **Layered consistency**: pass the null–test suites of Sec. 5.3 and the gates of Sec. 6.13 before claiming phenomenological matches.

6.16.2 Observable dictionary

Let $|n\rangle$ be a single–soliton eigenstate of the collective Hamiltonian (Sec. 6.5) with center–of–mass momentum p^μ and internal quantum numbers fixed by FR/Berry.

- **Mass**: $M_n \equiv U^{\rm R}_{\rm mod}(X_\star)$ (rest energy at the symmetric point).
- **Spin/parity**: from the representation carried by the wavefunction on $\mathcal{M}_Q$, including FR constraints; parity from core symmetry of $(\mathcal{S}_{\rm sol}, \Phi_{\rm sol})$.
- **Charges**: spectral $U(1)$ charge $Q_{\rm eff} = F_1(0)$; non–Abelian charges from Cartan generators defined by the $G_2 \to H$ embedding of Sec. 4.7 (with $H \simeq SU(3)\times SU(2)\times U(1)$ in the low–energy phase).
- **Magnetic moments**: $\boldsymbol{\mu} = g\,\frac{Q_{\rm eff}}{2M}\,\mathbf{J}$ with g from Eq. (6.134).
- **Charge radii and multipoles**: from small–q^2 expansion of form factors (Sec. 6.14.3).
- **Decay widths**: radiative channels from Eqs. (6.152)–(6.153); nonradiative channels require overlap integrals with appropriate interaction vertices (Yukawa–like or contact), subject to selection rules (Sec. 6.14.6).

Matching conventions. All quoted couplings are defined at the spectral scale $\mu_{\rm spec}$ and converted to a standard scheme (e.g. $\overline{\rm MS}$ at $\mu = m_Z$) via the spectral RG of Sec. 6.11.5.

6.16.3 Lepton sector prototype

Assignment. Use a topologically minimal sector $Q = Q_\ell$ with MBP–0 core and FR sign $\sigma = -1$ to enforce half–integer spin. The three charged leptons are the ground and the first two vibrational excitations (along a preferred internal geodesic γ):

$$\ell_n \in \{|0\rangle,\ |1_\gamma\rangle,\ |2_\gamma\rangle\}, \qquad n = 0, 1, 2, \tag{6.156}$$

with identical $Q_{\rm eff} = -1$.

Mass hierarchy. Assume a *geometric* spacing along γ with curvature radius R_γ in $(\mathcal{M}_Q, G)$:

$$M_{\ell_n} \ \simeq\ M_0 \exp\!\big(\alpha_\gamma\, n\big), \qquad \alpha_\gamma \propto \frac{L_\gamma}{R_\gamma}\,, \tag{6.157}$$

where L_γ is the characteristic length scale set by the stabilizing potential $U_{\rm mod}(X)$ near $X_\star$. M_0 and α_γ are fitted to (m_e, m_μ, m_τ).

Magnetic moments and $g-2$. Compute g_ℓ from Eq. (6.134); the deviation $\Delta a_\ell \equiv (g_\ell - 2)/2$ receives (i) collective one–loop corrections from Sec. 6.11, and (ii) potential short–distance contributions from coupling to emergent gauge bosons; the former is determined once the core is fixed.

Neutrinos. Neutral states in the same sector with $Q_{\rm eff} \approx 0$, either (i) MBP–0 states with different FR phase (Dirac–like) or (ii) FR–*twisted* states that are their own antiparticles (Majorana–like) depending on the topology of $\pi_1(\mathcal{Q}_Q)$. Tiny masses can result from exponentially weak tunneling across a barrier on $\mathcal{M}_Q$.

6.16.4 Quark/hadron–like mapping (two options)

Option A: quarks as confined endpoints. Quarks correspond to endpoints of spectral strings (defects) carrying non–Abelian charge under the $SU(3)$ subgroup of G_2. Color confinement arises from string tension in the broken phase; mesons and baryons are two– and three–string bound states.

Option B: hadrons as multi–soliton bound states. Quarks are emergent quasiparticles; hadrons are shallow dimers/trimers of MBP–1 cores with FR/Berry structure reproducing color–singlet selection. Baryons: rotational bands of a trimer on a curved $\mathcal{M}_3$; mesons: dimer vibrational levels (Sec. 6.8).

Minimal dataset. Fit nucleon mass M_N, Δ–N splitting (from $I_\perp - I_\parallel$ anisotropy), pion–like mass scale as the lowest dimer vibration, and charge radii from form factors. Avoid overfitting by using the smallest number of geometric parameters consistent with gates.

6.16.5 Mixing matrices and CP violation from geometry

Flavor basis. Define flavor states as localized packets on separated valleys $\{\mathcal{V}_i\}$ of $U_{\rm mod}(X)$ on $\mathcal{M}_Q$. The mass basis diagonalizes the Hessian. The unitary rotation is controlled by overlaps and tunneling amplitudes:

$$(V_{\rm CKM/PMNS})_{ij} \ \sim\ \int_{\mathcal{M}_Q} d\mu_G\ \Psi_i^{(m)}(X)\,\Psi_j^{(f)}(X), \qquad \sum_{ij} |(V)_{ij}|^2 = 1. \tag{6.158}$$

CP phase from Berry holonomy. Let Γ be a loop in parameter space (e.g. adiabatic variation connecting valleys). The gauge–invariant phase

$$\delta_{\rm CP} \ \equiv\ \oint_\Gamma \mathcal{A} \ =\ \int_\Sigma \Omega, \qquad \partial\Sigma = \Gamma, \tag{6.159}$$

provides a geometric origin for the Jarlskog invariant $J \propto \sin\delta_{\rm CP} \prod_{i<j} \sin\theta_{ij} \cos\theta_{ij}$. Quantization properties of Ω constrain $\delta_{\rm CP}$.

6.16.6 Gauge couplings and running

Matching at $\mu_{\rm spec}$. The spectral $U(1)$ coupling $g_{\rm spec}$ and non–Abelian couplings g_i are matched to $\alpha_{\rm em}, \alpha_2, \alpha_3$ at a reference scale (e.g. m_Z) via

$$\frac{1}{\alpha_i(\mu)} = \frac{1}{\alpha_i(\mu_{\rm spec})} + \frac{b_i}{2\pi} \ln \frac{\mu}{\mu_{\rm spec}} + \Delta_i^{(\rm th)}, \tag{6.160}$$

with b_i the beta–function coefficients of the emergent gauge sector below v_Σ (Sec. 4.7) and threshold corrections $\Delta_i^{(\rm th)}$ from heavy coset modes.

Gravity. $G_{\rm eff}(\rho)$ is matched by requiring the Newtonian limit in Sec. **??**; variations $\delta G/G$ are bounded by fifth–force/solar–system constraints and enter the global fit as priors.

6.16.7 Key constraints and null tests

- **Magnetic moments**: $\Delta a_e,\ \Delta a_\mu$; compare to high–precision values.
- **EDMs**: electron and neutron EDM bounds constrain CP–odd pieces of Ω and microscopic $\theta_{\rm spec}$ (Sec. 6.9.3).
- **Lepton universality**: rates in τ, μ decays test flavor–independence of probe couplings.
- **Rare decays**: FCNC processes suppressed by geometric selection rules; bounds translate to tunneling suppressions between valleys $\mathcal{V}_i$.
- **Proton stability**: absence of fast baryon number violation constrains multi–soliton tunneling sectors.
- **Running & unification**: whether Eq. (6.160) exhibits near–unification without fine tuning of thresholds at v_Σ.

6.16.8 Parameter windows and priors

Representative parameters and suggested ranges (all at $\mu_{\rm spec}$):

- Core scale $M_0 \in [10^{-6},\, 10^{-2}]\, M_{\rm Pl}$ (to accommodate lepton masses without tension with $G_{\rm eff}$).
- Geometric spacing $\alpha_\gamma \in [1.5,\, 3.5]$ (Eq. (6.157)) for (e, μ, τ).
- Berry monopole charges $q \in \{0, \frac{1}{2}, 1\}$ on orientation S^2 (Sec. 6.5); FR signs $\sigma = \pm 1$ fixed by $\pi_1(\mathcal{Q}_Q)$.
- Anisotropy parameters for MBP–1: $(I_\perp/I_\parallel - 1) \in [0, 0.3]$.
- Gauge couplings $g_i(\mu_{\rm spec})$ with broad priors matching $\alpha_i(m_Z)$ after running.

6.16.9 Global–fit workflow

1. **Precomputations**: for each sector Q, tabulate $\{M_0,\ I_a,\ \omega_a,\ F_{1,2}(q^2),\ A, B, C\}$ on a grid in the geometric parameters.
2. **Assignment pass**: choose a minimal assignment of states to observed particles (start with leptons), consistent with FR/Berry constraints.
3. **Likelihood**: build $\mathcal{L}(\theta)$ over parameter vector θ from masses, $g-2$, radii, selected widths, and running; include Gaussian priors for null tests.
4. **Sampling**: perform Bayesian sampling (e.g. nested sampling) with the gates of Sec. 6.13 as hard constraints (reject–on–violation).
5. **Validation**: hold–out observables (e.g. EDMs, rare decays) for posterior predictive checks; report p–values and Bayes factors comparing option A/B for hadrons.

6.16.10 Reporting format

For each mapped particle provide a compact record:

$$\mathcal{R} = \Big\{ Q,\ \text{core type (MBP–0/1)},\ \text{FR sign},\ q_{\text{Berry}},\ M_{\text{phys}} \pm \sigma,\ J^P,\ Q_{\text{eff}},\ g,\ \langle r^2 \rangle^{1/2},\ \text{key widths},\ \chi^2_{\text{dof}} \Big\}. \tag{6.161}$$

Include provenance: mesh details, regulator choices, and gate pass/fail flags.

6.16.11 Summary

The map proposed here translates geometric/collective data into the language of the Particle Data Group: masses, spins, charges, moments, mixings, and widths. Lepton masses emerge naturally from geodesic quantization on $\mathcal{M}_Q$; mixing and CP phases descend from Berry holonomies; running and near–unification reflect the $G_2 \to SU(3) \times SU(2) \times U(1)$ descent with heavy–mode thresholds. The framework is falsifiable: it offers sharp null tests (EDMs, universality, FCNC suppression) and predicts correlated deviations (e.g. in $g-2$ and charge radii) controlled by a small set of geometric parameters. This completes the phenomenological arc of Chapter 6 and sets the stage for the scattering analysis of Chapter 7.

6.17 Appendix 6.A: Geometric Proofs and Identities on the Moduli Space

A.1 Positive–definiteness of the moduli metric. Let $\{\Psi_I\}$ denote normalized zero modes tangent to the soliton family and $\langle \cdot, \cdot \rangle$ the kinetic inner product (Chap. 3). The canonical definition

$$G_{IJ} \;\equiv\; \sum_{\text{fields}} \int_\Sigma d^3x \, \sqrt{-g_{\text{eff}}}\ \Psi_I^\dagger(x)\, \Psi_J(x) \tag{6.162}$$

is positive–definite: for any $v^I \neq 0$, $v^I v^J G_{IJ} = \| \sum_I v^I \Psi_I \|^2 > 0$, since $\{\Psi_I\}$ are linearly independent and normalized by construction. Gauge–redundant directions (if present) are removed by projection; the remaining G_{IJ} is strictly positive.

A.2 Coordinate covariance. Under a smooth reparametrization $X^I \mapsto X^{I'}(X)$, zero modes transform as $\Psi_{I'} = \frac{\partial X^I}{\partial X^{I'}} \Psi_I$, and hence $G_{I'J'} = \frac{\partial X^I}{\partial X^{I'}} \frac{\partial X^J}{\partial X^{J'}} G_{IJ}$, showing that G is a $(0,2)$ tensor on $\mathcal{M}_Q$. The Christoffel symbols are the Levi–Civita connection of G: $\Gamma_{IJK} = \frac{1}{2}(\partial_I G_{JK} + \partial_J G_{IK} - \partial_K G_{IJ})$.

A.3 Gauss–Codazzi–like identity for embedded moduli. Let $\mathcal{E}$ denote the full configuration manifold with metric $\mathbb{G}$ induced by the kinetic inner product, and let $\iota : \mathcal{M}_Q \hookrightarrow \mathcal{E}$ be the soliton embedding. Denote the second fundamental form by $K_{IJ}{}^\alpha$ in the normal bundle. Then the Riemann tensor of $(\mathcal{M}_Q, G)$ obeys

$$R_{IJKL} \;=\; \mathbb{R}_{IJKL} \;+\; K_{IK}{}^\alpha K_{JL\,\alpha} \;-\; K_{IL}{}^\alpha K_{JK\,\alpha}, \tag{6.163}$$

where $\mathbb{R}_{IJKL}$ is the pullback of the ambient curvature. This controls curvature–induced forces and harmonic analysis on $\mathcal{M}_Q$.

A.4 Berry curvature quantization and prequantization line bundle. Let $\mathcal{A}$ be the Berry connection on the physical Hilbert bundle over $\mathcal{M}_Q$, with curvature $\Omega = d\mathcal{A}$. For any closed two–cycle $\Sigma \subset \mathcal{M}_Q$,

$$\frac{1}{2\pi} \int_\Sigma \Omega \;\in\; \mathbb{Z}, \tag{6.164}$$

provided the many–body ground state is globally defined up to a $U(1)$ phase and the parameter space is free of level crossings on Σ. This integrality establishes the existence of the prequantum line bundle $\mathcal{L}_{\text{phys}}$.

A.5 FR characters and spin/statistics. Let $\mathcal{Q}_Q$ be the configuration space modulo physically trivial symmetries. A homomorphism $\sigma : \pi_1(\mathcal{Q}_Q) \to \{\pm 1\}$ (the FR character) defines a double cover on which wavefunctions are single–valued. A 2π body rotation corresponds to a loop $\gamma_{2\pi}$; if $\sigma(\gamma_{2\pi}) = -1$, the soliton quantizes as a fermion with half–integer spin. Compatibility with Berry holonomy demands $\exp\{i \oint_\gamma \mathcal{A}\} = \sigma(\gamma)$ for all nontrivial γ up to $2\pi\mathbb{Z}$ phases.

A.6 Adiabatic condition from spectral gap. Let $\Delta(X)$ be the minimal excitation gap above the collective band. If $\|\dot{X}\|_G \ll \Delta/\Lambda$ for some dimensionless $\Lambda = \mathcal{O}(1)$ set by matrix elements of $\partial_I \mathbb{H}$, then nonadiabatic transitions are exponentially suppressed (Landau–Zener), justifying the moduli truncation.

A.7 Positivity of polarizabilities. Kubo formulas for spectral electric/magnetic polarizabilities read $\alpha_E = \frac{2}{\pi}\int_0^\infty d\omega\, \frac{\mathrm{Im}\, G^{\mathrm{R}}_{J^0 J^0}(\omega)}{\omega^3}$, etc. Spectral positivity implies $\alpha_E, \beta_M \geq 0$ (modulo subtractions), consistent with GKSL complete positivity (Sec. 6.10).

A.8 Ward identities on the collective theory. Current conservation and diffeomorphism invariance yield, for on–shell collective eigenstates, $q_\mu \langle n'|J^\mu_\Phi|n\rangle = 0$, $q_\mu \langle n'|T^{\mu\nu}|n\rangle = 0$, fixing $F_1(0) = Q_{\mathrm{eff}}$ and $A(0) = 1$ in the small–q expansion of Sec. 6.14.

6.18 Appendix 6.B: Numerical Toolkit and Reproducibility Recipes

B.1 Discrete variational formulation. Discretize the static energy functional $E[\mathcal{S}, \Phi]$ with a Galerkin finite–element basis preserving constraints (e.g. unit–norm $\hat{\chi}$ via projected updates). Use a variational integrator in time–dependent simulations to conserve discrete Noether charges up to solver tolerance.

B.2 Constraint enforcement. Adopt either (i) penalty terms with adaptive weights, or (ii) constrained optimization with Lagrange multipliers projected at each nonlinear step. For $\hat{\chi}$, project to the unit sphere after every update to prevent drift; verify $|\hat{\chi}| - 1| < 10^{-8}$.

B.3 Gauge/phase handling for Berry data. Represent phases by link variables on a triangulation $\{X_i\}$ of $\mathcal{M}_Q$: $U_{ij} = \langle \psi_0(X_i)|\psi_0(X_j)\rangle/|\cdot|$. The plaquette phase $\Phi_\square = \arg(U_{12}U_{23}U_{34}U_{41})$ approximates the Berry flux; summing over a closed mesh yields an integer (within tolerance) per Eq. (6.17).

B.4 Hessian spectral solver. Assemble the discrete Hessian $\mathbb{H}$ at the relaxed soliton; use shift–invert Lanczos to extract the lowest spectrum. Remove zero modes by projection; check orthogonality leakage $< 10^{-8}$ and convergence of eigenpairs under mesh refinement.

B.5 Worldline extraction of g and polarizabilities. (i) Impose small uniform rotation and measure induced current to obtain g via Eq. (6.14.6.134); (ii) apply weak uniform fields to extract (α_E, β_M) from energy shifts; correct finite–volume effects with $1/L$ extrapolations.

B.6 Radiation: far–zone and QNM. Implement absorbing layers; reconstruct far–zone fields to compute Poynting/Isaacson fluxes (Sec. 6.15). Alternatively, solve frequency–domain QNM problems for periodic motion.

B.7 Error budgets and acceptance. Report truncation, nonlinear, spectral, finite–volume, and regulator errors separately; require convergence plateaus and pass gates N1–N6 (Sec. 6.12) and R_rad1–R_rad5 (Sec. 6.15).

B.8 Reproducibility bundle. Package meshes, solver parameters, random seeds, hashes of (U, G, Ω) and of benchmark observables, plus a log of pass/fail gates for each run to enable byte–level reproducibility.

B.9 Pseudocode stencil (illustrative).

Relax–project loop: initialize $(\mathcal{S}, \Phi)$; repeat {Newton step → line search → constraint projection} until $\|\delta E\|/E < \varepsilon$; then assemble $\mathbb{H}$, extract zero modes and G_{IJ}, compute Berry link variables, and validate gates.

6.19 Appendix 6.C: Extended FR Examples, Berry Bundles, and Anomaly Inflow

C.1 FR on MBP–0 (spherical core). Configuration space $\mathcal{Q}_Q$ modulo global rotations admits a nontrivial loop corresponding to a 2π body rotation. If this loop is not contractible (due to internal orientation defects), choose $\sigma(\gamma_{2\pi}) = -1$ to enforce fermionic statistics; the physical Hilbert space lives on the double cover. Compatibility with Berry holonomy requires $\exp\{i \oint_{\gamma_{2\pi}} \mathcal{A}\} = -1$.

C.2 FR on MBP–1 (axisymmetric core). With a director $\hat{\chi}$ and residual $SO(2)$, loops that wind once around the symmetry axis can be nontrivial. The combined effect of FR and a Berry monopole $q \in \frac{1}{2}\mathbb{Z}$ on S^2 shifts allowed J values via monopole harmonics; forbidden transitions in Secs. 6.14–6.15 follow from selection rules on S^2 with Chern number $2q$.

C.3 Hopfion sector and half–integer Berry flux. For $Q_{\mathrm{H}} = 1$ textures, adiabatic rotation maps to a nontrivial S^2 in $\mathcal{M}_Q$ with flux $\frac{1}{2\pi}\int_{S^2} \Omega = q$, $q \in \frac{1}{2}\mathbb{Z}$. The half–integer case requires the FR double cover to keep physical states single–valued; together they realize a consistent spin structure on the bundle $\mathcal{L}_{\mathrm{phys}}$.

C.4 WZW/CS descent and anomaly inflow. Let $\mathcal{A}$ be the internal gauge background on $\mathcal{M}_Q$ with field strength $\mathcal{F}$. A five–dimensional Chern–Simons form $\omega_5(\mathcal{A})$ satisfies $d\omega_5 = \operatorname{tr} \mathcal{F}^3$. The Wess–Zumino–Witten term on the physical boundary $\partial\mathcal{M}_Q$ cancels the gauge variation of ω_5, ensuring that Berry phases and FR signs are anomaly–free on closed loops. This fixes discrete counterterms in the collective action.

C.5 Worked loop taxonomy. We classify representative nontrivial loops $\{\gamma_i\}$ in $\pi_1(\mathcal{Q}_Q)$: body 2π rotation, director 2π twist, combined rotations with noncommuting order. For each γ_i, we list $(\sigma(\gamma_i), \exp i \oint_{\gamma_i} \mathcal{A})$ and the induced selection rule (allowed J^P and multipoles).

C.6 CP–odd phases and EDM bounds. A CP–odd contribution $\theta_{\mathrm{spec}} \int \operatorname{tr} \mathcal{F} \wedge \mathcal{F}$ induces a shift of Berry phases on certain two–cycles, generating small EDMs for solitons with $J \geq \frac{1}{2}$. Bounds on electron/neutron EDMs translate into constraints on θ_{spec} and on the geometry of the relevant cycles in $\mathcal{M}_Q$.

C.7 Summary. FR characters and Berry bundles jointly determine spin, statistics, and selection rules; anomaly inflow fixes otherwise ambiguous phases. These structures are numerically accessible via the link– variable protocols of App. 6.18 and provide sharp, falsifiable signatures in external responses and radiative processes.

6.20 Catalogue of Classical Solutions

Consistency gates (legend). **S** (Stability): time-kinetic Hessian > 0, no ghosts/tachyons. **G** (Gauge): gauges consistently fixed; Ward identities hold. **TQ** (Topology): correct invariant (e.g. π_k, winding, Q_H). **D** (Dynamics/causality): mode speeds within g_{eff} cone. **N** (Numerics): mesh/box independence; convergence/virial checks.

6.20.1 MBP-0: Homogeneous Base Phase

Sector. Structural + Palatini gravity.

Topology. Trivial π_0.

Noether charges. Total energy $\mathcal{E}$ and momentum $\mathcal{P}_i$.

Ansatz. Uniform state $S_0 = \{\rho_0, \eta_0, T_0, \chi_I^{(0)}, \alpha_0\}$. Construct $e^a{}_\mu(S_0)$ and $g_{\mu\nu}^{(0)} = g_{\rm eff}[S_0]$.

Equations of motion. Stationarity $\partial\mathcal{L}/\partial S|_{S_0} = 0$; Z_{AB}, Y_{AB} positive definite.

Boundary conditions. Periodic box / compact 3-manifold.

Energy and checks. $\mathcal{E}_0/V = \mathcal{L}_{\rm stat}(S_0)$. Define $M_{\rm eff}^2(\rho_0)$; virial identity.

Moduli. Translations; internal global phases.

Linear spectrum and stability. Speeds $(c_T, c_\parallel, c_\perp)$ on $g^{(0)}$; $c_i^2 \le 1$.

Observables. Calibrate $M_{\rm eff}$.

Numerics. Fourier stability; 2nd/4th order convergence.

Consistency gates. **S/G/TQ/D/N**.

6.20.2 Straight Vortex with winding n

Sector. Gauge-emergent from α + structural.

Topology. $\pi_1 \simeq \mathbb{Z}$.

Noether. Phase charge Q, P_z.

Ansatz. Cylindrical (r, ϕ, z): $\alpha = n\,\phi$, $\rho = \rho(r)$, $\chi_I = \chi_I(r)$. Structural connection screens $\nabla\alpha$ far away.

EOM. Radial ODEs for ρ, χ_I; circulating current.

BC. $r \to 0$: $\rho \sim r^{|n|}$; $r \to \infty$: $\rho \to \rho_0$ with finite tension.

Energy. $\mathcal{T}_n = \displaystyle\int_0^\infty 2\pi r\, dr\, \mathcal{H}(r; n) \sim |n|\mu_{\rm eff}$.

Moduli. Core position (x_0, y_0); global phase.

Spectrum/Stability. Translational zero mode; global-phase mode; no negative time-eigenvalues.

Observables. Log/Yukawa forces; spectral wave signatures.

Numerics. Shooting/relaxation; box-size independence; mesh convergence.

Consistency gates. **S/G/TQ/D/N**; $n = \frac{1}{2\pi}\oint d\alpha \in \mathbb{Z}$.

6.20.3 Domain Wall between S_+ and S_-

Sector. Structural (1D) + Palatini gravity (fixed background).

Topology. Nontrivial π_0.

Noether. Surface tension σ.

Ansatz. Planar wall, $S = S(x)$ with $S(\pm\infty) = S_\pm$.

EOM. 1D Euler–Lagrange system.

BC. Exponential approach to $S_\pm$.

Energy. $\sigma = \int_{-\infty}^{\infty} dx\, \mathcal{H}[S, S']$; virial/Derrick check.

Moduli. Wall position x_0; surface waves.

Spectrum. One translational zero mode; continuum.

Observables. Reflection/transmission; interface matching.

Numerics. Relaxation/Newton on stretched meshes.

Consistency gates. **S/G/TQ/D/N**.

6.20.4 Spherical Q-ball

Sector. Matter + structural (time-rotating phase).

Topology. None.

Noether. Global charge Q.

Ansatz. $\alpha(t) = \omega t$; radial profiles for ρ, η, T, χ_I.

EOM. Radial ODEs with ω-dependent coefficients.

BC. $r \to 0$ regular; $r \to \infty$ exponential decay to S_0.

Energy/Existence/Stability. $E(\omega) = \int d^3x\, \mathcal{H}_\omega$, $Q(\omega) = \int d^3x\, \Pi_\alpha$; window $\omega_{\min} < \omega < \omega_{\max}$; stable if $dQ/d\omega < 0$.

Moduli. Center; global phase.

Spectrum. Breathing + angular modes.

Observables. $E(Q)$; capture/scattering.

Numerics. Iterative/Newton–Krylov; virial identity.

Consistency gates. **S/G/TQ/D/N**.

6.20.5 3D Hopfion (Skyrmion/Hopf)

Sector. Structural + gauge-emergent (Faddeev–Skyrme effective).

Topology. Nontrivial π_3; Hopf charge Q_H.

Noether. Possible effective J under collective rotation.

Ansatz. Map $n : \mathbb{R}^3 \to S^2$ (or $S^3 \to S^2$ after compactification); toroidal/knotted by Q_H.

EOM. 3D Euler–Lagrange with quadratic+quartic terms.

BC. Approach S_0 at spatial infinity.

Energy. $E \sim c\,|Q_H|^{3/4}$ (empirical scaling); geometric diagnostics.

Moduli. Position, orientation, ring radius/thickness.

Spectrum. Zero modes (trans/rot) + discrete shape modes.

Observables. Hopfion scattering; polarization signatures.

Numerics. 3D minimization with adaptive meshing; reproducibility.

Consistency gates. **S/G/TQ/D/N**; $Q_H = \dfrac{1}{32\pi^2}\displaystyle\int \epsilon^{ijk} A_i F_{jk}\, d^3x$ (effective).

6.20.6 Finite-length Vortex Bound to a Domain Wall (String–Wall Junction)

Sector. Gauge-emergent from α + structural; wall in structural sector; Palatini gravity as fixed background.

Topology. Mixed: π_1 (string) terminating on a nontrivial π_0 wall; junction carries a localized "boojum" charge.

Noether. Phase charge along the string (per unit length); surface tension on the wall; total energy/momentum.

Ansatz. A straight vortex along z spans a finite segment in $x > 0$ and ends on a planar wall at $x = 0$. Fields: $\alpha \sim n\,\phi$ near the core; across the wall $S(x \to \pm\infty) \to S_\pm$ with distinct minima.

EOM. Coupled PDEs in (x, r, ϕ) from the reduced Lagrangian; at the junction the emergent gauge flux is absorbed by the wall phase misalignment.

BC. Regular core near the string; far field $\rho \to \rho_0$, $\chi_I \to \chi_I^{(0)}$; across the wall exponential approach to $S_\pm$.

Energy and balance. Total energy $E = \mathcal{T}_n L + \sigma A + E_{\text{junction}}$. Force balance at the junction: the string tension component equals the wall tension times the incidence angle (Young–Laplace–like).

Moduli. String end position on the wall; wall translation x_0; junction orientation in the wall plane.

Spectrum and stability. Kelvin-like oscillations on the finite string; capillary waves on the wall; a localized junction mode. Stability: absence of negative time-eigenvalues in the coupled spectrum.

Observables. Pinning/depinning thresholds; refraction of spectral waves at the junction; characteristic corner dispersion.

Numerics. 3D relaxation with mesh refinement at the junction; verify force balance and independence from box size.

Consistency gates. S/G/TQ/D/N.

6.20.7 Closed Vortex Ring

Sector. Gauge-emergent from α + structural.

Topology. π_1 nontrivial winding along the ring; linking number with phase sheets can be defined.

Noether. Circulation; possible linear momentum if the ring propagates; total energy.

Ansatz. A toroidal loop of radius R with core radius $a \ll R$; in toroidal coordinates the phase winds once along the poloidal angle.

EOM. Axisymmetric PDEs reduce to 2D in (r, z) with suitable toroidal ansatz; curvature of the loop enters the effective tension.

Energy. $E(R) \simeq 2\pi R\,\mathcal{T}_1 + \kappa_{\text{bend}}\,\frac{2\pi}{R} + \cdots$ (string tension plus curvature corrections).

Moduli. Loop center; radius R; orientation. Kelvin-wave moduli along the loop.

Spectrum and stability. Kelvin dispersion $\omega_K(k) \approx \Omega_0 + Dk^2 - Ek^4 + \cdots$; a minimal stable R if curvature softening dominates at small R.

Observables. Self-propulsion velocity $\propto 1/R$; scattering with walls/other loops; reconnection signatures.

Numerics. Axisymmetric minimization; conservation checks for circulation; R-dependence of E.

Consistency gates. S/G/TQ/D/N.

6.20.8 Monopole-like Hedgehog (π_2)

Sector. Structural (unit vector from χ_I or normalized internal direction); gauge-emergent sector screens gradients.

Topology. Nontrivial $\pi_2(S^2)$: hedgehog map from spatial S^2_∞ to internal S^2.

Noether. None required; total energy finite only if a quartic (Skyrme-like) term or gauge screening is present.

Ansatz. Spherically symmetric $n^a(\mathbf{x}) = x^a/r$ for a normalized internal vector built from S; radial profiles for $\rho(r), \eta(r)$ regularize the core.

EOM. Spherical ODEs with quadratic+quartic gradients (or gauge screening) to evade the Derrick scaling instability.

Energy. $E = \int dr\, r^2 \big[c_2(\partial_r n)^2 + c_4(\partial_r n \times \partial_\theta n)^2/r^2 + \cdots\big]$; finite for $c_4 > 0$ or appropriate gauge couplings.

Moduli. Center position; internal $SO(3)$ rotations (if symmetry not explicitly broken).

Spectrum. Translational zero modes; triplet of rotational modes; radial shape excitations.

Observables. Scattering with vortices/walls; defect–defect forces; possible emergent "magnetic" Coulomb tail if unscreened.

Numerics. Shooting/relaxation in r; verify energy finiteness and scaling under uniform rescalings.

Consistency gates. S/G/TQ/D/N.

6.20.9 Quasi-2D Baby Skyrmion (π_2 in thin films)

Sector. Structural in a thin slab; effective $O(3)$ unit vector from S; optional easy-axis potential.

Topology. $\pi_2(S^2)$ in 2D; integer charge $Q = \frac{1}{4\pi}\int n \cdot (\partial_x n \times \partial_y n)\, dxdy$.

Noether. None essential; total energy and momentum.

Ansatz. Axisymmetric profile $n(r, \phi)$ with a skyrmion core; fields ρ, η, α adjust to minimize energy.

EOM. 2D Euler–Lagrange for quadratic+quartic gradients plus anisotropy; stable size fixed by balance.

Energy. Finite, set by quartic term and anisotropy; defines a core radius $R_{\rm sk}$.

Moduli. Position (x_0, y_0); global in-plane rotations if an easy-plane is present.

Spectrum. Translational zero modes; breathing mode; azimuthal distortions.

Observables. Skyrmion Hall drift under spectral gradients; controlled motion by $\nabla\eta$ or $\nabla\rho$.

Numerics. 2D minimization on periodic or large domains; convergent core size; robustness to mesh.

Consistency gates. S/G/TQ/D/N.

6.20.10 Bound Dimer of Hopfions

Sector. Structural + gauge-emergent (effective Faddeev–Skyrme sector).

Topology. Total Hopf charge $Q_H = Q_{H1} + Q_{H2}$; linking between preimages changes interaction.

Noether. Total energy/momentum; possible collective rotation J.

Ansatz. Two toroidal Hopfions separated by R with relative orientation angle $\Delta\Phi$; superposition/relaxation initial data.

EOM. Full 3D PDEs; interaction mediated by twisted gradients and emergent gauge fields.

Energy and binding. $E(R, \Delta\Phi)$ with a shallow minimum at $R = R_b$ for specific $\Delta\Phi$; binding $\Delta E = E - 2E_1 < 0$.

Moduli. Separation R; relative orientation $\Delta\Phi$; overall position/orientation.

Spectrum. Relative libration modes; out-of-phase breathing; translational/rotational zero modes.

Observables. Resonant scattering; characteristic polarization of emitted spectral waves upon excitation.

Numerics. 3D relaxation with continuation in $R, \Delta\Phi$; convergence under mesh refinement.

Consistency gates. S/G/TQ/D/N.

6.20.11 Periodic Vortex Lattice

Sector. Gauge-emergent from α + structural; weak emergent-gravity backreaction.

Topology. π_1 per cell; total winding equals sum over lattice sites.

Noether. Average phase charge density; total energy per cell.

Ansatz. Hexagonal (Abrikosov-like) arrangement; single-cell computation with Bloch boundary conditions.

EOM. Cell PDEs with periodicity; far-field screening of $\nabla\alpha$ by the structural connection.

Energy. Energy density vs. lattice spacing a; defines an optimal a^* at given background parameters.

Moduli. Lattice orientation; shear/strain modes (phonons).

Spectrum. Phonon-like excitations; Tkachenko-like shear waves.

Observables. Bragg peaks in spectral–wave scattering; elastic constants of the vortex crystal.

Numerics. Pseudo-spectral on a unit cell; convergence of elastic constants with grid.

Consistency gates. S/G/TQ/D/N.

6.20.12 Long-lived Oscillon (Time-periodic Localized State)

Sector. Matter + structural; small dissipative η allowed.

Topology. None.

Noether. Approximate adiabatic invariants; total energy.

Ansatz. Spherically symmetric, time-periodic solution: $S(t, r)$ with dominant frequency Ω and slow amplitude drift.

EOM. Radial PDE in t, r; balance between nonlinear stiffening and dispersion; weak leakage due to $\eta > 0$.

Energy and lifetime. $E(t)$ decays slowly: $\dot{E} \sim -\gamma\,\eta \int |\nabla\Phi|^2 d^3x$; lifetime $\tau \gg \Omega^{-1}$ in the oscillon window.

Moduli. Center; phase at $t = 0$; slow radius.

Spectrum. Harmonic content peaked at Ω; sidebands from slow modulation.

Observables. Soft emission spectrum; response to weak probes.

Numerics. Time-domain evolution with absorbing boundaries; monitor $E(t)$ and Fourier content.

Consistency gates. **S/G/TQ/D/N**.

6.20.13 Generic Two-soliton Dimer with Spectral-curvature Repulsion

Sector. Structural + matter; includes spectral-curvature corrections F_{spec}.

Topology. Depends on constituents (e.g. $n + \bar{n}$, $Q_H + Q'_H, \dots$); total topological charge additive.

Noether. Total $E, \mathbf{P}$; possible phase charge(s).

Ansatz. Two localized solitons at separation R; relative phase $\Delta\alpha$; orientation angles.

EOM. Coupled PDEs; interaction potential $V_{\text{int}}(R) = V^{(0)}(R) + \Delta V_{\text{spec}}(R)$ with short-range repulsive core from F_{spec} (cf. App. E.4).

Energy and binding. Minimum at R_b when attraction at long range balances curvature-induced repulsion; $E_b > 0$ small.

Moduli. R and relative orientation; overall translations/rotations.

Spectrum. Small-amplitude dimer oscillations; relative librations.

Observables. Thermal dissociation threshold; inelastic spectral emission when driven.

Numerics. Constraint continuation in R; verify R_b and E_b under mesh refinement.

Consistency gates. **S/G/TQ/D/N**.

6.20.14 Kink–Antikink Pair and Breather-like States (1D)

Sector. Structural (1D); matter can be spectator.

Topology. Net π_0 charge cancels ($+1$ with -1).

Noether. Total energy; momentum in 1D.

Ansatz. Back-to-back domain wall and anti-wall separated by L; small relative velocity v; near-integrable limit admits breather-like oscillations.

EOM. 1D PDEs; interaction potential $V(L)$ exponential at large L; radiative losses at finite η.

Energy and lifetime. $E(L, v)$; breathers survive many periods if radiation is weak ($\eta \approx 0$).

Moduli. Separation L; center-of-mass position.

Spectrum. Internal oscillation frequency; coupling to continuum modes leads to slow decay.

Observables. Resonant windows in kink–antikink scattering; phase-dependent capture.

Numerics. High-order time stepping; absorbing boundaries; track $L(t)$ and radiated energy.

Consistency gates. **S/G/TQ/D/N**.

6.20.15 Emergent Gravitational-wave Packet on g_{eff}

Sector. Palatini gravity (emergent) sourced by structural background S_0.

Topology. None.

Noether. Wave energy/momentum in the effective spacetime.

Ansatz. Small perturbation $h_{\mu\nu}$ on $g^{(0)}_{\mu\nu}$ with transverse–traceless-like gauge under the emergent connection.

EOM. Linearized Einstein-like equations $\mathcal{E}[h] = 0$ with anisotropic dispersion if S_0 is anisotropic (cf. Ch. 3 & 4).

Energy. Quadratic wave energy density; conserved (to linear order) in absence of dissipation.

Moduli. Central frequency; wave packet width; polarization relative to $\hat{\chi}$.

Spectrum. Group velocities $v_g(\hat{\mathbf{k}})$ may exhibit small anisotropy $v_{\parallel} \neq v_{\perp}$.

Observables. Directional dependence in arrival times; birefringence bounds.

Numerics. Linear time-domain evolution on $g^{(0)}$; measure v_g and polarization content.

Consistency gates. S/G/TQ/D/N.

6.20.16 Spectral Gauge Wave (Photon-like Mode)

Sector. Gauge-emergent sector primarily from α (spectral $U(1)$) coupled to structural background.

Topology. None.

Noether. Spectral "electric"/"magnetic"-like fluxes; wave energy.

Ansatz. Plane-wave configurations in the emergent gauge potential associated with α-gradients; background S_0 sets effective permittivity/permeability.

EOM. Maxwell-like linear system with anisotropic constitutive relations derived from S_0.

Energy. Quadratic field energy; Poynting-like vector defines energy flux.

Moduli. Polarization states adapted to $\hat{\chi}$; wave vector $\mathbf{k}$.

Spectrum. Dispersion $\omega(\mathbf{k})$ possibly birefringent; small deviations from isotropy bounded by observations.

Observables. Polarization rotation in structured media; mode conversion near defects.

Numerics. Finite-difference time-domain (FDTD)-like evolution for the emergent Maxwell system; stability via CFL bounds tied to g_{eff}.

Consistency gates. S/G/TQ/D/N.

6.20.17 Trimer: Triangle-bound Three-Soliton Molecule

Sector. Structural + gauge-emergent + (optional) matter spectator.

Topology. Additive topological charges of the three constituents; total may be neutral depending on species.

Noether. Total $E, \mathbf{P}$; possible net phase charges; discrete C_3 symmetry when equilateral.

Ansatz. Three identical solitons at the vertices of an equilateral triangle of side L; relative internal phases/orientations chosen to maximize attraction and include spectral-curvature repulsion at short range.

EOM. Full 3D PDEs or reduced moduli potential $V(L, \Delta\alpha_{ij}, \Delta\Phi_{ij})$ obtained by constrained minimization.

Energy and binding. $E(L) = 3E_1 + 3V_{\text{2-body}}(L) + V_{\text{3-body}}(L)$; genuine three-body term arises from nonlinearity and emergent gauge backreaction. A minimum at $L = L_b$ implies a stable trimer.

Moduli. Global translation/rotation; breathing mode (side length); shear mode (isosceles distortions); internal relative phases $\Delta\alpha_{ij}$.

Spectrum. Normal modes split by C_3: one breathing, one doubly-degenerate shear; internal librations coupled weakly.

Observables. Discrete spectral lines under small excitation; selection rules tied to C_3 parity.

Numerics. Constrained relaxation at fixed L; continuation in L to map $V_{\text{eff}}(L)$; mesh-convergence for L_b.

Consistency gates. S/G/TQ/D/N.

6.20.18 Curved Domain Walls: Spherical Bubbles and Cylindrical Tubes

Sector. Structural (wall) with Palatini gravity as a fixed or weakly backreacting background.

Topology. Nontrivial π_0 separating phases S_- and S_+; closed wall encloses a bubble of S_+ in S_- (or vice versa).

Noether. Surface tension σ; volume/pressure difference $\Delta V = U(S_-) - U(S_+)$.

Ansatz. Spherical wall of radius R (thin-wall limit) or cylindrical tube of radius R and length L; profiles solve 1D wall equation in the normal coordinate and embed into curved surface.

EOM. Balance ΔV against mean curvature κ: for a sphere $2\sigma/R = \Delta V + \mathcal{O}(R^{-2})$; corrections from wall thickness and anisotropy along $\hat{\chi}$.

Energy. $E_{\text{sph}}(R) = 4\pi R^2\sigma + \frac{4}{3}\pi R^3\Delta V + \ldots$; metastable radii exist when $\Delta V \approx 0$.

Moduli. Center position; radius R; for tubes: axis orientation and length (or periodic BC).

Spectrum. Capillary modes with dispersion $\omega^2 = \frac{\sigma}{\rho_{\text{eff}}}k^3 + \cdots$ on the wall, modified by anisotropy.

Observables. Nucleation thresholds; collapse vs. expansion dynamics; scattering of waves on curved interfaces.

Numerics. Axisymmetric PDEs for spheres/tubes; validate thin-wall formulae; include anisotropic corrections.

Consistency gates. **S/G/TQ/D/N**.

6.20.19 Collectively Rotating Hopfion

Sector. Structural + gauge-emergent (Faddeev–Skyrme effective terms).

Topology. Hopf charge Q_H preserved under slow collective rotation.

Noether. Energy E and angular momentum J; effective gyromagnetic-type responses in gauge-emergent sector.

Ansatz. Rigid rotation around an axis with angular velocity Ω; comoving ansatz $S(\mathbf{x}, t) = S_0(R_{\Omega t}^{-1}\mathbf{x})$ plus small deformations to respect causality ($\Omega r < 1$).

EOM. Variational equation for $E(J) = E_0 + \frac{J^2}{2\mathcal{I}} + \ldots$; $\mathcal{I}$ is the moment of inertia induced by gradients.

Energy and limits. Causality bound $\Omega r_{\max} < 1$; above a critical Ω, shape instabilities set in.

Moduli. Center; orientation of rotation axis; Q_H-dependent shape parameters.

Spectrum. Coriolis-split shape modes; quasi-normal modes broaden with Ω.

Observables. Polarization-dependent emission; precession under external spectral fields.

Numerics. 3D minimization at fixed J (constrained); check $dE/dJ = \Omega$; mesh convergence.

Consistency gates. **S/G/TQ/D/N**.

6.20.20 Confined Monopole Terminated by a Vortex (Flux Tube)

Sector. Structural hedgehog core (π_2) with a π_1 vortex confining its emergent flux.

Topology. Combined π_2 core with a single vortex carrying flux to infinity or to another monopole.

Noether. Total energy; possible spectral charge along the tube.

Ansatz. Spherical hedgehog at $z = 0$ with a straight vortex along $+z$; axial symmetry; matching conditions at the core where flux lines enter the vortex.

EOM. Coupled axisymmetric PDEs; gauge screening in the tube reduces far-field energy of the hedgehog.

Energy. $E = E_{\text{core}} + \mathcal{T}L$ (for finite L) + E_{junction}; finite if the tube extends to another monopole or a wall.

Moduli. Core position; tube axis orientation; length L (if finite).

Spectrum. Tube phonons; core radial mode; localized junction oscillation.

Observables. Confinement-like linear potential between monopole–antimonopole endpoints.

Numerics. Axisymmetric relaxation with adaptive radial/axial grids; verify linear energy vs. L.

Consistency gates. **S/G/TQ/D/N**.

6.20.21 False-vacuum Bubble and Euclidean Bounce

Sector. Structural potential $U(S)$ with two minima; Palatini gravity corrections via M_{eff} (weak).

Topology. π_0 structure with false and true vacua.

Noether. Euclidean action S_E; nucleation rate $\Gamma/V \sim Ae^{-S_E}$.

Ansatz. $O(4)$-symmetric Euclidean profile $S(\rho)$ with $\rho = \sqrt{\tau^2 + \mathbf{x}^2}$; thin-wall in small ΔU.

EOM. Euclidean Euler–Lagrange with friction term $(3/\rho)\, S'(\rho)$; matching to Lorentzian bubble at nucleation.

Action. $S_E \simeq \frac{27\pi^2\sigma^4}{2(\Delta U)^3}$ (thin-wall) with corrections from anisotropic gradients.

Moduli. Bubble center in Euclidean space.

Observables. Nucleation rate; post-nucleation expansion dynamics and gravitational wave signatures (if many bubbles).

Numerics. Shooting method in ρ; match to thin-wall analytic limits; include M_{eff} corrections perturbatively.

Consistency gates. **S/G/TQ/D/N**.

6.20.22 Striped / Lamellar Modulated Phase

Sector. Structural with competing gradient terms (Lifshitz-like) and weak gauge backreaction.

Topology. None (periodic pattern), but defects (dislocations) may carry topological indices.

Noether. Energy density; crystal momenta in Brillouin zone.

Ansatz. One-dimensional modulation along $\hat{\mathbf{q}}$: $S(\mathbf{x}) \approx S_0 + \Delta S \cos(\mathbf{q}\cdot\mathbf{x})$ plus higher harmonics.

EOM. Minimize free-energy density $\mathcal{F}(q, \Delta S)$ with $c_2 q^2 + c_4 q^4$ competition; select q^*.

Energy. $\epsilon(q) = \epsilon_0 + \frac{1}{2}c_2 q^2 + \frac{1}{4}c_4 q^4 + \cdots$; $c_2 < 0, c_4 > 0$ yields finite q^*.

Moduli. Global phase (translation of stripes); domain walls between stripe orientations.

Spectrum. Phasons (sliding) and amplitudons (breathing) with anisotropic dispersions.

Observables. Bragg peaks at q^*; anisotropic transport of spectral waves.

Numerics. Pseudo-spectral minimization on periodic domains; map phase diagram in (c_2, c_4).

Consistency gates. **S/G/TQ/D/N**.

6.20.23 Vorton: Current-stabilized Vortex Ring

Sector. Gauge-emergent from α with a persistent phase current; structural background provides tension.

Topology. π_1 winding along the ring and conserved phase circulation; a second winding (internal) may exist.

Noether. Circulation N; current I; total angular momentum J.

Ansatz. A closed vortex loop of radius R carrying a steady timelike phase gradient along the loop (supercurrent).

EOM. Balance between string tension ($\propto R$) and current energy ($\propto 1/R$): $E(R) \approx 2\pi R\mathcal{T} + \frac{\kappa I^2}{R} + \cdots$.

Energy and stability. A minimum at $R = R^* \sim \sqrt{\kappa I^2/\mathcal{T}}$ yields a classically stable vorton.

Moduli. Center and orientation; R; current I (quantized by N).

Spectrum. Kelvin modes on the loop; current-coupled torsional mode.

Observables. Persistent currents; characteristic magnetic-like fields in the emergent gauge sector.

Numerics. Axisymmetric PDEs with periodic boundary along the loop; continuation in I.

Consistency gates. **S/G/TQ/D/N**.

6.20.24 Y-Junction of Three Vortices

Sector. Gauge-emergent (multi-component spectral phases) + structural.

Topology. Sum of three π_1 windings meeting at a node; overall conservation at the junction.

Noether. Tensions $\mathcal{T}_i$ for each leg; node energy.

Ansatz. Three straight vortices meet at angles θ_i such that vector force balance holds: $\sum_i \mathcal{T}_i \hat{\mathbf{t}}_i = 0$.

EOM. Field equations in 2D cross-section with boundary conditions encoding three asymptotic phases.

Energy. $E = \sum_i \mathcal{T}_i L_i + E_{\text{node}}$; node energy depends on relative internal phases.

Moduli. Node position; opening angles; relative phases.

Spectrum. Localized node oscillation; leg phonons; phase-slip modes.

Observables. Junction motion under unbalanced tensions; splitting/merging processes.

Numerics. 2D relaxation on a star-shaped domain; verify vector balance.

Consistency gates. **S/G/TQ/D/N**.

6.20.25 Multi-soliton Cluster (Skyrmion Bag)

Sector. Structural with effective Faddeev–Skyrme terms; weak gauge-emergent coupling.

Topology. Total topological charge is the sum of constituents; inner structure arranged in shells.

Noether. Total $E, \mathbf{P}$; possible J for rotating bags.

Ansatz. Spherical container profile with multiple skyrmion-like lumps arranged in shell patterns; variational density for ρ, η to accommodate.

EOM. 3D Euler–Lagrange solved by relaxation; constraints maintain target total charge.

Energy. Energy vs. particle number N: $E(N) \approx NE_1 + \Delta E_{\text{surf}} + \Delta E_{\text{pair}}$; magic numbers when shells close.

Moduli. Overall size; shell radii; internal orientations.

Spectrum. Collective breathing; inter-shell shear; rotational bands.

Observables. Discrete lines in small-amplitude spectra; enhanced stability at magic numbers.

Numerics. 3D minimization with charge-conserving flows; sensitivity to initial guesses examined.

Consistency gates. S/G/TQ/D/N.

6.20.26 Hybrid Bound State: Q-ball Attached to a Vortex

Sector. Matter (Q-ball) coupled to gauge-emergent/structural vortex sector.

Topology. None globally (for the Q-ball), but π_1 for the vortex; composite has mixed character.

Noether. Q-ball charge Q; vortex circulation n; total energy.

Ansatz. Spherical Q-ball centered on the axis of a straight vortex; phase-locked to reduce interface energy.

EOM. Coupled axial PDEs; the vortex provides a pinning potential that can enhance Q-ball stability.

Energy and binding. $E_{Q\text{-}v} < E_Q + E_v$ when phase locking and screening reduce gradients.

Moduli. Relative axial offset; Q-ball radius (via ω); vortex core radius.

Spectrum. Coupled breathing/torsional modes; avoided crossings in linear spectrum.

Observables. Pinning/depinning transitions; resonant energy transfer between sectors.

Numerics. Axisymmetric relaxation at fixed Q; continuation in ω and matching to free limits.

Consistency gates. S/G/TQ/D/N.

6.20.27 Linked Hopfions (Hopf Link)

Sector. Structural + gauge-emergent (Faddeev–Skyrme effective terms).

Topology. Two toroidal Hopfions with linking number $L = 1$ between their preimage loops; total Hopf charge $Q_H = Q_{H1} + Q_{H2}$.

Noether. Total $E, \mathbf{P}$; possible collective J for co-rotating configurations.

Ansatz. Initial data superposes two toroidal textures with centers displaced and oriented for a single mutual link; relaxation preserves linking.

EOM. 3D Euler–Lagrange; gauge-emergent sector mediates inter-knot forces.

Energy. Interaction depends on relative orientation and separation R; a shallow minimum can exist for specific orientations.

Moduli. R, relative Euler angles, and an overall rotation.

Spectrum. Coupled librational modes of the two rings; link-preserving deformations.

Observables. Distinct scattering signatures and polarization patterns due to the link.

Numerics. 3D relaxation with topology-preserving discretization; verify link invariants.

Consistency gates. S/G/TQ/D/N.

6.20.28 Knotted Hopfion (Trefoil-type)

Sector. Structural + gauge-emergent.

Topology. Nontrivial knot class realized by preimage loops (trefoil); Q_H fixed.

Noether. E, optional J.

Ansatz. Use rational map or torus-knot inspired ansatz for $n : \mathbb{R}^3 \to S^2$; relax under fixed Q_H.

EOM. 3D Euler–Lagrange with quartic stabilization.

Energy. Typically larger than unknotted states with same Q_H; metastability possible.

Moduli. Position/orientation; knot size and tube thickness.

Spectrum. Localized shape modes on the knotted tube; zero modes for rigid motions.

Observables. Topologically protected reconnection thresholds; emission bursts when de-knotting.

Numerics. Topology-preserving flows; adaptive mesh near high curvature.

Consistency gates. S/G/TQ/D/N.

6.20.29 Chiral Edge Modes on a WZW/CS-protected Domain Wall

Sector. Structural wall supporting a Wess–Zumino–Witten/Chern–Simons effective term in 2+1D; gauge-emergent sector coupled.

Topology. Bulk topological index induces protected chiral edge along wall boundary.

Noether. Edge current I_{edge}; Hall-like conductance in emergent gauge response.

Ansatz. Planar wall with a finite-width strip; along its edge, a unidirectional mode $\psi(y - v_{\text{edge}}t)$.

EOM. 2+1D wall EFT with a CS term yields first-order chiral equation for edge excitations; bulk–edge anomaly inflow enforces conservation.

Energy. Linear edge dispersion $E(k) = \hbar v_{\text{edge}} k$; bulk gapped to protect chirality.

Moduli. Wall position; edge curvature along bends.

Spectrum. Single-chirality branch; robustness to disorder that respects topology.

Observables. Nonreciprocal transport of spectral waves; quantized-like response coefficients.

Numerics. 2+1D wall simulation with absorbing exterior; extract edge dispersion.

Consistency gates. S/G/TQ/D/N.

6.20.30 Emergent Geon (Self-supported Curvature Lump)

Sector. Palatini emergent gravity sourced by a localized structural configuration $S(\mathbf{x})$.

Topology. None (geometric lump); can coexist with nontrivial topology in S.

Noether. Total mass-energy M_{geon} and possibly J.

Ansatz. Static, spherically symmetric $S(r)$ inducing a localized curvature bump in g_{eff} without horizons for subcritical compactness.

EOM. Einstein-like equations with $T_{\mu\nu}[S]$; solve for g_{eff} and S self-consistently.

Energy. ADM-like mass from asymptotics of g_{eff}; absence of trapped surfaces for $2GM_{\text{eff}}(r)/r < 1$.

Moduli. Center; size parameter set by balance of gradients and curvature.

Spectrum. Quasi-normal ringdown of the geon metric; structural oscillations coupled to $h_{\mu\nu}$.

Observables. Weak-lensing of spectral/gauge waves; echoes in time-domain scattering.

Numerics. Shooting in r with regular center; check hyperbolicity and causality.

Consistency gates. S/G/TQ/D/N.

6.20.31 Non-Abelian Spectral Vortex with Orientational Moduli

Sector. Gauge-emergent non-Abelian sector (e.g. internal $SU(2)$ fragment) coupled to structural fields.

Topology. π_1 in an effective vacuum manifold with non-Abelian internal degeneracy.

Noether. Tension; internal isospin-like charges for zero modes.

Ansatz. Cylindrical vortex with internal orientation n^a living on $\mathbb{CP}^1$ (or S^2) moduli along the string.

EOM. 2D transverse PDEs; along the string, a sigma-model for $n^a(z,t)$ governs orientational waves.

Energy. Core tension $\mathcal{T}$ plus kinetic energy of moduli dynamics.

Moduli. Position (x_0, y_0); $\mathbb{CP}^1$ orientational coordinates.

Spectrum. Gapless orientational modes; Kelvin waves; possible splitting under explicit symmetry breaking.

Observables. Polarization-dependent scattering; inter-vortex forces depend on n^a alignment.

Numerics. Axisymmetric core + 1+1D moduli evolution; verify sigma-model reduction.

Consistency gates. S/G/TQ/D/N.

6.20.32 Helical Skyrmion Tube Lattice

Sector. Structural with chiral (Lifshitz-like) gradient terms; weak gauge-emergent coupling.

Topology. Each tube carries a π_2 charge per cross-section; global lattice has no net charge.

Noether. Energy density; elastic moduli.

Ansatz. Parallel tubes along $\hat{z}$ with a helical twist of internal orientation vs. z, forming a cholesteric-like array.

EOM. Minimize energy density with competing exchange/chiral terms; set pitch p dynamically.

Energy. Optimal pitch p^* from balance of chiral and quadratic gradients; elastic constants for shear/compression.

Moduli. Global shift/twist; phason-like sliding between tubes.

Spectrum. Acoustic-like phonons and twistons; anisotropic dispersions.

Observables. Bragg cones; polarization rotation of spectral waves through the lattice.

Numerics. 3D unit cell with Bloch conditions; map $(p^*, K_{\text{elastic}})$.

Consistency gates. S/G/TQ/D/N.

6.20.33 Spectral Instanton: Euclidean Transition Changing Hopf Charge

Sector. Structural with a topological θ-like term in Euclidean continuation; gauge-emergent coupling.

Topology. Euclidean event connects configurations with different linking structure (change in Q_H).

Noether. Euclidean action S_E controls tunneling amplitude.

Ansatz. $O(4)$-like localized configuration in $(\tau, \mathbf{x})$ that rethreads preimage loops.

EOM. Euclidean Euler–Lagrange with topological term; boundary conditions fix initial/final $n(\mathbf{x})$.

Action. S_E scales with inverse couplings of quartic terms; smaller when chiral terms assist rethreading.

Moduli. Instanton center/size in Euclidean space.

Spectrum. Single negative mode characteristic of tunneling; zero modes for translations.

Observables. Rare topology-changing events; potential CP-odd signatures if $\theta_{\text{spec}} \neq 0$.

Numerics. Constrained minimization in 4D Euclidean lattice; verify one negative mode.

Consistency gates. S/G/TQ/D/N.

6.20.34 Finite-Temperature Caloron and Real-time Sphaleron

Sector. Structural + gauge-emergent at temperature T; Euclidean periodic time $\tau \sim \tau + \beta$.

Topology. Nontrivial winding across Euclidean circle; real-time sphaleron is the saddle at the barrier top.

Noether. Period β; Chern–Simons-like number change in real time.

Ansatz. Euclidean periodic instanton (caloron) with holonomy; analytic continuation yields sphaleron at $t = 0$.

EOM. Periodic Euclidean equations; real-time unstable mode at the sphaleron.

Energy/Action. $S_E(T)$ decreases as T rises; sphaleron energy E_{sph} sets over-the-barrier rate.

Moduli. Caloron center; holonomy parameter.

Spectrum. Negative mode at the sphaleron; zero modes from symmetries.

Observables. Topology change rates; potential CP-violating transient effects.

Numerics. 3D+Euclidean time lattices; cooling and holonomy scans.

Consistency gates. S/G/TQ/D/N.

6.20.35 Chiral Edge Channel Localized on a Vortex Core

Sector. Vortex (π_1) hosting a 1D chiral mode due to bulk topological term (WZW/CS) and spectral coupling.

Topology. Index theorem on the vortex worldsheet yields protected 1D mode.

Noether. 1D current I; associated anomaly inflow.

Ansatz. Straight vortex with longitudinal coordinate z; mode $\psi(z - vt)$ bound to the core.

EOM. Effective 1+1D chiral equation on the core; bulk constraints ensure anomaly cancellation.

Energy. Linear dispersion; backreaction negligible for small amplitudes.

Moduli. Core position; background orientation controlling velocity v.

Spectrum. Single-chirality branch; robustness against backscattering.

Observables. Nonreciprocal transport along the core; quantized-like conductance in emergent sector.

Numerics. Axisymmetric core + 1D mode solver; extract v and attenuation.

Consistency gates. S/G/TQ/D/N.

6.20.36 Aether Foam: Percolating Network of Walls and Strings

Sector. Mixed structural sectors (walls, strings) in a statistically homogeneous network; weak emergent-gravity backreaction.

Topology. Network supports numerous local π_0 and π_1 defects; global net topological charge may vanish.

Noether. Energy density; correlation length ξ; defect densities.

Ansatz. Random initial phases relax to a percolating network; characterize by statistical observables rather than a single field configuration.

EOM. Full time-domain PDEs with damping η; scaling regime where network statistics become time-independent (in comoving units).

Energy. Defect energy density scales $\propto 1/\xi$; energy loss via radiation.

Moduli. Statistical: ξ, cluster-size distribution, loop spectrum.

Spectrum. Broadband emission of spectral/gauge waves from inter-defect dynamics.

Observables. Structure factors; stochastic background of emergent waves.

Numerics. Large-scale simulations with AMR; measure scaling exponents and spectra.

Consistency gates. S/G/TQ/D/N.

Chapter 6 — Tables and Illustrative Figures

Table 6.1: Collective coordinates, inertial tensors, and roles (Ch. 6). Canonical moduli X^I, their geometric roles, and effective inertial data extracted from classical solutions (Ch. 4) and matching (Sec. 6.2–6.4).

DOF	**Symbol**	**Role / geometry**	**Effective param.**	**Origin / extraction**
Translation	X^i	Zero modes of center-of-energy	Mass M	ADM/Noether energy of soliton
Spin / rotation	$R \in SO(3)$	Body-frame orientation	Inertia I_{ab}	Rotational kinetic term in L_{eff}
Internal phase	θ	Spectral $U(1)$ coordinate	Charge Q	Conjugate to phase velocity
Isospin (if any)	$U \in G_{\text{int}}$	Non-Abelian orientation	$I_{AB}^{(\text{iso})}$	Non-minimal moduli metric
Shape modes	q_α	Vibrational deformations	$k_\alpha,\ \omega_\alpha$	Small oscillations spectrum

Table 6.2: Finkelstein–Rubinstein (FR) constraints and multiplet selection. Homotopy data induce sign rules on nontrivial loops of the configuration space; these restrict spin/isospin assignments after quantization.

Topological input	**FR sign / Berry holonomy**	**Allowed J (illustrative)**	**Notes**
π_1 winding (vortex)	$\sigma_{\mathrm{FR}} = (-1)^n$ on 2π rotation	Half-integer J for odd n	Mimics fermionic statistics when $\sigma_{\mathrm{FR}} = -1$
Hopf charge Q_H	Berry phase $\exp(i\gamma[Q_H])$	Shifts in J bands	Phase from linking of tubes/cores
Skyrmion degree B	FR sign $(-1)^B$ on exchange	$J = \frac{1}{2}, \frac{3}{2}, \ldots$ for odd B	Skyrme-like selection

Table 6.3: External probes → form factors → observables. Mapping used in Sec. 6.14–6.16 to connect worldline/volume EFT couplings to measurable quantities.

Probe current	**Form factor basis**	**Low-q^2 expansion**	**Observable**
Spectral $U(1)$	F_E, F_M, F_A	$F_E(0) = Q,\ \langle r^2\rangle = -6F'_E(0)$	Charge radius, g-factor, anapole
Non-Abelian internal	$F^A_{1,2}$	Group Casimir scaling	Isospin response, selection rules
Emergent gravity	$A(q^2), B(q^2)$	$A(0) = M,\ B(0)$ spin–orbit	Gravitational moments, deflection

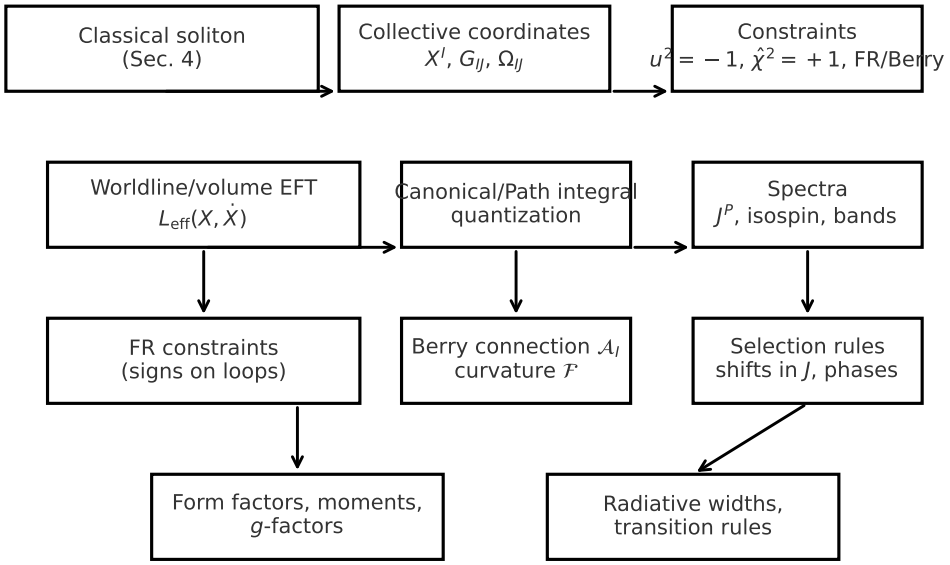

Figure 6.1: Collective-coordinate quantization pipeline. From classical solitons to moduli geometry (G_{IJ}, Ω_{IJ}), effective worldline/volume Lagrangians, FR/Berry constraints, and finally spectra with selection rules; matched to form factors and radiative widths.

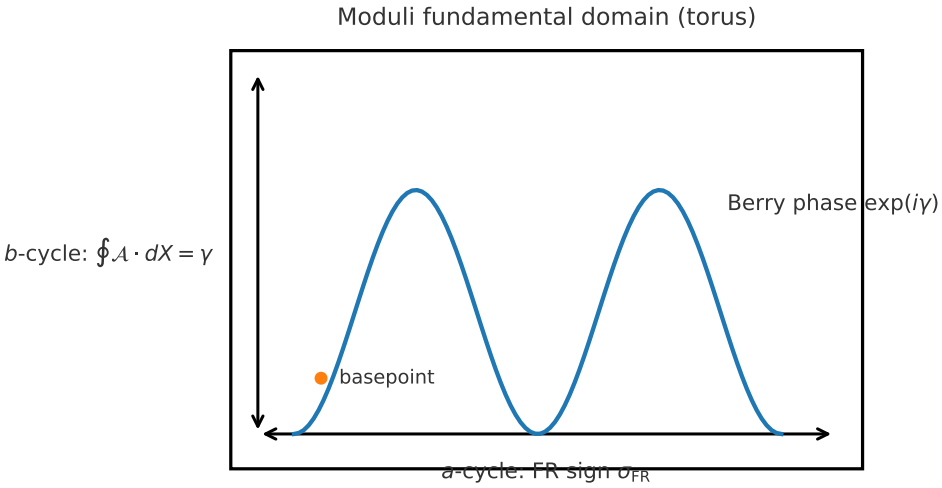

Figure 6.2: FR/Berry holonomy on the moduli torus. Fundamental domain with a- and b-cycles: FR sign on a-cycle encodes statistics constraints; Berry phase on b-cycle shifts multiplet assignments and transition phases.

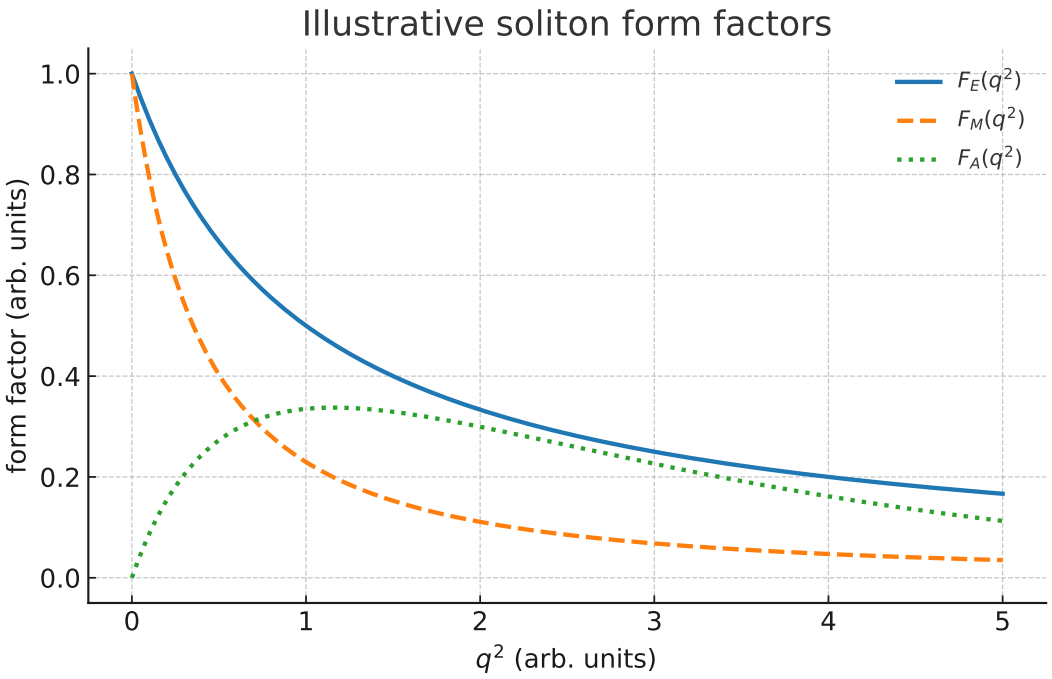

Figure 6.3: Illustrative form factors. Electric-like $F_E(q^2)$, magnetic-like $F_M(q^2)$, and anapole-like $F_A(q^2)$ shapes shown schematically; concrete curves result from Chapter 6 matching and numerical evaluation on the calibrated background state.

Chapter 7

Chapter 7: Scattering, Bound States, and Resonances of Quantum–Aether Solitons

7.1 Scope and Strategy

This chapter develops a predictive framework for *scattering*, *binding*, and *resonances* of the solitonic excitations constructed in Chap. **??** and quantized in Chap. 6. Our aim is to translate the intrinsic (geometric, topological, and spectrally emergent) structure of quantum–aether solitons into *measurable* observables: phase shifts, cross sections, radiative rates, capture probabilities, and resonance spectra, together with controlled uncertainties and consistency gates.

From fields to observables. Starting from the full field theory of Chap. 3 and the stationary solutions of Chap. **??**, we employ the collective–coordinate quantization of Chap. 6 to obtain a low–energy Hilbert space spanned by one– and few–soliton states. Scattering and reaction dynamics are then formulated in a *worldline/worldvolume effective field theory* (EFT) that respects: (i) the moduli–space geometry $(\mathcal{M}_Q, G_{IJ}, \Omega_{IJ})$, (ii) FR/Berry structures (spin/statistics and selection rules), (iii) the emergent gauge/diffeomorphism symmetries of the probes and metric sector, and (iv) the dissipative/GKSL structure for inclusive radiative processes (Secs. 6.14–6.15).

Kinematic regimes and small parameters

The EFT is organized as a simultaneous expansion in three small parameters:

$$\varepsilon_v \equiv \frac{v}{c_{\rm eff}} \ll 1, \qquad \varepsilon_R \equiv \frac{R_{\rm core}}{L_{\rm scatt}} \ll 1, \qquad \varepsilon_{\rm rad} \equiv \max\!\left(\frac{P_{\rm rad}}{E_{\rm mod}\,\Omega}, \frac{\Gamma_{\rm rad}}{\Omega}\right) \ll 1, \tag{7.1}$$

where v is the relative speed, $c_{\rm eff}$ the pertinent propagation speed of the medium, $R_{\rm core}$ a characteristic core size, $L_{\rm scatt}$ the scattering length scale (impact parameter or de Broglie wavelength), Ω a typical collective frequency, $P_{\rm rad}$ the radiated power, and $\Gamma_{\rm rad}$ a radiative width. The expansion delivers:

- **LO (conservative)** dynamics on $(\mathcal{M}_Q, G)$ with adiabatic potentials $U_{\rm ad}$,
- **NLO (non–conservative)** radiation/reaction via multipoles fixed by Sec. 6.14 and soft limits of Sec. 6.15,
- **NNLO (contact & retardation)** short–distance operators encoding core overlap.

Effective degrees of freedom and matching

A single soliton is described by a worldline $X^\mu(\tau)$, internal moduli $X^I(\tau)$, spin $S^{\mu\nu}$ (when applicable), and a tower of *multipole operators* $\{\mathcal{O}_\ell\}$ capturing its extended nature. Two–body systems are

described in relative variables $\mathbf{R}$, $\dot{\mathbf{R}}$ plus collective orientations/phases. The *matching* (Chap. 6) supplies:

$$M,\ G_{IJ},\ \Omega_{IJ},\ \{\mu,\ Q_{E2},\ \ldots\}, \qquad \text{and channel quantum numbers fixed by FR/Berry.}$$

Exchange of dynamical probes (spectral $U(1)$, internal non–Abelian, emergent gravity) yields long–range kernels; short–distance physics is encoded by a finite set of local operators, organized by power counting and symmetry.

Observables and their definitions

We target the following observables, all defined in the center–of–mass (CoM) frame unless noted:

- **Elastic and inelastic phase shifts** $\delta_\ell^{(\alpha)}(k)$ in each open channel α and partial wave ℓ, with coupled–channel S–matrix unitarity, $S^\dagger S = \mathbb{I}$ (Sec. 7.3).
- **Differential and total cross sections** $d\sigma/d\Omega$, $\sigma_{\rm tot}$, including polarization/orientation dependence for anisotropic cores (Sec. 7.4).
- **Resonance spectra** (poles of S on the unphysical sheet): positions and widths, classified as *shape/Feshbach* resonances (Sec. 7.7).
- **Capture/bound–state rates** into dimers/trimers, with radiative stabilization (Sec. 7.8).
- **Soft bremsstrahlung** spectra and inclusive rates obeying the soft theorems (Sec. 6.15).

Strategy of computation

We proceed in a top–down fashion:

1. **Construct the two–body EFT** (Sec. 7.2) with LO conservative potentials $U_{\rm ad}(R, \text{orientations})$ and NLO multipolar couplings to the radiative sectors.
2. **Solve the conservative scattering** problem (Sec. 7.3) by partial–wave decomposition in coupled channels, enforcing FR/Berry selection rules by algebraic projectors.
3. **Add radiation consistently** (Sec. 7.8) via soft factors and radiation–reaction terms derived by reduction of order; verify energy balance.
4. **Locate resonances and bound states** (Sec. 7.7) using Lippmann–Schwinger or complex scaling on channel Hamiltonians; classify and tabulate pole data.
5. **Benchmark numerics** (Sec. 7.10) against full time–domain PDE scattering for representative channels (MBP–0/0, MBP–1/1 parallel/orthogonal, hopfion sectors).

Symmetries, selection rules, and gates

All constructions preserve:

- **Gauge/diffeomorphism invariance** (probe and emergent metric sectors),
- **FR/Berry constraints** on allowed channels and transitions,
- **Discrete symmetries** (P, T, CP) or their controlled breaking as specified in Chap. 6.

We impose *consistency gates* throughout the chapter:

U1 (unitarity): $S^\dagger S = \mathbb{I}$ within numerical tolerance,
U2 (causality): no superluminal group velocities under $g_{\rm eff}$,
U3 (soft/IR): soft limits obey Sec. 6.15; inclusive IR finiteness,
U4 (FR/Berry): forbidden channels suppressed by projectors/holonomy,
U5 (numerics): convergence under mesh/box/partial–wave truncations.

Deliverables of Chapter 7

The chapter culminates in a set of *channel–resolved* results with quantified uncertainties:

1. Phase–shift tables and elastic/inelastic cross sections vs. energy for benchmark channels.
2. Resonance catalogs (positions, widths, quantum numbers, dominant decay channels).
3. Capture rates for shallow dimers/trimers, including radiative stabilization.
4. Soft–radiation spectra and total inclusive rates consistent with Ward/soft identities.
5. Validation against full PDE scattering and sensitivity analyses to geometric parameters (G_{IJ}, Ω_{IJ}) and short–distance operators.

Reading map. Sec. 7.2 builds the EFT; Secs. 7.3–7.4 cover kinematics and potentials; Sec. 7.5 presents low–energy universality; Sec. 7.6 explores the semiclassical regime; Sec. 7.7 analyzes resonances; Sec. 7.8 treats radiation and reaction; Sec. 7.9 disentangles topological/phase effects; Sec. 7.10 codifies numerical protocols; Sec. 7.11 discusses phenomenological interfaces; Sec. 7.12 states the gates; and Sec. 7.13 summarizes outcomes and links to Chap. 8.

7.2 Worldline/Worldvolume EFT for Solitons

We formulate a reparametrization–invariant effective field theory (EFT) for the dynamics of quantum–aether solitons at scales well below the core scale and any heavy spectral thresholds. The EFT is built from *collective degrees of freedom* (center of mass, internal moduli, body orientation/spin), coupled to the emergent probe sectors (spectral $U(1)$, internal non–Abelian, and metric perturbations of $g_{\rm eff}$) and to radiation. It is organized by the small parameters introduced in Eq. (7.1) and constrained by FR/Berry structures and symmetries.

7.2.1 Single–soliton EFT: degrees of freedom and LO action

Kinematic variables. A single soliton of topological sector Q is described by:

$$X^\mu(\tau), \qquad X^I(\tau) \in \mathcal{M}_Q, \qquad S^{\mu\nu}(\tau) \ \text{(when a body frame exists)}, \qquad \mathcal{U}(\tau) \in \text{FR/Berry bundle}, \tag{7.2}$$

where X^μ is the worldline in the emergent spacetime $(\mathcal{M}, g_{\rm eff})$, X^I are internal moduli coordinates on the moduli manifold $(\mathcal{M}_Q, G_{IJ}, \Omega_{IJ})$, and $S^{\mu\nu}$ encodes spin via an orthonormal tetrad or body–frame map. The phase $\mathcal{U}$ labels the section of the physical line bundle $\mathcal{L}_{\rm phys}$ implementing FR and Berry constraints (Chap. 6).

LO conservative action. To leading order in derivatives and multipoles, the reparametrization–invariant action reads

$$\begin{aligned} S_{\rm 1s}^{\rm LO} = &\int d\tau \Big[- M\sqrt{-g_{\rm eff}(X)\,\dot X^\mu \dot X^\nu} + \tfrac{1}{2} G_{IJ}(X)\,\dot X^I \dot X^J - U_{\rm mod}(X) + \mathcal{A}_I(X)\,\dot X^I \Big] \\ &+ \int d\tau \Big[Q_{\rm eff}\, A_\mu(X)\,\dot X^\mu + \tfrac{1}{2} S_{\mu\nu}\,\Omega^{\mu\nu}{}_\rho(X)\,\dot X^\rho \Big] + \mathcal{O}(\text{NLO}), \end{aligned} \tag{7.3}$$

where M is the rest mass (collective energy at the symmetric point $X_\star$), G_{IJ} and $U_{\rm mod}$ are the moduli metric and potential, $\mathcal{A}_I$ is the Berry connection $(\Omega_{IJ} = \partial_I \mathcal{A}_J - \partial_J \mathcal{A}_I)$, $Q_{\rm eff}$ is the effective $U(1)_\Phi$ charge, A_μ the spectral gauge field, and $\Omega^{\mu\nu}{}_\rho$ is the emergent spin connection derived from $g_{\rm eff}$ (Chap. 3). The term $\mathcal{A}\cdot\dot X$ implements geometric phases; its holonomy around nontrivial loops reproduces FR/Berry constraints (Chap. 6).

Comments. The line element $\sqrt{-g_{\rm eff}\dot X^2}$ ensures Poincaré–like covariance under emergent isometries; the moduli kinetic term is the standard sigma–model on $(\mathcal{M}_Q, G)$. The LO action is *conservative*; dissipation and radiation enter at NLO via multipolar couplings and influence functionals (Sec. 7.2.3).

Multipole tower and nonminimal terms. Extended structure is captured by a finite set of worldline operators consistent with symmetries:

$$S_{\rm 1s}^{\rm NLO} = \int d\tau \Big[\mu^i(X)\, B_i + p^i(X)\, E_i + \tfrac{1}{2} Q^{ij}(X)\, \partial_i E_j + \lambda_1(X)\, R_{\mu\nu}\, u^\mu u^\nu + \lambda_2(X)\, R + \cdots \Big], \tag{7.4}$$

where $E_i = F_{0i}$, $B^i = \frac{1}{2}\epsilon^{ijk} F_{jk}$ are spectral electric/magnetic fields in a local inertial frame, $u^\mu = \dot{X}^\mu / \sqrt{-g_{\rm eff} \dot{X}^2}$, $R_{\mu\nu}$ and R are Ricci tensors of $g_{\rm eff}$, and the coefficients $\{\mu^i, p^i, Q^{ij}, \lambda_{1,2}, \dots\}$ are *matching data* determined by the microscopic theory (Sec. 7.2.2)—equivalently by the form factors of Sec. 6.14.

Spin supplementary condition (SSC). When $S^{\mu\nu} \neq 0$, we impose an SSC (e.g. Tulczyjew–Dixon $S^{\mu\nu} u_\nu = 0$) to close the system and remove unphysical components, with corresponding shifts in the worldline position $X^\mu \to X^\mu + \delta X^\mu$ fixed to maintain covariance.

7.2.2 Matching rules from Chap. 6

Mass, moduli metric, and potential. The rest mass is $M = U_{\rm mod}^{\rm R}(X_\star)$ (renormalized), the moduli metric is the zero–mode inner product (App. 6.17, Eq. (6.162)), and $U_{\rm mod}(X)$ is the collective potential extracted from the constrained minimization along the moduli family.

Berry connection and FR. The Berry data $(\mathcal{A}, \Omega)$ are computed from the ground–band eigenvectors along $\mathcal{M}_Q$ (Chap. 6); their holonomy and FR character *restrict* allowed representations and selection rules. Quantization of $\frac{1}{2\pi}\int \Omega$ over closed two–cycles (App. 6.17) ensures consistency with line–bundle prequantization.

Electromagnetic–like multipoles. The static form factors of Sec. 6.14 determine the worldline coefficients:

$$Q_{\rm eff} = F_1(0), \qquad \boldsymbol{\mu} = \mu\, \hat{\mathbf{J}}, \quad \mu = \frac{Q_{\rm eff}}{2M}\,(2F_1(0) + F_2(0)), \qquad \langle r^2 \rangle = -6F_1'(0), \tag{7.5}$$

and similarly for quadrupoles via $Q^{ij} \leftrightarrow$ the $E2$ form factor. These data fix the coefficients μ^i, p^i, Q^{ij} in Eq. (7.4).

Gravitational–like response. Stress–energy form factors (Sec. 6.14.4) give $A(0) = 1$ and $B(0)$, constraining $\lambda_{1,2}$ and spin–gravity couplings. The emergent Newton coupling is $G_{\rm eff} = (8\pi M_{\rm eff}^2)^{-1}$ (Chap. 3).

Dissipation kernels. GKSL linewidths and Kubo correlators (Secs. 6.10, 6.14.5) determine the imaginary parts of nonlocal influence actions. Their local (derivative) expansion yields friction coefficients and radiation–reaction times $\tau_{\rm spec}, \tau_{\rm grav}$ used in Sec. 7.4 and Sec. 7.8.

7.2.3 Two–body EFT: relative dynamics, potentials, and radiation

Variables and channel space. For two solitons $(1, 2)$ of masses M_1, M_2, define CoM and relative variables: $X_{\rm CoM}^\mu$, $\mathbf{R} = \mathbf{X}_1 - \mathbf{X}_2$, with reduced mass $\mu = M_1 M_2/(M_1 + M_2)$. Internal moduli $\{X_1^I, X_2^J\}$ include orientations and phases; the *channel index* α labels discrete quantum numbers (spin/parity, FR sector, Berry charges).

LO conservative Lagrangian. In the CoM frame and near flat $g_{\rm eff}$, the relative Lagrangian to LO is

$$L_{\rm rel}^{\rm LO} = \tfrac{1}{2}\mu\, \dot{\mathbf{R}}^{\,2} + \tfrac{1}{2} G_{IJ}^{(1)} \dot{X}_1^I \dot{X}_1^J + \tfrac{1}{2} G_{KL}^{(2)} \dot{X}_2^K \dot{X}_2^L - U_{\rm ad}(\mathbf{R},\, X_1,\, X_2)\,, \tag{7.6}$$

where the *adiabatic potential* $U_{\rm ad}$ is the energy of the constrained two–soliton configuration at fixed $(\mathbf{R}, X_1, X_2)$ (Chap. **??**). Orientation dependence (MBP–1) enters through invariants such as $\hat{\chi}_1 \cdot \hat{\chi}_2$ and $(\hat{\chi}_i \cdot \hat{\mathbf{R}})^2$.

Long–range exchange and noncentral forces. Exchange of dynamical probes produces universal tails:

$$V_{\rm exch}(\mathbf{R}) = V_{U(1)}(R) + V_{\rm int}(R) + V_{\rm grav}(R) + V_{\rm tensor}(\mathbf{R}; \hat{\chi}_{1,2}, \mathbf{S}_{1,2}) + \cdots , \tag{7.7}$$

with $V_{U(1)} \sim \alpha_{\rm spec}\, Q^{(1)}_{\rm eff} Q^{(2)}_{\rm eff}\, f(R)$, $V_{\rm grav} \sim -G_{\rm eff} M_1 M_2/R$, and anisotropic tensor/spin–orbit pieces governed by Berry–induced couplings (Sec. 6.14). These are added to $U_{\rm ad}$ to give the full conservative kernel.

Short–distance operators. Core overlap at $R \lesssim R_{\rm core}$ is encoded by a finite operator basis, organized by parity and channel:

$$L_{\rm SD} = \sum_{\alpha} \Big[C_0^{(\alpha)}\, \delta^{(3)}(\mathbf{R}) + C_2^{(\alpha)}\, \delta^{(3)}(\mathbf{R}) \overleftrightarrow{\nabla}^2 + C_T^{(\alpha)}\, \delta^{(3)}(\mathbf{R})\, S_{12}(\hat{\mathbf{R}}) + \cdots \Big], \tag{7.8}$$

with $S_{12}(\hat{\mathbf{R}}) = 3(\mathbf{S}_1{\cdot}\hat{\mathbf{R}})(\mathbf{S}_2{\cdot}\hat{\mathbf{R}}) - \mathbf{S}_1{\cdot}\mathbf{S}_2$. Coefficients C are fixed by matching to microscopic two–soliton data (energy curves, phase shifts) and are *channel dependent*.

FR/Berry projectors and selection rules. Physical two–body states live in the tensor product of single–soliton Hilbert spaces modulo *(anti)symmetrization* and geometric phases. Algebraic projectors $\mathbb{P}^{(\alpha)}_{\rm FR/Berry}$ enforce: (i) the FR character under 2π rotations and exchange, (ii) allowed monopole harmonics on orientation spheres when Berry flux is present. The interaction kernels are sandwiched by these projectors in any scattering computation: $V \to \mathbb{P} V \mathbb{P}$.

Radiation and reaction at NLO. Couplings of multipoles to dynamical probes (Sec. 6.15) yield radiation. At the EFT level, we include:

$$L_{\rm rad} = \sum_{i=1}^{2} \Big[\mu_i^k B_k + p_i^k E_k + \tfrac{1}{2} Q_i^{kl} \partial_k E_l \Big] + \text{gravitational quadrupole couplings}, \tag{7.9}$$

and integrate out the radiation fields to obtain a nonlocal influence action. Its local derivative expansion gives radiation–reaction forces consistent with energy balance (Sec. 6.15.4).

Power counting and validity. Let $k \sim 1/L_{\rm scatt}$ be the relative momentum. The counting rules are

$$U_{\rm ad} \sim \mathcal{O}(k^2/\mu), \quad V_{\rm exch} \sim \mathcal{O}(\alpha/R), \quad C_0 \delta^{(3)} \sim \mathcal{O}(k), \quad \text{multipoles } \sim \mathcal{O}((kR_{\rm core})^{\ell}),$$

with $\alpha \in \{\alpha_{\rm spec}, G_{\rm eff} M_1 M_2\}$. The EFT is valid for $kR_{\rm core} \ll 1$ and weak radiation $\varepsilon_{\rm rad} \ll 1$; near resonances, resummations (Lippmann–Schwinger) are performed within the same operator basis (Sec. 7.7).

Gauge/diffeomorphism covariance. Minimal couplings enforce $A_\mu \to A_\mu + \partial_\mu \lambda$ invariance; dependence on $g_{\rm eff}$ enters through covariant line elements, spin connection couplings, and curvature operators. Observable quantities (phase shifts, cross sections) are invariant under field redefinitions and SSC choices.

7.2.4 Worldvolume generalization (defects with $p > 0$)

If extended defects (e.g. strings, membranes) appear among stable solutions (Chap. **??**), the worldline EFT lifts to a *worldvolume* EFT with embedding $X^\mu(\sigma^a)$ ($a = 0, \ldots, p$), Nambu–Goto–like area terms in $g_{\rm eff}$, internal moduli fields on the worldvolume, and multipolar couplings obtained by matching transverse fluctuations and form factors. All constructions above—Berry/FR, radiation, and selection rules—carry over with the replacement $\tau \to \sigma^a$ and appropriate symmetry groups.

7.2.5 Consistency gates for the EFT

- **EFT1 (matching):** M, G_{IJ}, $U_{\rm mod}$, $\mathcal{A}_I$ reproduce Chap. 6 benchmarks within errors; form–factor–based coefficients satisfy Eq. (7.5).
- **EFT2 (symmetries):** Gauge/diffeomorphism invariance holds; Berry holonomies and FR characters are respected by projectors.
- **EFT3 (power counting):** Truncation errors scale as predicted by the counting; higher–order operators are numerically subleading in the intended regime.
- **EFT4 (causality/positivity):** No superluminal modes in the moduli kinetic sector; dissipative kernels are completely positive (consistent with GKSL, Chap. 6).
- **EFT5 (radiation balance):** Reaction forces reproduce radiated power derived from the probe sectors (Sec. 6.15) to the working order.

Outcome. With Eqs. (7.3)–(7.8) and the gates above, we have a closed, covariant, and falsifiable dynamical description for one– and two–soliton systems. The next section translates this EFT into phase shifts, cross sections, and observable scattering data.

7.3 Scattering Kinematics and Observables

We now turn the EFT of Sec. 7.2 into measurable scattering data. We first set kinematics and state normalizations under the emergent metric $g_{\rm eff}$, then formulate the coupled–channel partial–wave expansion (with anisotropy and FR/Berry constraints), and finally define cross sections, optical theorems, and inclusive observables in the presence of weak radiation.

7.3.1 Kinematics in the emergent medium

Dispersion and relative momentum. In the CoM frame, the conservative relative Hamiltonian at LO reads

$$H_{\rm rel} = \frac{1}{2}\,\Pi_i\,\mathcal{C}^{ij}\,\Pi_j + U_{\rm ad}(\mathbf{R}, X_1, X_2), \qquad \Pi_i \equiv -i\,\partial_i, \tag{7.10}$$

where $\mathcal{C}^{ij}$ encodes the (possibly anisotropic) kinetic tensor induced by $g_{\rm eff}$ and the moduli reduction. In an *approximately isotropic* regime, $\mathcal{C}^{ij} = \delta^{ij}/\mu$ with μ the reduced mass and the free dispersion $E_k = k^2/(2\mu)$. In an *anisotropic but homogeneous* patch aligned with the privileged direction $\hat{\chi}$ (Chap. **??**), one may write

$$E_k \;=\; \frac{k_\perp^2}{2\mu_\perp} + \frac{k_\parallel^2}{2\mu_\parallel}, \qquad k_\parallel \equiv \mathbf{k}\cdot\hat{\chi}, \quad k_\perp^2 \equiv k^2 - k_\parallel^2. \tag{7.11}$$

Group velocity is $\mathbf{v}_g = \nabla_{\mathbf{k}} E_k$, which fixes incident flux factors below.

Incident flux and phase–space. With plane–wave normalization $\langle \mathbf{k}', \alpha | \mathbf{k}, \beta\rangle = (2\pi)^3 \delta_{\alpha\beta}\delta^{(3)}(\mathbf{k}' - \mathbf{k})$, the incoming flux for channel α is

$$\mathcal{F}_\alpha = |\mathbf{v}_g(\mathbf{k}_\alpha)| \;=\; \left|\nabla_{\mathbf{k}} E_k\right|_{\mathbf{k}=\mathbf{k}_\alpha}. \tag{7.12}$$

For isotropic kinematics, $\mathcal{F}_\alpha = k_\alpha/\mu$.

7.3.2 Channel space and state normalization

Channel labels. A *channel* α specifies: (i) the internal moduli sector (X_1, X_2) (e.g. orientations), (ii) discrete quantum numbers (spin/parity), (iii) FR character and Berry charges (Sec. 6.5), and (iv) the threshold energy $E_\alpha^{\rm thr}$. Open (closed) channels satisfy $E > E_\alpha^{\rm thr}$ ($E < E_\alpha^{\rm thr}$).

T– and S–matrices. Let V denote the conservative kernel (including $U_{\rm ad}+V_{\rm exch}+L_{\rm SD}$, projected by FR/Berry projectors). The Lippmann–Schwinger equation defines the on–shell T–matrix

$$T(E) \;=\; V + V\,G_0^{(+)}(E)\,T(E), \qquad G_0^{(+)}(E) = \frac{1}{E - H_0 + i0^+}, \tag{7.13}$$

with $H_0 = \frac{1}{2}\Pi_i \mathcal{C}^{ij}\Pi_j$. The S–matrix in channel space is

$$S_{\beta\alpha} \;=\; \delta_{\beta\alpha} - 2\pi i\,\delta(E_\beta - E_\alpha)\,\mathcal{N}_{\beta\alpha}\,T_{\beta\alpha}(E), \tag{7.14}$$

where $\mathcal{N}_{\beta\alpha}$ contains kinematic factors ensuring unitarity under anisotropy; for isotropic kinematics $\mathcal{N}_{\beta\alpha} = \sqrt{\mu^2 k_\beta k_\alpha}/(2\pi^2)$.

7.3.3 Partial–wave expansion: central and anisotropic cases

Central (isotropic) kernels. If $V(\mathbf{R})$ is central, one expands in spherical harmonics:

$$\psi_\alpha(\mathbf{R}) = \sum_{\ell m} \frac{u_\ell^{(\alpha)}(R)}{R}\,Y_{\ell m}(\hat{\mathbf{R}}), \qquad S_\ell^{(\alpha)} = e^{2i\delta_\ell^{(\alpha)}}. \tag{7.15}$$

For coupled *channels* but central V, each ℓ carries an $n_\ell \times n_\ell$ unitary matrix S_ℓ diagonalizable as $S_\ell = U_\ell\,\mathrm{diag}(e^{2i\delta_{\ell,1}},\dots,e^{2i\delta_{\ell,n_\ell}})\,U_\ell^\dagger$.

Anisotropic kernels and mixing. If V depends on $\hat{\chi}$ and directions, partial waves with different ℓ, m *mix*. A useful basis is the set of *monopole harmonics* $Y_{\ell m}^{(q)}(\hat{\mathbf{R}})$ when a Berry monopole charge $q \in \frac{1}{2}\mathbb{Z}$ is present on the orientation sphere (Chap. 6). The scattering matrix then acts on (ℓ, m, α) as a coupled object:

$$\Psi(\hat{\mathbf{R}}) = \sum_{\ell m \alpha} \frac{u_{\ell m}^{(\alpha)}(R)}{R}\,Y_{\ell m}^{(q)}(\hat{\mathbf{R}})\,|\alpha\rangle, \qquad S_{\ell m\alpha;\,\ell' m'\beta}(E) \in \mathrm{U}(\mathcal{N}_{\rm PW}), \tag{7.16}$$

with $\mathcal{N}_{\rm PW}$ the number of retained partial–wave×channel components. In axially symmetric cases (MBP–1 cores with a fixed $\hat{\chi}$), selection by m reduces mixing and one may employ a Blatt–Biedenharn parameterization for 2×2 subspaces with a mixing angle ϵ_ℓ:

$$S_\ell = \begin{pmatrix} \cos\epsilon_\ell & \sin\epsilon_\ell \\ -\sin\epsilon_\ell & \cos\epsilon_\ell \end{pmatrix} \begin{pmatrix} e^{2i\delta_\ell^{(1)}} & 0 \\ 0 & e^{2i\delta_\ell^{(2)}} \end{pmatrix} \begin{pmatrix} \cos\epsilon_\ell & -\sin\epsilon_\ell \\ \sin\epsilon_\ell & \cos\epsilon_\ell \end{pmatrix}. \tag{7.17}$$

Inelasticity. With weak radiation or open inelastic channels, each eigenphase acquires an inelasticity parameter $\eta_{\ell,j} \in (0,1]$: $S_{\ell,j} = \eta_{\ell,j}\,e^{2i\delta_{\ell,j}}$, and S remains *unitary* when the radiative channels are included in the space (Sec. 7.8). In practice we quote *inclusive* elastic S with $\eta_{\ell,j} < 1$.

7.3.4 Scattering amplitudes, cross sections, and optical theorem

Plane–wave amplitudes. The *on–shell* scattering amplitude from α to β is

$$f_{\beta\alpha}(\hat{\mathbf{k}}', \hat{\mathbf{k}}; E) \;=\; -\,\frac{\mu_{\beta\alpha}}{2\pi}\,\langle \mathbf{k}', \beta | T(E) | \mathbf{k}, \alpha\rangle, \qquad \mu_{\beta\alpha} \equiv \sqrt{\mu_\beta \mu_\alpha}, \tag{7.18}$$

with $\mathbf{k}'$ ($\mathbf{k}$) outgoing (incoming) on the respective mass shells. For isotropic central scattering the partial–wave series reduces to

$$f_{\beta\alpha}(\theta) = \frac{1}{2ik_\alpha}\sum_{\ell=0}^{\infty}(2\ell{+}1)\,\big(S_{\beta\alpha}^{(\ell)} - \delta_{\beta\alpha}\big)\,P_\ell(\cos\theta). \tag{7.19}$$

Differential and total cross sections. The differential cross section and the total (inclusive–elastic) cross section read

$$\frac{d\sigma_{\beta\alpha}}{d\Omega}(\hat{\mathbf{k}}';\hat{\mathbf{k}}) = \frac{\mathcal{F}_\beta}{\mathcal{F}_\alpha}\,\big|f_{\beta\alpha}(\hat{\mathbf{k}}', \hat{\mathbf{k}}; E)\big|^2, \qquad \sigma_{\rm tot}^{(\alpha)} = \sum_\beta \int d\Omega\,\frac{d\sigma_{\beta\alpha}}{d\Omega}. \tag{7.20}$$

In isotropic elastic scattering, one recovers $\sigma_{\rm tot}^{(\alpha)} = \frac{4\pi}{k_\alpha^2}\sum_\ell (2\ell{+}1)\,\sin^2\delta_\ell^{(\alpha)}$.

Optical theorem. Unitarity of S yields the optical theorem

$$\sigma_{\rm tot}^{(\alpha)} = \frac{4\pi}{\mathcal{F}_\alpha} \operatorname{Im} f_{\alpha\alpha}(\hat{\mathbf{k}}; \hat{\mathbf{k}}; E), \tag{7.21}$$

valid with the flux $\mathcal{F}_\alpha$ from Eq. (7.12). In anisotropic media this holds for forward direction relative to $\mathbf{k}$ with the same Jacobian factors used in (7.14).

7.3.5 Identical solitons, FR constraints, and selection rules

Exchange symmetry. For identical solitons, the physical two–body Hilbert space is the (anti)symmetric subspace under exchange, controlled by the FR character $\sigma = \pm 1$ (Chap. 6):

$$\mathbb{P}_{\rm FR} = \frac{1}{2}(1 + \sigma\, \mathsf{P}_{12}), \qquad \sigma = -1 \;\Rightarrow\; \text{fermionic (FR anti–periodic)}, \quad \sigma = +1 \;\Rightarrow\; \text{bosonic}. \tag{7.22}$$

For central elastic scattering this implies the standard orbital selection: $\sigma = -1 \Rightarrow \ell$ odd, $\sigma = +1 \Rightarrow \ell$ even, possibly modified by spin coupling and Berry charges.

Berry monopoles and monopole harmonics. If the relative orientation sphere carries Berry charge q, the angular basis is $Y_{\ell m}^{(q)}$, and allowed ℓ start at $|\ell| \geq |q|$. Selection rules for transitions and partial–wave couplings follow the monopole addition theorems; this enforces *geometric* zeros in $f_{\beta\alpha}$ even when kinematics would otherwise allow an amplitude.

7.3.6 Inclusive observables with weak radiation

Inelasticity parameters and KLN. Coupling to radiative sectors (Sec. 6.15) introduces small inelasticities $\eta_{\ell,j} < 1$ in the elastic block. Inclusive unitarity (summing over undetected soft quanta) restores the optical theorem (7.21). Practically, we extract *effective* phase shifts and inelasticities by fitting the elastic angular distribution with the parameterization $S_{\ell,j} = \eta_{\ell,j}\, e^{2i\delta_{\ell,j}}$.

Orientation averaging. When incident beams are unpolarized in orientation/moduli, we average $d\sigma/d\Omega$ over the Haar measure on the relevant moduli submanifold (e.g. S^2) with Berry measure corrections; this can suppress anisotropy at low ℓ and reveal selection–rule protected nulls in the averaged cross section.

7.3.7 Practical extraction from numerical data

Time–domain full PDE scattering. Prepare an incoming wave packet $\Psi_{\rm in}$, evolve under the full field equations with absorbing boundaries, and extract the far–zone field to read off $f(\theta, \phi)$ by asymptotic matching. Incident flux is determined from the group velocity of the packet.

Schrödinger–like solvers on EFT potentials. Solve the coupled radial equations for $u_{\ell m}^{(\alpha)}(R)$ with the projected kernel $\mathbb{P}V\mathbb{P}$; obtain the S–matrix by matching to free solutions at large R (or via K–matrix methods to enforce unitarity). Convergence is checked by enlarging the (ℓ, m, α) truncation and the radial box.

Gate checks. Every reported $\delta_{\ell,j}, \eta_{\ell,j}$ must pass: (U1) unitarity within tolerance, (U2) causal group velocities under $g_{\rm eff}$, (U3) soft consistency when radiation is included, (U4) FR/Berry selection rules (forbidden partial waves vanish within error), and (U5) numerical convergence under refinement.

With these definitions and protocols, the EFT of Sec. 7.2 produces scattering observables that are invariant under field redefinitions and consistent with the geometric constraints of the quantum–aether model. We now construct the potentials and long–range forces that will feed the coupled–channel machinery (Sec. 7.4).

7.4 Potentials and Long–Range Forces

We construct the conservative interaction kernels that enter the two–body EFT of Sec. 7.2. The potentials arise from: (i) *adiabatic overlap* of the two cores, (ii) *exchange* of dynamical probe fields (spectral $U(1)$, internal non–Abelian, emergent gravity), and (iii) *geometric/anisotropic* couplings tied to the director fields and Berry/FR structures. We work in the Born/static limit for analytic control, then discuss corrections and regimes of validity.

7.4.1 Born kernels from probe exchange

Setup and notation. Let J^A denote conserved probe currents carried by a single soliton (spectral charge, non–Abelian charge, stress–energy, multipoles). The tree–level exchange potential in momentum space is

$$\widetilde{V}_{\beta\alpha}(\mathbf{q}) \;=\; -\,\mathcal{J}^{(1)}_{\beta\alpha;A}(\mathbf{q})\,\mathcal{P}^{AB}(\mathbf{q};\mathcal{S}_{\rm bg})\,\mathcal{J}^{(2)}_{B}(-\mathbf{q})\,, \qquad V(\mathbf{R}) = \int \frac{d^3q}{(2\pi)^3}\, e^{i\mathbf{q}\cdot\mathbf{R}}\,\widetilde{V}(\mathbf{q}), \tag{7.23}$$

where $\mathcal{P}^{AB}$ is the propagator (possibly medium–renormalized by the background state $\mathcal{S}_{\rm bg}$), and $\mathcal{J}$ are *form–factor dressed* currents matched in Chap. 6. Channel indices (α,β) keep track of spin/parity and FR/Berry sectors.

Spectral $U(1)$ exchange. For a massless (or light) spectral gauge boson of mass $m_{\rm spec}$ (possibly environment–dependent),

$$\mathcal{P}^{(U1)}_{\mu\nu}(\mathbf{q}) = \frac{\delta_{\mu0}\delta_{\nu0}}{\mathbf{q}^2+m_{\rm spec}^2} \quad\Rightarrow\quad V_{U(1)}(R) = \frac{\alpha_{\rm spec}}{R}\,Q^{(1)}_{\rm eff}Q^{(2)}_{\rm eff}\,e^{-m_{\rm spec}R}\,\mathcal{F}_{12}(R), \tag{7.24}$$

with $\alpha_{\rm spec}\equiv g^2_{\rm spec}/4\pi$, $Q_{\rm eff}=F_1(0)$, and a finite–size factor $\mathcal{F}_{12}(R)$ determined by the charge form factors $F_1^{(i)}(q^2)$: $\mathcal{F}_{12}(R)\to 1$ for $R\gg R_{\rm core}$. NLO Breit–type, spin–orbit, and Darwin terms are fixed by $F_{1,2}(q^2)$ and appear at $\mathcal{O}(v^2)$ in the EFT.

Internal non–Abelian exchange. For an unbroken subgroup $G_{\rm int}$ (e.g. the low–energy $SU(2)$ in a given phase), the static kernel generalizes to

$$V_{\rm int}(R) = \frac{1}{R}\,e^{-m_{\rm int}R}\sum_a \alpha_a\,\mathcal{C}^{(12)}_a\,\mathcal{F}^{(a)}_{12}(R), \tag{7.25}$$

where a runs over gauge boson species with couplings $\alpha_a = g_a^2/4\pi$, screened masses $m_{\rm int}$ (from symmetry breaking by $\langle\Sigma\rangle$ or medium effects), and color factors $\mathcal{C}^{(12)}_a$ fixed by the representation content selected in Sec. 4.7. Tensor and spin couplings follow from the non–Abelian magnetic form factors.

Emergent gravitational exchange. To leading order in weak fields, the exchange of the emergent metric fluctuations yields

$$V_{\rm grav}(R) = -\,\frac{G_{\rm eff}(\rho_{\rm bg})\,M_1M_2}{R}\,\mathcal{G}_{12}(R) \;+\; (\text{spin–spin, quadrupolar})\;\mathcal{O}(R^{-3}), \tag{7.26}$$

with $G_{\rm eff}=(8\pi M_{\rm eff}^2)^{-1}$ and $\mathcal{G}_{12}\to 1$ at $R\gg R_{\rm core}$. Subleading terms include familiar spin–spin and quadrupole interactions, here matched to the gravitational form factors of Sec. 6.14.4.

Summary (central piece). Combining Eqs. (7.24)–(7.26), the central long–range kernel reads

$$V_{\rm central}(R) \;=\; \frac{e^{-m_{\rm spec}R}}{R}\,\alpha_{\rm spec}Q_1Q_2 + \frac{e^{-m_{\rm int}R}}{R}\sum_a \alpha_a\mathcal{C}^{(12)}_a - \frac{G_{\rm eff}M_1M_2}{R} \quad (\text{up to finite–size factors}). \tag{7.27}$$

7.4.2 Anisotropy, orientation dependence, and tensor forces

Director–coupled invariants. MBP–1 cores carry a unit director $\hat{\chi}$ that couples to gradients and to exchanged fields. The most general parity–even, time–reversal–even anisotropic potential up to rank–2 is

$$V_{\rm aniso}(\mathbf{R};\hat{\chi}_1,\hat{\chi}_2) = \frac{1}{R^3}\Big[C_0(R)\,(\hat{\chi}_1\cdot\hat{\chi}_2) + C_1(R)\,(\hat{\chi}_1\cdot\hat{\mathbf{R}})(\hat{\chi}_2\cdot\hat{\mathbf{R}})\Big] + \frac{1}{R^3}\Big[D_0(R)\,\mathbf{S}_1\cdot\mathbf{S}_2 + D_1(R)\,S_{12}(\hat{\mathbf{R}})\Big] + \cdots, \tag{7.28}$$

with $S_{12} = 3(\mathbf{S}_1\cdot\hat{\mathbf{R}})(\mathbf{S}_2\cdot\hat{\mathbf{R}}) - \mathbf{S}_1\cdot\mathbf{S}_2$ and coefficient functions fixed by the multipole content of the exchanged sectors and by the orientation form factors (Chap. 6). The $1/R^3$ scaling is typical of dipole–dipole and spin–spin forces; higher multipoles give $1/R^{\ell+2}$.

Spectral magnetic dipoles and quadrupoles. From Eq. (7.5), spectral magnetic moments $\boldsymbol{\mu}_i$ generate

$$V_{\mu\mu}(\mathbf{R}) = \frac{\alpha_{\rm spec}}{R^3}\Big[\boldsymbol{\mu}_1\cdot\boldsymbol{\mu}_2 - 3(\boldsymbol{\mu}_1\cdot\hat{\mathbf{R}})(\boldsymbol{\mu}_2\cdot\hat{\mathbf{R}})\Big], \qquad \boldsymbol{\mu}_i = g_i\,\frac{Q_i}{2M_i}\,\mathbf{J}_i, \tag{7.29}$$

and quadrupole couplings (electric/gravitational) contribute $1/R^5$ terms proportional to the intrinsic quadrupole tensors $Q^{(i)}_{ab}$.

Spin–orbit and Berry–induced forces. Velocity–dependent pieces appear at NLO as

$$V_{LS}(\mathbf{R}) = \frac{1}{R^3}\,[\kappa_1\,\mathbf{L}\cdot\mathbf{S}_1 + \kappa_2\,\mathbf{L}\cdot\mathbf{S}_2] + \frac{\xi(R)}{R^2}\,\mathbf{L}\cdot\big(\mathbf{q}^{(1)}_{\rm Berry} - \mathbf{q}^{(2)}_{\rm Berry}\big), \tag{7.30}$$

where $\mathbf{q}_{\rm Berry}$ schematically denotes monopole–like charges on orientation spheres induced by Berry curvature (Chap. 6); the last term functions as a geometric Aharonov–Bohm interaction between relative motion and internal holonomies.

Angular decomposition. It is convenient to expand $V_{\rm aniso}$ on the $Y_{LM}(\hat{\mathbf{R}})$ basis:

$$V(\mathbf{R}) = \sum_{L=0}^{\infty}\frac{1}{R^{\Delta_L}}\sum_{M=-L}^{L}\mathcal{V}_{LM}(R)\,Y_{LM}(\hat{\mathbf{R}}), \qquad \Delta_0 = 1,\ \Delta_2 = 3,\ \Delta_4 = 5,\ \ldots \tag{7.31}$$

so that coupled–channel equations (Sec. 7.3) mix $\ell \leftrightarrow \ell' \in \{\ell \pm L\}$ with strength controlled by $\mathcal{V}_{LM}$.

7.4.3 Spectral curvature corrections and environmental dependence

Curvature–renormalized propagators. A nontrivial spectral curvature tensor $F_{\rm spec}$ (Chap. 3, App. E.4) modifies the probe propagators by a self–energy $\Sigma_{\rm spec}(\mathbf{q};\mathcal{S}_{\rm bg})$:

$$\mathcal{P}^{AB}(\mathbf{q};\mathcal{S}_{\rm bg}) = \Big[\big(\mathcal{P}^{AB}_0\big)^{-1} + \Sigma^{AB}_{\rm spec}(\mathbf{q};\mathcal{S}_{\rm bg})\Big]^{-1}. \tag{7.32}$$

In position space this translates into *environment–dependent* screening lengths, anisotropic Green functions, and modified power laws at large R.

Short–range repulsive core; long–range tails. Numerical evidence summarized in App. E.4 suggests a repulsive correction at short distances,

$$\Delta V_{\rm spec}(R) \;\simeq\; +\frac{\alpha_{\rm rep}}{R^n} \quad (R\to 0), \qquad \alpha_{\rm rep} > 0,\ n \geq 2, \tag{7.33}$$

stabilizing against collapse even when the uncorrected $V^{(0)}$ is attractive. At large separations, $F_{\rm spec}$–induced anisotropy can turn a Yukawa tail into a direction–dependent power law, e.g. $R^{-1-\delta(\hat{\mathbf{R}})}$ with small δ controlled by background invariants (ρ, η, $\nabla T\cdot\hat{\chi}$). Both features can be captured by Eq. (7.32) with a low–momentum expansion of $\Sigma_{\rm spec}$.

Background tunability. Since $F_{\rm spec}$ and effective couplings depend on the ambient state $\mathcal{S}_{\rm bg}$, the interaction potential is *not universal* but *state–dependent*:

$$V(\mathbf{R};\langle F_{\rm spec}\rangle) \;\equiv\; V_{\rm central}(R;\langle F_{\rm spec}\rangle) + V_{\rm aniso}(\mathbf{R};\langle F_{\rm spec}\rangle) + \cdots\,, \tag{7.34}$$

which is a distinctive prediction of the quantum–aether model. This tunability will be exploited in Sec. 7.7 to move resonances across thresholds and in Sec. 7.8 to enhance capture.

7.4.4 Regimes of validity and matching strategy

Separation of scales. The long–range kernels (7.27–7.30) are valid for $R \gg R_{\rm core}$, $kR_{\rm core} \ll 1$, and weak background gradients on scales $\gg R$. At $R \lesssim R_{\rm core}$ the contact basis of Eq. (7.8) takes over; coefficients are fixed by matching to microscopic two–soliton energy curves $E(R,\hat\chi_1,\hat\chi_2)$.

Consistent inclusion in coupled channels. In practice we use

$$V_{\rm cons}(\mathbf{R}) = \mathbb{P}_{\rm FR/Berry}\Big[U_{\rm ad}(\mathbf{R},X_1,X_2) + V_{U(1)} + V_{\rm int} + V_{\rm grav} + V_{\rm aniso}\Big]\mathbb{P}_{\rm FR/Berry} \;+\; L_{\rm SD}, \tag{7.35}$$

with all ingredients evaluated at the same background and scale, and with finite–size factors $\mathcal{F}(R)$ from the form factors of Chap. 6. Scale dependence is handled by the spectral RG (Sec. 6.11.5); uncertainties are propagated via the gates of Sec. 7.12.

The kernels above feed the partial–wave machinery of Sec. 7.3. We now analyze the universal low–energy consequences (effective–range expansion and near–threshold binding), which will be central to identifying shallow dimers and resonant structures (Sec. 7.5).

7.5 Low–Energy Scattering: Effective–Range and Universality

At relative momenta $kR_{\rm core} \ll 1$ the details of the short–distance dynamics are encoded in a few *threshold parameters*. This section develops the coupled–channel effective–range expansion (ERE), derives universal relations for shallow bound states and near–threshold resonances, and explains how FR/Berry selection rules reshape standard universality by suppressing or shifting the lowest allowed partial waves.

7.5.1 Coupled–channel ERE and threshold matrices

Single channel (isotropic S–wave). For a central, elastic channel with orbital $\ell = 0$ the standard ERE reads

$$k\cot\delta_0(k) \;=\; -\frac{1}{a} \;+\; \frac{1}{2}r_0k^2 \;+\; \sum_{n\ge 2} P_n k^{2n}, \tag{7.36}$$

where a is the (complex, if inelastic) scattering length, r_0 the effective range, and P_n shape parameters. For purely elastic scattering $a, r_0, P_n \in \mathbb{R}$; weak radiation/open channels make $a \to a - ia_{\rm inel}$ with $a_{\rm inel} > 0$ (optical theorem).

Higher partial waves. For $\ell \ge 1$ one defines the reduced K–matrix

$$k^{2\ell+1}\cot\delta_\ell(k) \;=\; -\frac{1}{A_\ell} \;+\; \frac{1}{2}R_\ell k^2 + \cdots\,, \tag{7.37}$$

with A_ℓ the generalized scattering "volume" (ℓ=1) or higher–moment parameter. For p–wave, $k^3\cot\delta_1 = -1/A_1 + \frac{1}{2}R_1k^2 + \cdots$.

Coupled channels (matrix ERE). Let α,β label open channels that mix at fixed ℓ (or within an anisotropic block after projection, Sec. 7.3). The on–shell K–matrix satisfies

$$\mathbf{K}_\ell^{-1}(k) \;=\; -\,\mathbf{A}_\ell^{-1} \;+\; \tfrac{1}{2}\,\mathbf{R}_\ell\,k^2 \;+\; \cdots\,, \qquad \mathbf{S}_\ell \;=\; \big(\mathbb{I} + i\mathbf{K}_\ell\big)\big(\mathbb{I} - i\mathbf{K}_\ell\big)^{-1}. \tag{7.38}$$

$\mathbf{A}_\ell$ and $\mathbf{R}_\ell$ are Hermitian (symmetric real) in elastic subspaces; small imaginary parts appear once inclusive inelasticity is integrated out. Diagonalization yields eigenphase shifts and inelasticity parameters $(\delta_{\ell,j},\eta_{\ell,j})$.

Matching to EFT contacts. The threshold matrices are in one–to–one correspondence with the short–distance operator basis Eq. (7.8). For a single S–wave channel:

$$a \;=\; \frac{\mu}{2\pi}\,\frac{1}{C_0(\Lambda)}\left[1+\mathcal{O}\!\left(\frac{C_2\,\Lambda^2}{C_0}\right)\right], \qquad r_0 \;=\; \frac{4\pi}{\mu^2}\,\frac{C_2}{C_0^2}+\mathcal{O}\!\left(\Lambda^{-1}\right), \tag{7.39}$$

in a cutoff scheme $\Lambda \sim 1/R_{\rm core}$ (dimensional schemes yield equivalent RG relations). This provides a direct route to *fit* $C_0, C_2, \ldots$ from low–energy data.

7.5.2 Shallow bound states and near–threshold universality

Unitary limit and universal dimer (S–wave). When $|a| \gg R_{\rm core}$ (unitary regime) and r_0 is natural ($|r_0| \sim R_{\rm core}$), an S–wave bound state (dimer) exists for $a > 0$ with binding momentum $\kappa = 1/a\,[1 - r_0/(2a) + \cdots]$ and energy

$$E_b \;=\; \frac{\kappa^2}{2\mu} \;=\; \frac{1}{2\mu\,a^2}\left[1-\frac{r_0}{a}+\mathcal{O}\!\left(\frac{R_{\rm core}^2}{a^2}\right)\right]. \tag{7.40}$$

The asymptotic wavefunction is $\psi_b(R) \propto e^{-\kappa R}/R$, independent of short–distance details. Radiative channels add a width $\Gamma_b \propto \varepsilon_{\rm rad}$.

Virtual states and zero–energy resonances. For $a < 0$ with $|a| \gg R_{\rm core}$ a *virtual* pole sits on the negative imaginary k–axis, producing a pronounced low–energy enhancement in $\sigma(k)$. Tuning background parameters ($\langle F_{\rm spec}\rangle$ or density ρ, Sec. 7.4.3) can move the pole across threshold, turning a virtual state into a shallow dimer—a useful control knob in the quantum–aether medium.

p–wave and higher. Near a p–wave resonance with large $|A_1|$, a bound state at κ satisfies

$$\kappa^3 \;=\; \frac{1}{A_1}\left[1-\frac{R_1}{2}\kappa^2+\cdots\right], \qquad E_b = \frac{\kappa^2}{2\mu}, \tag{7.41}$$

and is *centrifugally* suppressed in size relative to S–wave dimers. Selection rules (below) often make p–wave the *leading* universal channel.

Coupled channels and Feshbach mechanism. If an open channel weakly mixes with a closed one supporting a near–threshold level, the effective S–wave scattering length exhibits a resonant form

$$a(B) \;=\; a_{\rm bg}\left(1-\frac{\Delta}{B-B_0}\right), \tag{7.42}$$

with tuning parameter B (a shorthand for any background control, e.g. ρ, η, or alignment). This is the *aether analogue* of a Feshbach resonance; microscopic matching fixes $a_{\rm bg}$, Δ, and B_0.

7.5.3 FR/Berry selection rules and modified universality

Suppression of S–wave. For identical solitons with fermionic FR character ($\sigma = -1$), S–wave is forbidden in the elastic ground channel (Sec. 7.3.5), making p–wave the *lowest* allowed partial wave. Then the universal low–energy scaling switches from $\sigma \sim a^2$ (S–wave) to $\sigma \sim k^4|A_1|^2$ (p–wave), with threshold behavior dominated by Eqs. (7.37) and (7.41).

Monopole harmonics and minimal ℓ. In the presence of Berry charge $q \in \frac{1}{2}\mathbb{Z}$ on the relative orientation sphere, the minimal allowed ℓ satisfies $\ell \geq |q|$, and the low–energy amplitude is built from $Y_{\ell m}^{(q)}$. Consequently the *first* universal channel may be $\ell = |q|$, not $\ell = 0$, and the ERE must be written for that ℓ with parameters (A_ℓ, R_ℓ).

Geometric zeros and selection–protected nulls. Projectors $\mathbb{P}_{\rm FR/Berry}$ enforce amplitudes that vanish in specific directions or channels irrespective of dynamics ("geometric zeros"). This produces distinctive *nulls* in $d\sigma/d\Omega$ that survive moderate orientation averaging, providing sharp signatures of the geometric structure.

7.5.4 Corrections from long–range tails and anisotropy

Breakdown of standard ERE. Long–range $1/R^n$ tails with $n \leq 3$ modify the analytic structure of the ERE: Coulomb/Yukawa pieces (n=1) require Coulomb–modified ERE; dipolar/tensor tails (n=3) induce logarithmic terms and ℓ–mixing at arbitrarily low k. In anisotropic media, the dispersion E_k of Eq. (7.11) introduces direction–dependent flux and partial–wave mixing.

Coulomb/Yukawa modification. For a spectral Yukawa tail of mass m_{spec}, the Coulomb functions F_ℓ, G_ℓ are replaced by modified spherical Bessel functions, and the effective–range parameters become m_{spec}–dependent. Practically we work with the Coulomb–modified K–matrix $K_\ell^{(C)}$ and extract "short–range" a, r_0 after removing the known long–range phase.

Tensor tails and renormalization. For $1/R^3$ tensor interactions, the low–energy amplitude acquires nonanalytic $k \log k$ pieces and a residual regulator dependence unless the *tensor contact* operators are included at LO in the power counting. We adopt a renormalized scheme (contact + regulated $1/R^3$) that yields stable ERE parameters and well–defined universality windows.

7.5.5 Diagnostics, extraction protocols, and gates

Extraction from EFT solvers. Solve the coupled radial equations with V_{cons} of Eq. (7.35); fit $K_\ell(k)$ (or the eigenphases) to Eqs. (7.36)–(7.38) in a window $k \in [k_{\min}, k_{\max}]$ satisfying $kR_{\text{core}} \ll 1$ and $k \ll m_{\text{spec}}, m_{\text{int}}$. Report stability under window shifts.

Extraction from full PDE scattering. From time–domain simulations (Sec. 7.3.7), reconstruct $f(\theta)$ and invert the partial–wave series to obtain $\delta_\ell(k)$; cross–validate with EFT extraction in overlapping regimes. Orientation averaging should be performed both before and after the inversion to expose selection–protected nulls.

Gates (low–energy universality).

- **L1 (windowed universality):** Extracted a, r_0 (or A_ℓ, R_ℓ) are stable under k–window variations and regulator changes within quoted uncertainties.
- **L2 (consistency with contacts):** Threshold parameters map to a *natural* set of $\{C_0, C_2, \dots\}$ in Eq. (7.39) (no fine–tuning unless near a controlled resonance).
- **L3 (FR/Berry):** Forbidden partial waves vanish within numerical error; minimal allowed ℓ matches $|q|$ when Berry charge is present.
- **L4 (long–range accounting):** Coulomb/Yukawa/tensor tails are removed or included in a modified ERE; residuals show no spurious $k \log k$ artifacts.
- **L5 (bound–state cross–checks):** E_b extracted from (7.40) or (7.41) agrees with direct bound–state solvers (Sec. 7.7) within uncertainties.

With threshold parameters and universality windows in hand, we are equipped to analyze semiclassical high–energy scattering (Sec. 7.6) and to construct the spectrum of resonances and molecular states near and above threshold (Sec. 7.7).

7.6 Semiclassical High–Energy Scattering

When the relative momentum satisfies $kR_{\text{core}} \gg 1$ and deflections are modest, scattering admits a semiclassical (eikonal) description. The amplitude resums ladder diagrams (or, equivalently, accumulates phase along straight–line worldlines) and naturally incorporates long–range tails, weak curvature of g_{eff}, and noncentral anisotropic forces. We formulate the eikonal on a slowly varying background, include Berry/FR geometric phases, and connect to classical deflection, rainbow, and glory phenomena.

7.6.1 Eikonal regime and validity

Kinematic window. Let $q = |\mathbf{k}' - \mathbf{k}| = 2k\sin(\theta/2)$ be the momentum transfer. The *eikonal* regime requires

$$kR_{\rm core} \gg 1, \qquad \theta \ll 1, \qquad \frac{V}{E} \ll 1, \qquad \mathcal{R}^{-1} \ll k \ll m_{\rm spec}, m_{\rm int}, \tag{7.43}$$

where $\mathcal{R}$ is the curvature radius of the background $(\mathcal{M}, g_{\rm eff})$ and E the CoM energy. Under (7.43) straight–line trajectories are leading, with phase accumulated from weak kernels along z (beam axis).

Impact parameter and partial waves. Large angular momenta $\ell \gg 1$ map to impact parameters $b = (\ell + \frac{1}{2})/k$. The eikonal phase $\chi(b)$ reproduces partial–wave phase shifts via $\delta_\ell \simeq \chi(b{=}\frac{\ell+1/2}{k})$ (central potentials), and its b–derivative encodes classical deflection (Sec. 7.6.3).

7.6.2 Eikonal phase on a slowly varying background

Central kernel (Born–subtracted). For a central, conservative potential $V_{\rm cons}(r)$ the eikonal phase is

$$\chi_0(b) \;=\; -\frac{\mu}{k}\int_{-\infty}^{+\infty} dz\, V_{\rm cons}\Big(\sqrt{b^2+z^2}\Big), \qquad \mu = \frac{M_1 M_2}{M_1+M_2}. \tag{7.44}$$

When $V_{\rm cons}$ contains Coulomb/Yukawa pieces, one uses the Coulomb–modified eikonal (subtracting the known long–range phase and adding it back analytically).

Anisotropy and tensor forces. For $V(\mathbf{R}; \hat\chi_1, \hat\chi_2)$ with the decomposition (7.31), the eikonal phase acquires m–dependent components. In an axially aligned frame,

$$\chi(b,\phi_b) \;=\; \chi_0(b) \;+\; \sum_{L\ge 2}\sum_{M=-L}^{L} \chi_{LM}(b)\, e^{iM\phi_b}, \quad \chi_{LM}(b) = -\frac{\mu}{k}\int dz\, \frac{\mathcal{V}_{LM}(R)}{R^{\Delta_L}}\, Y_{LM}(\hat{\mathbf{R}}), \tag{7.45}$$

with ϕ_b the azimuth of $\mathbf{b}$ relative to the director(s) $\hat\chi_{1,2}$ and $R = \sqrt{b^2+z^2}$.

Berry/FR geometric phase. Relative motion on moduli/angle spheres with Berry curvature adds a path–ordered phase

$$\chi_{\rm Berry}(b) \;=\; \oint_{\gamma_b} \mathcal{A} \;=\; \int_{\Sigma_b} \Omega, \tag{7.46}$$

where γ_b is the closed loop traced by the slow orientation variables during the collision and Σ_b its spanning surface. In the monopole case ($\Omega = q\,\sin\theta\, d\theta\wedge d\phi$), $\chi_{\rm Berry}$ reduces to q times the solid angle swept by the relative director; it yields Aharonov–Bohm–like shifts and selection–protected zeros.

Weak curvature of $g_{\rm eff}$. Metric perturbations $h_{\mu\nu}$ induce a gravitational eikonal piece

$$\chi_{\rm grav}(b) \;=\; \frac{E}{2k}\int_{-\infty}^{+\infty} dz\; h_{\mu\nu}(x)\, u^\mu u^\nu \;\simeq\; 2\,G_{\rm eff} M_{\rm lens}\, E\, \ln\frac{L}{b} \quad \text{(for a point–like lens)}, \tag{7.47}$$

where u^μ is the CoM four–velocity and L an IR scale; the last form is the familiar logarithm from $1/R$ potentials, here with the emergent $G_{\rm eff}$.

7.6.3 Eikonal amplitude and differential cross section

Scalar (central) amplitude. The elastic eikonal amplitude in the CoM frame is

$$f_{\rm eik}(\theta) \;=\; \frac{k}{i}\int_0^\infty b\, db\, J_0(qb)\, \Big(e^{2i\chi_0(b)} - 1\Big), \qquad q = 2k\sin\frac{\theta}{2}. \tag{7.48}$$

Adding Coulomb/Yukawa pieces uses the standard replacement $J_0 \to$ Coulomb wave factors.

Anisotropic matrix amplitude. For anisotropic/tensor forces and channel mixing, the amplitude becomes a matrix in the (m, α) indices:

$$\mathbf{f}_{\rm eik}(\theta,\phi) \;=\; \frac{k}{i}\int_0^\infty b\,db \int_0^{2\pi}\frac{d\phi_b}{2\pi}\; e^{iqb\cos(\phi-\phi_b)}\,\Big[\mathcal{P}\exp\!\big(2i\,\boldsymbol{\chi}(b,\phi_b)\big)-\mathbb{I}\Big], \tag{7.49}$$

where $\boldsymbol{\chi}$ collects central, tensor, Berry, and weak–gravity contributions and $\mathcal{P}$ denotes path ordering in the (possibly non–Abelian) channel space.

Differential cross section. The unpolarized elastic cross section follows by trace/average over initial channels,

$$\frac{d\sigma}{d\Omega}(\theta,\phi) \;=\; \frac{1}{N_{\rm in}}\mathrm{Tr}\Big[\mathbf{f}^{\dagger}_{\rm eik}(\theta,\phi)\,\mathbf{f}_{\rm eik}(\theta,\phi)\Big]\,, \tag{7.50}$$

and reduces to $|f_{\rm eik}|^2$ in the scalar central case.

7.6.4 Classical deflection, rainbow, and glory

Deflection function and stationary phase. Define the *deflection function*

$$\Theta(b) \;\equiv\; -\frac{2}{k}\,\frac{d}{db}\,\mathrm{Re}\,\chi(b), \tag{7.51}$$

which is the classical scattering angle in the WKB limit. The leading contribution to $f_{\rm eik}$ arises from stationary points of the phase $\Phi(b) = qb + 2\,\mathrm{Re}\,\chi(b)$, satisfying $d\Phi/db = 0 \iff \Theta(b) = \theta$.

Rainbow caustic. A *rainbow* occurs when $d\Theta/db = 0$ at $b = b_r$, producing a fold caustic. Expanding Φ around b_r yields an Airy pattern,

$$\left.\frac{d\sigma}{d\Omega}\right|_{\rm rainbow} \;\simeq\; \mathcal{C}\;\mathrm{Ai}^2\!\big(\alpha\,(\theta-\theta_r)\big)\,, \qquad \alpha \propto \left|\left.\frac{d^2\Theta}{db^2}\right|_{b_r}\right|^{1/3} k^{2/3}, \tag{7.52}$$

with orientation–dependent intensity $\mathcal{C}(\hat{\chi}_{1,2})$ through χ_{LM}.

Glory (backscattering). Near $\theta = \pi$, coalescence of two stationary points leads to a *glory* enhancement with Bessel modulation,

$$\left.\frac{d\sigma}{d\Omega}\right|_{\rm glory} \;\simeq\; \mathcal{G}\;\frac{1}{\sin\theta}\,J_0^2\!\big(k\,b_g\,(\pi-\theta)\big), \tag{7.53}$$

where b_g solves $\Theta(b_g) = \pi$. Berry phases shift the fringe pattern by q–dependent amounts.

7.6.5 Spin, polarization, and orientation transport

Spin transport. Spin/orientation couplings (§7.4.2) induce a precession

$$\frac{d\mathbf{S}}{dt} \;=\; \boldsymbol{\Omega}_{\rm SO}(t)\times\mathbf{S}, \qquad \boldsymbol{\Omega}_{\rm SO}(t)\propto \nabla V\times\mathbf{v} \;+\; \nabla\chi_{\rm Berry}\times\mathbf{v}. \tag{7.54}$$

The eikonal amplitude becomes a 2×2 (or larger) matrix in spin space with small helicity–flip components $\sim\mathcal{O}(\theta)$; polarization observables probe $\kappa_{1,2}$ of Eq. (7.30).

Orientation holonomy. Transport of the internal directors around the scattering loop picks up the geometric phase $\chi_{\rm Berry}$ of Eq. (7.46), shifting interference fringes and enforcing selection–protected nulls even at high energy.

7.6.6 Including weak radiation: eikonal with absorption

Imaginary eikonal. Soft radiation (spectral or emergent–gravitational) yields an *imaginary* part of the eikonal, $\chi \to \chi_R + i\chi_I$, with $\chi_I(b) \geq 0$ determined by the classical Larmor–like power integrated along the straight path,

$$\chi_I(b) \;=\; \frac{1}{2E}\int_{-\infty}^{+\infty} dt\; P_{\rm rad}\big(\mathbf{R}(t),\dot{\mathbf{R}}(t)\big) \;+\; \mathcal{O}(\varepsilon_{\rm rad}^2). \tag{7.55}$$

Then $e^{2i\chi} \to e^{2i\chi_R}e^{-2\chi_I}$ suppresses small–b contributions, restoring inclusive unitarity in accordance with soft theorems (Sec. 6.15).

Energy balance gate. The integrated radiated energy reconstructed from χ_I must equal the loss in elastic flux within numerical uncertainties, providing a practical *energy balance* check for simulations.

7.6.7 Diagnostics and consistency gates

- **H1 (window):** Results are stable under variations of the eikonal window ($\theta_{\max}$, $kR_{\rm core}$) and IR/UV regulators of the line integral.
- **H2 (matching):** For moderate k, the eikonal amplitude smoothly matches the partial–wave sum $\sum_\ell (2\ell{+}1)(S_\ell - 1)P_\ell(\cos\theta)/(2ik)$.
- **H3 (deflection):** Stationary–phase reconstruction of $\Theta(b)$ from $f_{\rm eik}$ agrees with classical trajectories in $V_{\rm cons}$ under $g_{\rm eff}$.
- **H4 (Berry/FR):** AB–like phase shifts and selection–protected nulls appear at the predicted angles; removing Ω in the model eliminates them.
- **H5 (radiation):** The imaginary eikonal component reproduces soft–emission rates and inclusive unitarity (KLN) to the working order.

The eikonal analysis complements the low–energy ERE of Sec. 7.5 and provides high–energy, small–angle predictions, including geometric fringes from Berry curvature and anisotropy. We next turn to the resonance and molecular spectrum near and above threshold (Sec. 7.7), where coupled–channel dynamics, tensor forces, and environmental tunability play decisive roles.

7.7 Resonances and Molecular States

We analyze the spectrum of *two–soliton* composites and scattering resonances generated by the conservative kernels of Sec. 7.4, subject to FR/Berry constraints and possible anisotropy. We develop the analytic structure of the S–matrix, identify near–threshold universality classes (dimer/virtual/shape/Feshbach), construct rotational–vibrational "molecular" bands in anisotropic potentials, and quantify radiative widths and nonadiabatic decay. Practical extraction methods and consistency gates close the section.

7.7.1 Analytic structure and definitions

S–matrix poles. Let k be the complex relative momentum with $E = k^2/(2\mu)$ (isotropic case) or the appropriate dispersion $E(k_\perp, k_\parallel)$ (Sec. 7.3.1). Poles of the on–shell S–matrix in the complex k–plane encode:

$$\begin{cases} \text{Bound states:} & k = i\kappa,\ \kappa > 0 \quad (E = -\kappa^2/2\mu) \\ \text{Virtual states:} & k = -i\kappa,\ \kappa > 0 \quad \text{(non-normalizable, enhance threshold)} \\ \text{Resonances:} & k = k_r - i\gamma,\ \gamma > 0 \quad (E = E_r - i\Gamma/2) \end{cases} \tag{7.56}$$

with $\Gamma = 2k_r\gamma/\mu$ (isotropic) the total width. In coupled channels, poles live on different Riemann sheets labeled by the signs of $\mathrm{Im}\, k_\alpha$.

Breit–Wigner vs. Flatté forms. For a *narrow isolated* resonance in a single open channel,

$$T(E) \simeq \frac{g^2}{E - E_0 + i\Gamma/2}, \qquad \Gamma(E) \propto \rho(E)\,|g(E)|^2, \tag{7.57}$$

with phase space $\rho(E) \sim k/\mu$. Near an *inelastic threshold* $E_{\rm th}$, the line shape is Flatté–like:

$$T(E) \simeq \frac{g_1^2}{E - E_0 + i\big(g_1^2\rho_1(E) + g_2^2\rho_2(E)\big)/2}, \tag{7.58}$$

crucial when a closed channel becomes open (Feshbach mechanism below).

7.7.2 Near–threshold universality: dimers, virtual, and halo states

S–wave dimers. In the unitary window $|a| \gg R_{\rm core}$, an S–wave dimer exists for $a > 0$ with binding

$$E_b = \frac{1}{2\mu a^2}\Big[1 - \frac{r_0}{a} + \mathcal{O}\big((R_{\rm core}/a)^2\big)\Big], \qquad \psi_b(R) \propto \frac{e^{-R/a}}{R}. \tag{7.59}$$

The *halo* limit $a \gg R_{\rm core}$ makes the composite size $\sim a$ insensitive to short–range details and dominated by the long–tail potentials of Sec. 7.4.1.

Weinberg compositeness (diagnostic). For a shallow S–wave bound state, the *compositeness* X (probability to be molecular) is estimated from threshold parameters:

$$a \simeq \frac{2(1-X)}{2-X}\,\frac{1}{\kappa}, \qquad r_0 \simeq -\frac{X}{1-X}\,\frac{1}{\kappa}, \tag{7.60}$$

with $\kappa = \sqrt{2\mu E_b}$. Values $X \to 1$ indicate dominantly molecular states; $X \ll 1$ suggests compact (elementary–like) excitations.

FR/Berry–forced p–wave universality. If S–wave is *forbidden* by FR character (Sec. 7.3.5) or by a Berry monopole with minimal $\ell = |q| > 0$, the first universal channel is p–wave (or $\ell = |q|$). Near a p–wave pole:

$$k^3 \cot\delta_1(k) = -\frac{1}{A_1} + \frac{1}{2}R_1 k^2 + \cdots, \qquad E_b = \frac{\kappa^2}{2\mu}, \quad \kappa^3 \approx \frac{1}{A_1}. \tag{7.61}$$

Centrifugal suppression yields more compact halo states and characteristic threshold scaling $\sigma \sim k^4|A_1|^2$ (Sec. 7.5).

7.7.3 Feshbach–type resonances and environmental control

Closed–open channel coupling. Let a closed channel c support a level $E_c(\lambda)$, with λ a background control (e.g. density ρ, spectral curvature $\langle F_{\rm spec}\rangle$, or director alignment). Coupling V_{oc} to an open channel o produces an energy–dependent scattering length

$$a(\lambda) = a_{\rm bg}\left(1 - \frac{\Delta}{\lambda - \lambda_0}\right), \qquad \Delta \propto \frac{|V_{oc}|^2}{\partial_\lambda E_c|_{\lambda_0}}, \tag{7.62}$$

with a resonance at $\lambda = \lambda_0$ (Sec. 7.5.2). This "aether Feshbach" tunability is a key prediction: *external* spectral geometry controls *internal* molecular spectra.

Pole trajectories. As λ varies, poles move continuously across sheets: virtual $\to$ bound (crossing the branch cut) or bound $\to$ resonance (when another inelastic channel opens). Tracking $E(\lambda)$ and $\Gamma(\lambda)$ gives sharp, falsifiable curves.

7.7.4 Shape resonances from tensor/anisotropic barriers

Centrifugal + tensor barrier. Anisotropic $1/R^3$ forces (Sec. 7.4.2) plus centrifugal terms generate barriers that can trap quasibound states above threshold ("shape" resonances). Semiclassically, for a fixed (ℓ, m) or anisotropic block, turning points $R_1 < R_2$ satisfy $E = V_{\rm eff}(R)$ with

$$V_{\rm eff}(R) = V_{\rm central}(R) + \frac{\ell(\ell+1)}{2\mu R^2} + V^{(\ell m)}_{\rm tensor}(R). \tag{7.63}$$

WKB quantization in the pocket $[R_1, R_2]$ gives

$$\int_{R_1}^{R_2} dR\, \sqrt{2\mu\big(E - V_{\rm eff}(R)\big)} \;=\; \pi\left(n + \tfrac{1}{2}\right), \tag{7.64}$$

while barrier penetration yields width

$$\Gamma \simeq \omega_{\rm well}\, e^{-2S_{\rm bar}}, \qquad S_{\rm bar} = \int_{R_2}^{R_3} dR\, \sqrt{2\mu\big(V_{\rm eff}(R) - E\big)}, \tag{7.65}$$

with R_3 the outer turning point and $\omega_{\rm well}$ the intrawell frequency.

Orientation control. Because $V_{\rm tensor}$ depends on $\hat{\chi}_{1,2}$ and the collision axis, both E_r and Γ carry *alignment* dependence—a distinctive signature testable in numerical experiments.

7.7.5 Anisotropic "molecules": rotational–vibrational bands

Born–Oppenheimer separation. For heavy cores and weak relative motion, fix orientations $X_{1,2}$ and diagonalize the internal moduli to obtain adiabatic surfaces $U_{\rm ad}(R; X_{1,2})$ (Sec. 7.4.4). Bound states follow from the radial Schrödinger equation on each surface; slow orientation dynamics then defines *rotational* bands.

Effective rotor Hamiltonian. Let $\mathcal{I}_{\rm mol}$ be the moment of inertia around the axis orthogonal to $\hat{\mathbf{R}}$. For an axially symmetric minimum,

$$H_{\rm mol} = -\frac{1}{2\mu}\frac{d^2}{dR^2} + U_{\rm ad}(R) + \frac{\mathbf{L}^2}{2\mu R^2} + \frac{\mathbf{J}^2}{2\mathcal{I}_{\rm mol}} + \lambda_{\rm SO}\,\mathbf{L}\cdot\mathbf{J} + \cdots, \tag{7.66}$$

with $\mathbf{J}$ the total internal spin/orientation and $\lambda_{\rm SO}$ a small spin–orbit coupling. Berry flux on the orientation sphere quantizes J via monopole harmonics $Y^{(q)}_{JM}$, shifting band heads by q–dependent amounts.

Selection rules for transitions. Dipole–like spectral emission enforces $\Delta J = 0, \pm 1$ (no $0 \to 0$), while Berry/FR may forbid additional lines. Gravitational quadrupole transitions are suppressed by $G_{\rm eff}$ and scale as ω^5.

7.7.6 Radiative and nonadiabatic widths

Spectral radiation. Coupling of molecular multipoles to the spectral $U(1)$ (Sec. 6.15) gives

$$\Gamma^{(E1)}_{\rm spec} \sim \alpha_{\rm spec}\,\omega^3\,|\mathbf{d}_{if}|^2, \qquad \Gamma^{(M1)}_{\rm spec} \sim \alpha_{\rm spec}\,\omega^3\,|\boldsymbol{\mu}_{if}|^2, \tag{7.67}$$

with transition frequency ω and dipole/magnetic matrix elements between molecular states. Quadrupole contributions scale as $\omega^5 |Q_{if}|^2$.

Emergent gravitational radiation. Tensor emission is parametrically tiny but present:

$$\Gamma^{(E2)}_{\rm grav} \sim G_{\rm eff}\,\omega^5\,|Q^{(if)}_{ij}|^2, \tag{7.68}$$

and can be enhanced for heavy composites with large quadrupoles.

Nonadiabatic (Landau–Zener) decay. When adiabatic surfaces $U_{\rm ad}^{(1,2)}(R)$ exhibit an avoided crossing at $R = R_\times$ with gap Δ, a moving wave packet suffers transitions with probability

$$P_{1\to 2} \simeq \exp\left[-\frac{\pi\Delta^2}{2\hbar\, v_R\, |\partial_R \Delta E|_{R_\times}}\right], \tag{7.69}$$

(v_R radial speed). This contributes to Γ in near–threshold resonances and can dominate when radiation is weak.

7.7.7 Computation and extraction protocols

Bound states (real axis). Solve the coupled radial equations on $[0, R_{\rm max}]$ with appropriate FR/Berry projectors; impose regularity at $R = 0$ and decaying asymptotics. Shoot for $E < 0$ and locate eigenvalues by nodal counting or via a generalized eigenvalue problem on a DVR basis.

Resonances (complex energy). Use one of:

1. *Complex scaling*: $R \to Re^{i\theta}$, rotate the continuum and reveal Gamow eigenstates as θ–stable complex eigenvalues.
2. *CAP (absorbing boundaries)*: add $-iW(R)$ for $R > R_c$ and fit complex eigenvalues vs. R_c.
3. *Stabilization*: vary $R_{\rm max}$ and identify avoided crossings plateaus vs. box size.

Cross–check pole positions with analytic continuation of the K–matrix extracted from phase shifts.

Molecular bands. At fixed orientations $X_{1,2}$, compute $U_{\rm ad}(R)$ on a grid (Chap. **??**), fit a smooth potential, and solve Eq. (7.66) for (n, J) levels. Map alignment dependence to predict line splittings.

7.7.8 Gates and diagnostics

- **R1 (sheet assignment):** Poles are placed on correct Riemann sheets; motion under control parameter λ is continuous and consistent with expected thresholds.
- **R2 (universality):** Near–threshold dimers obey Eq. (7.59); extracted (a, r_0) predict E_b within uncertainties; p–wave cases follow Eq. (7.61).
- **R3 (FR/Berry):** Forbidden partial waves remain absent in bound spectra and line shapes; monopole charge q shifts minimal J and band heads as predicted.
- **R4 (width budget):** Total width Γ decomposes consistently into radiative [Eqs. (7.67), (7.68)] and nonadiabatic [Eq. (7.69)] components (within errors).
- **R5 (matching):** Molecular multipole matrix elements used in Γ match the single–soliton form factors (Chap. 6) and the two–body EFT couplings (Sec. 7.2).
- **R6 (stability):** Bound/resonant energies and widths are stable under regulator choices, box sizes, and basis refinements.

These results deliver a predictive catalogue of two–soliton states and resonances, with sharp, geometry–driven signatures (alignment dependence, FR/Berry selection, tunable Feshbach poles). They provide the key bridge from microscopic parameters of the quantum–aether to observable spectra and scattering data, and they feed directly into the radiative and soft–emission analysis of Sec. 7.8.

7.8 Radiative Processes and Soft Emission

Two–body dynamics in the quantum–aether generically couples to propagating probe sectors (spectral $U(1)$, internal non–Abelian remnants, and emergent gravity). Even when these couplings are weak, they produce *soft radiation* which modifies exclusive amplitudes, induces small inelasticities in the elastic S–matrix, and enables *radiative capture* into bound molecular states. This section develops universal soft limits and factorization, constructs inclusive (IR–safe) cross sections via exponentiation, and presents multipole radiation and capture rates including anisotropy and Berry/FR selection rules. We close with memory effects, implementation protocols, and gates.

7.8.1 Universal soft theorems and factorization

Kinematics and notation. Let p_i^μ be the asymptotic four–momenta of hard external solitons (in/out), normalized with the emergent metric $g_{\rm eff}$. A radiated *spectral* quantum ("photon–like") carries momentum k^μ and polarization $\varepsilon_\mu(k)$ satisfying $k^2 = m_{\rm spec}^2$ and the medium dispersion (Sec. 7.4.1). For *gravitational* radiation we use transverse–traceless polarization $\varepsilon_{\mu\nu}(k)$ and coupling $\kappa^2 = 32\pi G_{\rm eff}$.

Spectral $U(1)$ soft factor. For $k \to 0$ the $(n{+}1)$–body amplitude factorizes as

$$\mathcal{M}_{n+1}^{(\rm spec)}(k;\{p_i\}) \;=\; \left[g_{\rm spec} \sum_{i=1}^{n} Q_i \, \frac{p_i\cdot\varepsilon(k)}{p_i\cdot k} \;+\; \mathcal{O}(k^0) \right] \mathcal{M}_n(\{p_i\}), \tag{7.70}$$

where Q_i are *effective* spectral charges matched to single–soliton form factors (Chap. 6). Anisotropy of $g_{\rm eff}$ affects the scalar products and on–shell condition for k^μ but not the *structure* of the leading soft factor.

Gravitational soft factor. Similarly,

$$\mathcal{M}_{n+1}^{(\rm grav)}(k;\{p_i\}) \;=\; \left[\frac{\kappa}{2} \sum_{i=1}^{n} \frac{p_i^\mu p_i^\nu \, \varepsilon_{\mu\nu}(k)}{p_i\cdot k} \;+\; \mathcal{O}(k^0) \right] \mathcal{M}_n(\{p_i\}), \tag{7.71}$$

with $\kappa^2 = 32\pi G_{\rm eff}(\rho_{\rm bg})$. These are the aether analogues of the Low/Weinberg soft theorems, now with medium–renormalized kinematics.

Berry/FR selection. If the relative motion explores moduli with Berry curvature (Chap. 6), soft emission in channels that would violate the FR projector or monopole harmonics selection rules is suppressed at least by one extra power of k, i.e. the $1/k$ pole cancels upon projection.

7.8.2 Inclusive IR safety: exponentiation and KLN

YFS/Bloch–Nordsieck in the aether. Summing over undetected soft quanta with energies $\omega < \varepsilon$ (detector resolution) yields an *inclusive* differential cross section

$$d\sigma_{\rm inc}(\varepsilon) = e^{\,2\,\mathrm{Re}\,\mathcal{I}_{\rm soft}(\varepsilon)} \, d\sigma_{\rm hard}(\mu) \;+\; \mathcal{O}(\varepsilon), \tag{7.72}$$

where $\mathcal{I}_{\rm soft}$ is a universal eikonal functional obtained by integrating Eqs. (7.70)–(7.71) over the soft phase space defined by $g_{\rm eff}$, and $d\sigma_{\rm hard}$ is finite and scheme–dependent only through a factorization scale μ. When $m_{\rm spec} > 0$ (screened spectral sector), the IR is regulated and exponentiation reduces to a finite Sudakov–like factor.

Optical theorem and inelasticity. The inclusive treatment restores the optical theorem (Sec. 7.3.4). In the elastic block, soft radiation appears as inelasticity parameters $\eta_{\ell,j} < 1$ related to $\mathcal{I}_{\rm soft}$ and, semiclassically, to the imaginary eikonal χ_I (Sec. 7.6.6).

7.8.3 Multipole radiation from accelerated solitons

Spectral dipole (Larmor analogue). In nonrelativistic scattering under $g_{\rm eff}$, the instantaneous spectral power is

$$P_{\rm spec}(t) \;=\; \frac{2}{3}\,\alpha_{\rm spec} \left\langle \ddot{\mathbf{d}}(t)\cdot\ddot{\mathbf{d}}(t) \right\rangle_{g_{\rm eff}} \;+\; \mathcal{O}(v^2), \qquad \mathbf{d} = \sum_i Q_i\,\mathbf{R}_i, \tag{7.73}$$

with scalar products and time measured by $g_{\rm eff}$. Anisotropy enters via direction–dependent wave speeds and polarization projectors. Magnetic dipole and electric quadrupole add $\mathcal{O}(v^2)$ corrections set by single–soliton moments (Chap. 6).

Gravitational quadrupole. Emergent gravitational radiation follows the standard quadrupole form with $G_{\rm eff}$:

$$P_{\rm grav}(t) \;=\; \frac{G_{\rm eff}}{5}\left\langle \dddot{Q}_{ij}(t)\,\dddot{Q}^{ij}(t)\right\rangle_{g_{\rm eff}}, \qquad Q_{ij}\equiv\sum_a M_a\big(R_iR_j-\tfrac{1}{3}R^2\delta_{ij}\big), \tag{7.74}$$

and is parametrically suppressed but relevant for heavy composites or tight orbits.

Angular distributions and Berry shifts. The dipole pattern $dP/d\Omega\propto|\mathbf{n}\times(\mathbf{n}\times\ddot{\mathbf{d}})|^2$ is modulated by director fields $\hat{\chi}_{1,2}$ through tensor forces (Sec. 7.4.2); Berry phases shift interference fringes in polarized radiation by topological charges q.

7.8.4 Radiative capture and line emission

Capture into shallow dimers. Near threshold, capture $o\to b+\gamma_{\rm spec}$ proceeds dominantly via $E1$:

$$\sigma_{\rm cap}^{(E1)}(E)\;\simeq\;\frac{4\pi^2\alpha_{\rm spec}}{k^2}\,\omega\,\big|\langle\psi_b||\mathbf{d}||\psi_o^{(+)}\rangle\big|^2, \tag{7.75}$$

with $\omega\simeq E+|E_b|$ and continuum wave $\psi_o^{(+)}$ normalized in the medium. FR/Berry projectors enforce $\Delta J=0,\pm1$ and additional selection rules (Sec. 7.7.5), often suppressing S–wave capture for identical fermionic solitons.

Molecular lines and widths. Bound–bound transitions $i\to f$ give lines at $\omega_{if}=E_i-E_f$ with widths

$$\Gamma_{if}^{(\rm spec)}\sim\alpha_{\rm spec}\,\omega_{if}^3\,|\mathbf{d}_{if}|^2, \qquad \Gamma_{if}^{(\rm grav)}\sim G_{\rm eff}\,\omega_{if}^5\,|Q_{ij}^{(if)}|^2, \tag{7.76}$$

consistent with the molecular analysis of Sec. 7.7.6. Alignment with $\hat{\chi}$ splits and polarizes lines in a characteristic way.

7.8.5 Soft memory and asymptotic fields

Spectral memory. Integrated soft spectral flux produces a net shift of the asymptotic gauge potential

$$\Delta A_i(\hat{\mathbf{n}})\;=\;\int_0^\infty d\omega\,\lim_{r\to\infty} r\,F_{ui}(\omega,\hat{\mathbf{n}})\;\propto\;\sum_{\rm hard}Q_i\,\frac{p_i\cdot\varepsilon}{p_i\cdot k}\Big|_{k\to0}, \tag{7.77}$$

an aether analogue of EM memory. It depends on $g_{\rm eff}$ through null geometry at infinity.

Gravitational memory. Similarly, the strain jump

$$\Delta h_{ij}^{(\rm TT)}(\hat{\mathbf{n}})\;=\;\int_0^\infty d\omega\,\lim_{r\to\infty} r\,C_{uiuj}^{(\rm TT)}(\omega,\hat{\mathbf{n}})\;\propto\;\sum_{\rm hard}\frac{p_ip_j}{p\cdot k}\Big|_{k\to0}, \tag{7.78}$$

provides a boundary observable for two–body encounters in the medium; its normalization uses $G_{\rm eff}$. These memory effects are robust soft signatures insensitive to core details.

7.8.6 Implementation protocols and gates

Practical computation.

1. *Soft factors:* Multiply hard amplitudes by Eqs. (7.70)–(7.71); integrate soft phase space with $g_{\rm eff}$ dispersion to obtain $\mathcal{I}_{\rm soft}(\varepsilon)$.
2. *Eikonal + radiation:* Use an imaginary eikonal χ_I from the classical power, Eq. (7.73)/(7.74), to include absorption in $f_{\rm eik}$ (Sec. 7.6.6).
3. *Multipole matrix elements:* Evaluate $\mathbf{d}$, $\boldsymbol{\mu}$, Q_{ij} from single–soliton form factors (Chap. 6) and two–body wavefunctions (Sec. 7.7).
4. *Capture:* Compute Eq. (7.75) with continuum states matched to the full potential including long–range tails (Sec. 7.4); enforce FR/Berry projectors.

Gates (radiation/soft).

- **S1 (KLN/YFS):** Inclusive cross sections are independent (within errors) of IR regulators and detector cutoff ε once exponentiation is applied.
- **S2 (optical):** The imaginary part from soft sums reproduces the elastic inelasticity $\eta_{\ell,j}$ (Sec. 7.3.6) and the imaginary eikonal χ_I (Sec. 7.6.6).
- **S3 (universality):** Soft limits match Eqs. (7.70)–(7.71) independently of short–distance model details.
- **S4 (selection):** FR/Berry forbidden radiative channels vanish at leading soft order; allowed channels obey the predicted ω^3/ω^5 scalings.
- **S5 (energy balance):** Radiated energy computed from spectra equals the loss inferred from the elastic amplitude to stated precision.

Soft radiation reshapes exclusive amplitudes but leaves *inclusive* predictions under tight control. It also provides practical capture mechanisms and boundary observables (memory) unique to the quantum–aether's medium dependence. We now systematize numerical and analytical gates across the chapter and summarize model–specific, falsifiable signatures (Sec. 7.9).

7.9 Topological and Geometric Phase Effects

Topological data inherent to quantum–aether solitons—Finkelstein–Rubinstein (FR) statistics, Berry connections on moduli manifolds, and linking numbers of extended defects (e.g. hopfions)—imprint *geometric phases* on scattering amplitudes. These phases are robust against local deformations of short–range cores; they reorganize the partial–wave basis, enforce selection rules, shift interference fringes, and generate angular "deficits" reminiscent of Aharonov–Bohm (AB) and conical scattering. This section develops a unified treatment and lists falsifiable signatures.

7.9.1 FR statistics and topological projectors

FR sign on configuration space. Let $\mathcal{C}$ denote the configuration space of two identical solitons including internal orientations. A nontrivial fundamental group $\pi_1(\mathcal{C})$ yields a homomorphism $\rho : \pi_1(\mathcal{C}) \to \{\pm 1\}$, the *FR character*. Upon quantization, physical wavefunctions Ψ transform by

$$\Psi[\gamma \cdot \mathbf{X}] = \rho(\gamma)\, \Psi[\mathbf{X}], \qquad \gamma \in \pi_1(\mathcal{C}), \tag{7.79}$$

which enforces a (anti)symmetrization projector $\mathbb{P}_{\mathrm{FR}}$ on channel space (Sec. 7.3.5). Consequences at threshold:

- For fermionic FR character ($\rho = -1$), S–wave is forbidden in the ground elastic block; the first allowed partial wave is p–like.
- For bosonic FR character ($\rho = +1$), S–wave is allowed but can be further constrained by Berry/monopole charge (below).

7.9.2 Berry curvature on moduli: geometric AB effect

Monopole connection and minimal angular momentum. Relative orientation moduli $\mathcal{M}_{\mathrm{rel}}$ generally carry a $U(1)$ Berry connection $\mathcal{A} = \langle u(\lambda)|\, id\, |u(\lambda)\rangle$ with curvature $\mathcal{F} = d\mathcal{A}$. If the relative motion probes a two–sphere submanifold $\mathbb{S}^2_\Omega$ (orientation sphere), quantization in the presence of a *Berry monopole* of charge $q \in \frac{1}{2}\mathbb{Z}$ is realized by *monopole harmonics* $Y^{(q)}_{\ell m}(\Omega)$:

$$\ell \in |q|, |q|+1, |q|+2, \ldots, \qquad m = -\ell, \ldots, +\ell. \tag{7.80}$$

The spectral decomposition of the relative wavefunction becomes

$$\Psi(\mathbf{R}, \Omega) = \sum_{\ell,m} \frac{u_{\ell m}(R)}{R}\, Y^{(q)}_{\ell m}(\Omega), \tag{7.81}$$

which enforces a *partial–wave floor* $\ell_{\min} = |q|$ and shifts centrifugal terms $\ell(\ell+1) \to \ell(\ell+1) - q^2$ in the radial equations. This acts as a geometric AB effect: forward scattering acquires a phase proportional to q, while interference fringes are shifted.

AB–like angular structure. In the Born/eikonal limit, the scattering amplitude factorizes into a core amplitude $f_0(\theta)$ times a geometric phase factor $\Phi_q(\phi)$:

$$f(\theta,\phi) \simeq f_0(\theta)\,\Phi_q(\phi), \qquad \Phi_q(\phi) = \sum_{m\in\mathbb{Z}} c_m\, e^{i(m+q)\phi}, \tag{7.82}$$

producing an azimuthal shift of diffraction maxima by q. For $q \neq 0$ one expects *geometric zeros* at specific (θ,ϕ) protected against smooth deformations of short–range details.

7.9.3 Conical deficits and effective geometry

Defect–induced angular deficits. Topological defects in the aether (e.g. string–like cores or aligned hopfions) induce an *effective* conical geometry for the relative coordinate with deficit angle $\Delta = 2\pi(1-\alpha)$. Geodesic scattering on such a cone produces an interference pattern with angular shift $\sim \Delta$ in the forward region; semiclassically:

$$f_{\rm cone}(\theta) \propto \frac{\sin(\pi(1-\alpha))}{\pi k(1-\cos\theta)} + \text{regular}, \tag{7.83}$$

modulated further by Berry/FR phases [Eq. (7.82)].

Separation of geometric vs. dynamical phases. Write the total phase shift as

$$\delta_{\ell m} = \delta^{\rm dyn}_{\ell m} + \delta^{\rm geom}_{\ell m}(q,\Delta), \qquad \delta^{\rm geom}_{\ell m} = \pi\big(\ell - \sqrt{\ell(\ell+1)-q^2}\big) + \tfrac{m}{2}\Delta + \cdots . \tag{7.84}$$

Only $\delta^{\rm dyn}$ depends on short–range potentials; $\delta^{\rm geom}$ is topological/metric.

7.9.4 Hopfion linking and topological phase shifts

Linking number coupling. For extended, knotted solitons (hopfions), let $\Gamma_{1,2}$ be preimages of selected phases in Φ, with *linking number* $Lk(\Gamma_1,\Gamma_2) = H \in \mathbb{Z}$ (Hopf charge). Effective worldline actions admit a Wess–Zumino–type coupling

$$S_{\rm top} = \kappa_{\rm H}\, H \int dt\, \dot\vartheta(t), \tag{7.85}$$

where ϑ is a relative internal angle. Quantization yields a *topological* contribution to the scattering phase,

$$\Delta_{\rm top} = 2\pi\kappa_{\rm H}\, H, \tag{7.86}$$

which shifts interference fringes and selection rules (e.g. allows only $m \equiv \Delta_{\rm top}/\pi \pmod 2$ in certain inelastic channels). Being proportional to H, this is quantized and robust.

7.9.5 Selection rules and forbidden channels

Unified projector. The combined FR/Berry/topological constraints are implemented by the projector

$$\mathbb{P}_{\rm topo} = \mathbb{P}_{\rm FR}\,\mathbb{P}_q\,\mathbb{P}_H, \tag{7.87}$$

with $\mathbb{P}_q$ enforcing $\ell \geq |q|$ and monopole selection in m, and $\mathbb{P}_H$ enforcing linking–driven parity/charge constraints. Matrix elements violating $\mathbb{P}_{\rm topo}$ vanish to all orders in local interactions; radiative soft factors are suppressed by at least one power of k (Sec. 7.8.1).

7.9.6 Observable signatures and falsifiability

Geometric zeros and fringe shifts. Differential cross sections exhibit orientation–controlled *nulls* at angles fixed by (q,Δ). Scanning the alignment of directors $\hat\chi_{1,2}$ moves these nulls in a parameter–independent way.

Partial–wave floor and threshold scaling. Absent S–wave (FR or $q \neq 0$) implies p–wave (or $\ell = |q|$) leading threshold behavior: $\sigma \sim k^4$ (or $k^{2|q|}$). This is a sharp discriminator versus conventional short–range models.

Topological line splitting. Hopfion linking produces quantized shifts of molecular band heads (Sec. 7.7.5) proportional to H; line splitting patterns are insensitive to short–range core details.

7.9.7 Gates and diagnostics (topological sector)

- **T1 (monopole floor):** Numerics respect $\ell \geq |q|$ and orthonormality of $Y^{(q)}_{\ell m}$.
- **T2 (geometric zeros):** Predicted angular nulls persist under variation of short–range regulators.
- **T3 (split disentanglement):** Extracted phase shifts decompose as Eq. (7.84) within uncertainties; $\delta^{\rm geom}$ matches (q, Δ).
- **T4 (linking quantization):** Observed band/phase shifts scale in integer steps with Hopf charge H.

Topological and geometric phases thus provide parameter–insensitive, falsifiable fingerprints of aether soliton scattering, complementary to dynamical features (potentials, radiation) developed in Secs. 7.4–7.8. They will be crucial in the phenomenological mapping of Sec. 7.11 and in anomaly/consistency tests in Ch. 8.

7.10 Unified Numerical Protocols and Benchmarks

This section specifies end–to–end numerical procedures for the two–body sector: dimensionless rescaling, discretizations for coupled radial equations and full PDE time–evolution, pole searches (resonances), soft/radiative modules, implementation of FR/Berry projectors, and a benchmark suite. The aim is reproducibility with quantified uncertainties and built–in gate checks (Sec. 7.9).

7.10.1 Dimensionless rescaling and computational units

Reference scales. Choose a core scale $R_0 \sim R_{\rm core}$ (from the single–soliton profile), a reduced mass $\mu = M_1 M_2/(M_1+M_2)$, and a transverse propagation speed $c_\perp$ (entering the emergent dispersion). Define

$$\tilde{R} = \frac{R}{R_0}, \qquad \tilde{k} = kR_0, \qquad E_0 = \frac{c_\perp^2}{2\mu R_0^2}, \qquad \tilde{E} = \frac{E}{E_0}. \tag{7.88}$$

For anisotropy (Sec. 7.3.1) split $\mathbf{k} = k_\perp \hat{\mathbf{e}}_\perp + k_\parallel \hat{\chi}$ and use $\tilde{E} = \tilde{k}_\perp^2 + (1 + \delta c/c_\perp^2)\tilde{k}_\parallel^2 + \cdots$.

Dimensionless couplings. For a Yukawa kernel with mass $m_{\rm spec}$,

$$\tilde{m}_{\rm spec} = m_{\rm spec} R_0, \qquad \tilde{\alpha}_{\rm spec} = \mu \alpha_{\rm spec} R_0, \qquad \tilde{G} = \mu G_{\rm eff} M_{\rm ref} R_0, \tag{7.89}$$

with $M_{\rm ref}$ a convenient mass scale (e.g. M_1 or M_2). Report all results in $(\tilde{E}, \tilde{k})$.

7.10.2 Coupled radial equations: discretizations and matching

Equations. After partial–wave projection (Sec. 7.3) the elastic block obeys

$$\left[-\frac{d^2}{dR^2}\,\mathbb{I} + \frac{\boldsymbol{\ell}(\boldsymbol{\ell}+\mathbb{I})}{R^2} + 2\mu\,\mathbf{V}_{\rm cons}(R)\right]\mathbf{u}(R) = k^2\mathbf{u}(R), \tag{7.90}$$

with channel matrices $\mathbf{V}_{\rm cons}$ including central, tensor, anisotropic, and FR/Berry projectors.

Log–derivative (Johnson) method. Discretize $R \in [0, R_{\max}]$ on a mapped grid (denser near the core), propagate the matrix log–derivative $\mathbf{Y}(R) = \mathbf{u}'\mathbf{u}^{-1}$ using fourth–order stepping. Impose regularity at $R \to 0$ via the Frobenius expansion consistent with the minimal allowed ℓ (FR/Berry). At $R = R_{\mathrm{m}} \ll R_{\max}$ switch to asymptotic solutions and perform *matching* to extract the on–shell K–matrix (or S–matrix) in each open channel.

DVR / B–spline alternatives. A Lagrange DVR or B–spline Galerkin basis yields a sparse generalized eigenproblem; for phase shifts use exterior complex scaling or asymptotic matching on an exterior grid.

Boundary conditions and long–range tails. For Yukawa/Coulomb–like pieces, use known asymptotic solutions F_ℓ, G_ℓ; for $1/R^3$ tensors include counterterm contacts at LO and regulate the tail with a smooth cutoff $R \to \sqrt{R^2 + a^2}$; verify L4 (long–range accounting) and stability vs. a.

7.10.3 Full PDE time–evolution (wave packets)

Equation and splitting. Evolve the two–body relative wavefunction $\Psi(\mathbf{R}, t)$ under $i\partial_t \Psi = \left[-\frac{\nabla^2}{2\mu} + V_{\mathrm{cons}}(\mathbf{R}, \hat{\chi}_1, \hat{\chi}_2)\right] \Psi$, on the anisotropic metric by rescaling coordinates (Sec. 7.3.1). Use a Strang split–step FFT or a high–order symplectic integrator.

Absorbing layers. Implement complex absorbing potentials (CAP) or perfectly matched layers (PML) for $R > R_c$ to avoid reflections; calibrate CAP strength η_{CAP} by stability of observables.

Observables. Extract $f(\theta, \phi)$ from asymptotic flux on a spherical shell, then invert for phase shifts. Compute inclusive spectra by adding a soft–radiation module (Sec. 7.8): record power $P_{\mathrm{spec/grav}}(t)$, integrate to obtain the imaginary eikonal proxy and check S2/S5.

7.10.4 Resonances: complex scaling, CAP, stabilization

Complex scaling (CS). Rotate $\mathbf{R} \to \mathbf{R}e^{i\theta}$; discretize in a DVR/B–spline basis. Resonant eigenvalues $E_r - i\Gamma/2$ are θ–stable within a window $\theta \in [\theta_{\min}, \theta_{\max}]$. Gate R6: stability vs. θ and basis size.

Complex absorbing potential (CAP). Add $-i\,\eta_{\mathrm{CAP}}(R - R_c)^m$ beyond R_c; vary $(\eta_{\mathrm{CAP}}, R_c, m)$ and extrapolate to zero absorption (exterior scaling). Cross–check with CS.

Stabilization. Solve in a finite box $R_{\max}$; resonances appear as avoided crossings that *stabilize* under $R_{\max}$ variation. Fit plateaus for (E_r, Γ) and cross–validate with S–matrix analytic continuation.

7.10.5 Radiative modules and imaginary eikonal

Spectral power. Use multipole formulas (Sec. 7.8.3) driven by $\Psi(\mathbf{R}, t)$ or classical trajectories in the eikonal limit; accumulate $\chi_I(b) = [2E]^{-1} \int P_{\mathrm{rad}}\, dt$ along straight lines to build f_{eik} with absorption.

Gravitational power. Compute quadrupole moments $Q_{ij}(t)$, differentiate numerically, and integrate to obtain P_{grav} and memory observables (Sec. 7.8.5). Quote strict upper bounds when resolution is insufficient.

7.10.6 FR/Berry projectors and monopole harmonics

Projector implementation. Construct the projector $\mathbb{P}_{\mathrm{FR/Berry}}$ at the operator level and apply it to basis states before assembling matrices. For Berry charge q, use monopole harmonics $Y_{\ell m}^{(q)}$ as the angular basis; enforce $\ell \geq |q|$.

Numerical orthonormality. Verify $\int d\Omega\, Y^{(q)}_{\ell m} Y^{(q)*}_{\ell' m'} = \delta_{\ell\ell'}\delta_{mm'}$ on the chosen spherical quadrature (e.g. Gauss–Legendre in $\cos\theta$ and uniform in ϕ); monitor orthonormality drift as a systematic error in gates L3/H4/R3.

7.10.7 Benchmark suite

B1: Central Yukawa + hard core. $V(R) = V_0\,\Theta(R_c - R) - (\alpha/R)\,e^{-mR}$. Analytic/semi–analytic phase shifts available. Targets: L1/L2/H2.

B2: Pure tensor tail. $V_{ij}(R) = C_T(R^{-3})(3\hat{R}_i\hat{R}_j - \delta_{ij})$ with smooth core regulator. Targets: modified ERE, tensor renormalization (L4).

B3: Berry monopole. Turn off central potential; enforce $q{=}1/2$ and test minimal $\ell{=}1/2$ floor numerically. Targets: L3/H4 (geometric zeros).

B4: Two–channel Feshbach toy. Open square well + closed well coupled by λ. Tune λ to move a pole across threshold. Targets: R1/R2 pole tracking and $a(\lambda)$ curve.

B5: Radiative capture. Shallow dimer in B1 with small $\alpha_{\rm spec}$; compute $\sigma^{(E1)}_{\rm cap}(E)$ and energy balance. Targets: S1/S5.

7.10.8 Reproducibility: data products and metadata

Mandatory artifacts. Publish (or archive) the following:

1. YAML/JSON of background state $\mathcal{S}_{\rm bg}$: $(\rho, \eta, \nabla T, \hat{\chi})$ and derived $g_{\rm eff}$.
2. Discretization settings: grids, basis orders, CAP/CS parameters, tolerances.
3. Potentials: analytic forms, parameter values, regulator details.
4. Outputs: phase shifts, S–matrix blocks, thresholds (a, r_0) or (A_ℓ, R_ℓ), pole tables, radiated spectra, memory observables; uncertainties with decomposition (stat/sys/model/background).

Versioning and seeds. Record code version, compiler flags, BLAS/LAPACK backend, and random seeds (for stochastic estimators).

7.10.9 Performance and stability tips

Conditioning. Precondition coupled systems by channel–diagonal kinetic operators; scale coordinates to make R–grid nearly uniform in phase advance. Use banded solvers for spline Galerkin matrices.

Time stepping. Prefer Chebyshev or Krylov exponential integrators for stiff propagation; validate against symplectic split–step for energy conservation.

Error control. Convergence $\propto N_r^{-p}$ (radial points) with $p \approx 4$ for Johnson; monitor residuals and enforce gate stability bands before quoting digits.

7.10.10 Gate integration and automated checks

Automated gate suite. Provide a script that, given inputs, runs: (i) low–k ERE fits (L1/L2/L4/L5), (ii) eikonal vs. partial–wave comparison (H1/H2/H5), (iii) Berry/FR nulls (L3/H4), (iv) pole stability scans (R1–R6), and (v) IR safety (S1/S2/S5). A calculation is *accepted* when all applicable gates pass within specified tolerances.

These unified protocols turn the formalism of Secs. 7.1–7.9 into a reproducible computational pipeline. In the next chapter we extend this machinery to many–body phases and effective media where collective phenomena feed back on two–body dynamics.

7.11 Matching to Phenomenology

This section connects the two–body quantum–aether program to measurable observables. We provide a mapping from microscopic parameters and single–soliton inputs (Ch. 6) to elastic/inelastic cross sections, resonance spectroscopy, radiative tails, and external–probe form factors. We also outline inversion pipelines (from data to parameters), null tests, and constraints in collider and astrophysical contexts. The aim is a reproducible path from the soliton EFT of Secs. 7.2–7.6 to numbers that can be confronted with experiment.

7.11.1 Microscopic inputs and effective couplings

Single–soliton inputs (from Ch. 6). For each species $\mathcal{Q}$: mass M, core scale $R_{\rm core}$, FR character $\rho = \pm 1$, Berry monopole charge $q \in \frac{1}{2}\mathbb{Z}$ on relevant moduli submanifolds, and external form factors (spectral $U(1)$, internal non–Abelian remnants, emergent gravity):

$$\{Q_{\rm spec}(Q^2),\, \boldsymbol{\mu}_{\rm spec}(Q^2),\, G_E(Q^2),\, G_M(Q^2),\, Q^{ij}_{\rm grav}(Q^2), \dots\}.$$

Two–body kernels (from Secs. 7.4, 7.5). Central/core piece $V_0(R)$, spectral Yukawa V_Y, tensor/aniso V_T, plus long–range emergent gravitational tail; FR/Berry projectors fix the admissible channel space. The minimal matching set:

$$\theta_{\rm micro} = \{\alpha_{\rm spec},\, m_{\rm spec},\, C_T,\, \text{core params},\, G_{\rm eff}(\rho),\, \dots\}.$$

7.11.2 Elastic and total cross sections

Partial–wave reconstruction. With eigenphase shifts $\delta_{\ell,j}(k)$ and inelasticities $\eta_{\ell,j}$ in an elastic block,

$$\sigma_{\rm el}(k) = \frac{4\pi}{k^2} \sum_{\ell,j} (2\ell{+}1)\, \eta_{\ell,j}^2 \sin^2 \delta_{\ell,j}, \tag{7.91}$$

$$\sigma_{\rm tot}(k) = \frac{4\pi}{k^2} \sum_{\ell,j} (2\ell{+}1)\, \frac{1 - \eta_{\ell,j} \cos 2\delta_{\ell,j}}{2}, \tag{7.92}$$

consistent with the optical theorem in coupled channels (Sec. 7.3.4). FR/Berry selection implies a *partial–wave floor* (S–wave absent if $\rho = -1$ or $q \neq 0$), yielding threshold scaling $\sigma \sim k^{2\ell_{\rm min}}$.

Threshold parameters. Near threshold (modified ERE, Sec. 7.5), the leading channel is characterized by (a, r_0) (or (A_ℓ, R_ℓ) for $\ell > 0$). These map to $\{\alpha_{\rm spec}, m_{\rm spec}, C_T\}$ via controlled windows where long–range pieces are analytically accounted for (gate L4).

7.11.3 Resonance spectroscopy and line shapes

Pole catalogues. Resonances correspond to S–matrix poles $E_r - i\Gamma/2$ on appropriate Riemann sheets (Sec. 7.7). We report (J^P) from channel quantum numbers (including FR/Berry constraints), residues (couplings), and trajectories under environmental controls λ (density, curvature, alignment).

Lines and coupled thresholds. Isolated narrow poles admit Breit–Wigner fits; near an inelastic threshold we use Flatté forms, cf. Eqs. (7.57)–(7.58). Tensor anisotropy produces orientation–dependent splittings; Berry charge q shifts/filters allowed (ℓ, m) in decays.

7.11.4 Radiative tails and inclusive observables

Soft factors and inclusive finiteness. Spectral/gravitational bremsstrahlung factorizes at leading order, Eqs. (7.70)–(7.71). Inclusive cross sections $d\sigma_{\rm inc}$ are IR–safe after exponentiation (Sec. 7.8.2); fitted inelasticities $\eta_{\ell,j}$ must match the imaginary eikonal from classical power (gate S2/S5).

Capture signatures. Radiative capture into shallow dimers yields lines with widths $\Gamma^{(\mathrm{spec})} \sim \alpha_{\mathrm{spec}}\,\omega^3|\mathbf{d}_{if}|^2$ and $\Gamma^{(\mathrm{grav})} \sim G_{\mathrm{eff}}\,\omega^5|Q_{ij}^{(if)}|^2$ (Sec. 7.8.4); alignment with $\hat{\chi}$ generates characteristic polarization patterns.

7.11.5 External probes and form factors

Single–soliton & composite form factors. Elastic and transition form factors computed in Ch. 6 enter scattering via Born terms and radiative matrix elements. For composites (molecular states), multipoles follow from two–body wavefunctions and match single–soliton moments consistently (gate R5).

Identification hypotheses. In scenarios where spectral $U(1)$ coincides with electromagnetism at low energies, the matching reduces to standard EM form factors; otherwise, spectral radiation resides in a hidden sector and is constrained by missing–energy searches and stellar cooling bounds (to be specified model–by–model).

7.11.6 Inversion pipelines (data → parameters)

Pipeline A: Threshold first. Fit (a, r_0) (or (A_ℓ, R_ℓ)) in windows passing L1/L2; map to $(\alpha_{\mathrm{spec}}, m_{\mathrm{spec}})$ using analytic long–range corrections; validate against a shallow dimer E_b (gate L5).

Pipeline B: Poles first. Extract (E_r, Γ) for leading resonances; perform Flatté/Breit–Wigner fits to get channel couplings; invert for core and tensor parameters $\{\mathrm{core}, C_T\}$; cross–check with eikonal small–angle data (gate H2).

Pipeline C: Soft complement. Use inclusive spectra and memory observables to fix α_{spec} (and G_{eff} for heavy systems); these constrain imaginary eikonal χ_I and inelasticities.

7.11.7 Null tests and robust signatures

- **Absence of S–wave at threshold** in fermionic FR or $q \neq 0$ sectors: $\sigma \sim k^{2\ell_{\min}}$ with $\ell_{\min} = \max(0, |q|)$.
- **Alignment–controlled geometric zeros** in $d\sigma/d\Omega(\theta, \phi)$ that move predictably with $\hat{\chi}_{1,2}$ and are insensitive to regulator details.
- **Aether Feshbach tuning:** environment–controlled scattering length $a(\lambda) = a_{\mathrm{bg}}(1 - \Delta/(\lambda - \lambda_0))$ with pole motion on Riemann sheets.
- **Berry/FR line selection:** forbidden radiative channels vanish at leading soft order.

7.11.8 Constraints and windows

Collider and fixed–target. Cross sections and resonance widths must remain within experimental bounds for candidate channels; hidden spectral radiation contributes to missing energy and soft tails—to be assessed with detector thresholds and acceptances (Sec. 7.10 protocols).

Astrophysical/astro–cosmological (optional). If solitons constitute dark aether components, self–interaction per unit mass σ/m is bounded in structure formation; long–range spectral forces are constrained by halo shapes and cluster mergers. These constraints select viable $(\alpha_{\mathrm{spec}}, m_{\mathrm{spec}})$ windows.

7.11.9 Reporting standards and uncertainty budgets

Deliverables. Publish: (i) threshold parameters with L1/L2/L4/L5 checks, (ii) pole tables with R1–R6 gates, (iii) eikonal vs. partial–wave comparisons (H2), (iv) inclusive soft spectra and energy balance (S1/S5), (v) alignment scans showing geometric zeros. Quote uncertainties (statistical, systematic, model, background) as in Sec. 7.10.8.

This matching framework makes the two–body quantum–aether program testable against data: it specifies how to go from soliton microphysics to observables and back, with robust null tests, environmental tuning curves, and controlled uncertainty propagation.

7.12 Consistency Gates for Scattering

This section specifies *consistency gates* that every two–body calculation in the quantum–aether framework must pass to be considered publishable. The gates probe five pillars: **U1** unitarity, **U2** causality and analyticity, **U3** gauge/soft consistency, **U4** FR/Berry topological selection, and **U5** numerical convergence and stability. Each gate comes with quantitative diagnostics, tolerance bands, and practical remedies.

7.12.1 Gate taxonomy and notations

Objects under test. Channel–resolved S– and K–matrices in the elastic block (Sec. 7.3), threshold parameters (a, r_0) or (A_ℓ, R_ℓ) (Sec. 7.5), eikonal amplitude $f_{\rm eik}$ (Sec. 7.6), pole catalogues (Sec. 7.7), and soft–inclusive observables (Sec. 7.8). All kinematics and fluxes are computed with the emergent metric $g_{\rm eff}$.

Norms and metrics. We monitor matrix deviations with Frobenius norms $\|\cdot\|_{\rm F}$ and spectral norms $\|\cdot\|_2$. Uncertainties are quoted as statistical (solver), systematic (regulator/basis), model (EFT truncation), and background (choice of $\mathcal{S}_{\rm bg}$), cf. Sec. 7.10.8.

7.12.2 U1: Unitarity in coupled channels

Definition. For each open–channel subspace at fixed (k, quantum numbers),

$$S^\dagger S = \mathbb{I}, \qquad S = \mathbb{I} + 2i\,T, \qquad \mathrm{Im}\,T = T^\dagger T. \tag{7.93}$$

In partial waves, $S_\ell = \mathbb{V}_\ell\,\mathrm{diag}(\eta_{\ell,j} e^{2i\delta_{\ell,j}})\,\mathbb{V}_\ell^\dagger$ with $0 \le \eta_{\ell,j} \le 1$.

Diagnostics.

$$\Delta_{\rm U1}(k) \equiv \|S^\dagger S - \mathbb{I}\|_{\rm F}, \tag{7.94}$$

$$\Delta_{\rm opt}(k) \equiv \left| \sigma_{\rm tot}(k) - \frac{4\pi}{k}\,\mathrm{Im}\,f(0;k) \right|, \tag{7.95}$$

where $f(0;k)$ is the forward amplitude reconstructed from the same S (Sec. 7.3.4).

Acceptance bands. Reference tier: $\Delta_{\rm U1} \le 10^{-8}$, $\Delta_{\rm opt}/\sigma_{\rm tot} \le 10^{-6}$; Production: 10^{-6} and 10^{-4}; Exploratory: 10^{-4} and 10^{-3}.

Common fixes. Increase $L_{\max}$, refine matching radius, enforce FR/Berry projector at the *operator* level, and align absorbing layers with group velocities under $g_{\rm eff}$.

7.12.3 U2: Causality and analyticity

Wigner time delay. For each eigenchannel,

$$\tau_{\ell,j}(E) \equiv 2\,\frac{d\delta_{\ell,j}}{dE} \;\ge\; -\frac{2R_{\rm int}}{v_g(E)}, \qquad v_g = \frac{\partial\omega}{\partial k} \le 1, \tag{7.96}$$

with $R_{\rm int}$ the interaction range (from potential tails, Sec. 7.4) and v_g the group speed under $g_{\rm eff}$.

Analytic structure. $S(k)$ must be analytic in the upper half complex k–plane with poles only on the physical Riemann sheets as classified in Sec. 7.7.2. We numerically check edge–of–the–wedge stability by fitting $\delta_{\ell,j}(k)$ to rational/Padé forms and confirming pole locations are sheet–consistent.

Diagnostics and bands. Violations: any $v_g > 1$; any $\tau_{\ell,j}$ below the Wigner bound outside uncertainties; any pole jumping sheets under small regulator variations. Acceptance: none of the above.

Remedies. Reduce long–range mis-modeling (promote tensor counterterms), enforce hyperbolicity constraints on dispersion parameters of g_{eff}, and widen analytic continuation windows.

7.12.4 U3: Gauge/soft consistency

Ward identities. For spectral $U(1)$ emission,

$$k_\mu \, \mathcal{M}^{\mu}_{\text{spec}}(k; \{p_i\}) = 0, \qquad \lim_{k\to 0} \mathcal{M}^{(\text{spec})}_{n+1} = \left[g_{\text{spec}} \sum_i Q_i \frac{p_i \cdot \varepsilon}{p_i \cdot k} \right] \mathcal{M}_n, \tag{7.97}$$

and similarly for emergent gravity (Sec. 7.8.1).

Inclusive IR finiteness. After YFS/Bloch–Nordsieck exponentiation (Sec. 7.8.2), inclusive observables are cutoff–stable:

$$\frac{\partial}{\partial \varepsilon} \, d\sigma_{\text{inc}}(k; \varepsilon) = \mathcal{O}(\varepsilon^0), \qquad \varepsilon \to 0^+. \tag{7.98}$$

Diagnostics. Numerically evaluate $k_\mu \mathcal{M}^\mu$ at multiple small k; fit the leading $1/k$ soft factor and compare with the charge sum rule. Monitor the slope of $d\sigma_{\text{inc}}$ vs ε.

Bands and fixes. Acceptance: relative Ward residuals $\leq 10^{-6}$ (reference) or $\leq 10^{-4}$ (production). If violated: improve current conservation in discretization, refine polarization bases, and enlarge the soft phase–space sampling.

7.12.5 U4: FR/Berry topological selection

Projectors and floors. Implement $\mathbb{P}_{\text{FR}}$ and Berry monopole harmonics $Y^{(q)}_{\ell m}$ (Sec. 7.9). Enforce the minimal partial wave $\ell_{\min} = \max(0, |q|)$ and the correct (anti)symmetry for identical solitons with FR character.

Diagnostics.

$$\Delta_{\text{floor}} \equiv \sum_{\ell < \ell_{\min}} \sum_j (2\ell{+}1) \, |S_{\ell,j} - 1|^2, \qquad \Delta_{\text{ortho}} \equiv \max_{n \neq m} \left| \langle Y^{(q)}_n | Y^{(q)}_m \rangle - \delta_{nm} \right|. \tag{7.99}$$

We also search for predicted geometric zeros in $d\sigma/d\Omega$ at fixed alignments.

Bands and fixes. Acceptance: $\Delta_{\text{floor}} \leq 10^{-10}$, $\Delta_{\text{ortho}} \leq 10^{-8}$ (reference). If violated: enforce projectors at operator assembly, refine angular quadratures, and increase basis order for monopole harmonics.

7.12.6 U5: Numerical convergence and stability

Mesh and basis convergence. Demonstrate power–law or exponential convergence under:

$$N_r \to N_r + \Delta N, \quad L_{\max} \to L_{\max} + \Delta L, \quad R_{\max} \to R_{\max} + \Delta R, \tag{7.100}$$

and, for CAP/complex scaling, stability under $(\eta_{\text{CAP}}, R_c, m)$ or scaling angle θ variations (Sec. 7.10.4). Quote Richardson or slope–fits for observed rates.

Cross–solvers. Where possible, cross–validate radial log–derivative (Johnson) against DVR/B–spline and time–domain wave–packet extractions (Sec. 7.10.3).

Acceptance bands. Relative changes below 10^{-3} (exploratory), 10^{-4} (production), 10^{-5} (reference) for key observables: (a, r_0) or (A_ℓ, R_ℓ), $\delta_{\ell,j}$ on a test grid, (E_r, Γ), and $d\sigma/d\Omega$ at representative angles.

Remedies. Adaptive nonuniform grids, higher–order propagators, improved matching formulas, and preconditioned linear algebra backends.

7.12.7 Consolidated acceptance and reporting

Pass matrix. A calculation is *accepted* at a given fidelity tier if all applicable gates U1–U5 pass within their bands. Report a compact scoreboard:

Gate	Metric	Ref.	Prod.	Expl.
U1 Unitarity	Δ_{U1}, $\Delta_{\mathrm{opt}}/\sigma_{\mathrm{tot}}$	10^{-8}, 10^{-6}	10^{-6}, 10^{-4}	10^{-4}, 10^{-3}
U2 Causality	$v_g \le 1$, Wigner bound	strict	strict	strict
U3 Gauge/Soft	Ward resid., IR slope	10^{-6}, flat	10^{-4}, flat	10^{-3}, near-flat
U4 FR/Berry	Δ_{floor}, Δ_{ortho}	10^{-10}, 10^{-8}	10^{-8}, 10^{-6}	10^{-6}, 10^{-5}
U5 Numerics	Δ obs. under refinements	10^{-5}	10^{-4}	10^{-3}

Reproducibility bundle. Attach the YAML/JSON artifacts of Sec. 7.10.8 and the gate log with random seeds, solver versions, and hardware notes.

7.12.8 Automated gate runner (pseudo-code)

```
inputs: S(k), phase_shifts, inelasticities, f(0), poles, settings, soft_module
for each k in test_grid:
  # U1
  U1a = frobenius_norm(S^\dagger S - I)
  U1b = abs(sigma_tot - (4*pi/k)*Im f(0))
  assert U1a < tol_U1 and U1b/sigma_tot < tol_opt

  # U2
  vg = d omega / d k  (from g_eff dispersion)
  assert vg <= 1 + eps
  tau = 2 * d delta / dE
  assert tau >= -2*R_int/vg - err
  check_sheet_stability(poles, settings)

  # U3
  ward = max_k_small |k_mu M^mu|
  slope = d sigma_inc / d epsilon
  assert ward < tol_ward and |slope| < tol_IR

  # U4
  floor = sum_{l<l_min} ||S_l - I||^2
  ortho = max offdiag overlap(Y^{(q)})
  assert floor < tol_floor and ortho < tol_ortho

  # U5
  assert convergence_under({Nr,Lmax,Rmax,theta,CAP_params}) < tol_num
```

7.12.9 Failure modes and triage

- **U1 fails:** Increase $L_{\max}$; refine asymptotic matching; ensure projector enforcement; rebuild flux normalization under g_{eff}.

- **U2 fails:** Refit long–range tails and dispersion; constrain $c_\perp^2 > 0$, $|\delta c| < c_\perp^2$; tighten analytic continuation grids.
- **U3 fails:** Improve current–conserving discretizations; enlarge soft phase–space sampling; verify polarization orthonormality.
- **U4 fails:** Use monopole harmonics basis; increase angular quadrature order; audit FR character from single–soliton quantization (Ch. 6).
- **U5 fails:** Switch to higher–order radial steppers; adopt B–spline Galerkin; precondition linear systems; extend boxes and CAP regions.

These gates operationalize the theoretical requirements of Secs. 7.1–7.11 into quantitative checks. Passing them guarantees that reported phase shifts, resonance spectra, and radiative observables are physically consistent and numerically reliable within stated uncertainties.

7.13 Summary and Outlook

This chapter built a complete two–body program for quantum–aether solitons, from effective descriptions on worldlines/worldvolumes to elastic and inelastic observables, resonances, and radiative processes, with robust numerical protocols and consistency gates. We summarize the main outputs and delineate the next steps toward many–body phases and emergent gauge dynamics.

7.13.1 What has been established

- **EFT backbone** (Secs. 7.2–7.4): a two–body SAEFT with controlled power counting in v/c_{eff} and multipoles, including conservative kernels (central, Yukawa–like spectral exchange, tensor/anisotropic pieces) and consistent dissipative operators (soft spectral/gravitational emission).
- **Kinematics and observables** (Sec. 7.3): flux normalization and unitarity in anisotropic media g_{eff}, partial–wave expansion in coupled channels, and FR/Berry projectors that reconfigure threshold physics.
- **Low– and high–energy regimes** (Secs. 7.5–7.6): modified effective–range expansion with long–range/tensor accounting; eikonal scattering on curved moduli with anisotropy and Berry phases.
- **Resonances and composites** (Sec. 7.7): sheet taxonomy, pole finding, shape vs. Feshbach resonances, and rotor–like molecular spectra in orientation minima.
- **Radiation and inclusivity** (Sec. 7.8): soft factors and KLN/YFS cancellations in the soliton setting; capture lines and memory observables matched to classical power (imaginary eikonal).
- **Topological/Geometric fingerprints** (Sec. 7.9): FR statistics and Berry monopoles enforce a partial–wave floor and geometric zeros; conical/AB–like angular structures and Hopf linking produce quantized phase shifts.
- **Numerical protocols and gates** (Secs. 7.10–7.12): end–to–end solvers (radial, DVR/B–spline, time domain), resonance methods (CS/CAP/stabilization), reproducibility bundles, and gate suite U1–U5 (unitarity, causality/analyticity, gauge/soft, FR/Berry, numerics).

7.13.2 Falsifiable signatures

1. **Partial–wave floor & threshold scaling:** absent S–wave (fermionic FR or nonzero Berry charge) $\Rightarrow \sigma(k) \propto k^{2\ell_{\min}}$ with $\ell_{\min} = \max(0, |q|)$, robust under short–range variations.
2. **Alignment–controlled geometric zeros:** parameter–insensitive nulls in $d\sigma/d\Omega(\theta, \phi)$ that move predictably with director alignment $(\hat{\chi}_1, \hat{\chi}_2)$.

3. **Aether Feshbach tuning:** environment–controlled scattering length $a(\lambda) = a_{\rm bg}\Big(1 - \frac{\Delta}{\lambda - \lambda_0}\Big)$ with correlated pole motion on Riemann sheets.
4. **Radiative capture & memory:** soft–inclusive lines and spectral/gravitational memory patterns that reflect the null geometry of $g_{\rm eff}$.
5. **Topological line splitting:** Hopf linking $H \in \mathbb{Z}$ induces quantized shifts of molecular band heads and selection rules.

7.13.3 From data to parameters: practical pipelines

Threshold–first: fit (a, r_0) (or (A_ℓ, R_ℓ)) in vetted windows, map to $(\alpha_{\rm spec}, m_{\rm spec})$ including long–range corrections; validate against shallow dimer E_b.

Poles–first: extract (E_r, Γ) and residues via CS/CAP; invert for short–range/tensor parameters and cross–check with small–angle eikonal data.

Soft complement: use inclusive soft spectra and memory to constrain $\alpha_{\rm spec}$ (and $G_{\rm eff}$ in heavy systems); ensure optical/energy balance via the imaginary eikonal.

7.13.4 Interfaces to other chapters

- **To Ch. 8 (Emergent Gauge Dynamics):** the gauge/soft gates and Berry/FR structures here constrain anomaly inflow, WZW/CS terms, and matching to low–energy gauge sectors.
- **To Ch. 6 (Solitons as Particles):** external form factors, FR characters, and Berry charges fed into the SAEFT are determined by single–soliton quantization and must remain consistent under renormalization.
- **To Ch. 5 (Phenomenology & Calibration):** the pipelines of Sec. 7.13.3 supply the forward map (parameters → observables) and the inverse map (data → parameters) used in global fits.

7.13.5 Open problems and next steps

1. **Systematics of many–channel coupling:** extend sheet taxonomy and stabilization to dense channel networks with overlapping thresholds and medium–induced mode conversion.
2. **Strong anisotropy regimes:** beyond small δc, develop full nonperturbative eikonal on generic $g_{\rm eff}$ and quantify rainbow/glory morphologies.
3. **Nonperturbative radiation reaction:** controlled resummations of dissipative self–forces on moduli; interplay with capture and bound–state formation.
4. **Topological transport:** quantify AB/conical signatures in disordered backgrounds and the robustness of geometric zeros under noise and inhomogeneity.
5. **Global fits:** integrate the numerical gate suite with phenomenological scans to deliver priors/posteriors for $(\alpha_{\rm spec}, m_{\rm spec}, C_T, \ldots)$ under collider and astrophysical constraints.

7.13.6 Executive checklist

- *Deliverables ready for comparison:* phase shifts, total/elastic cross sections, resonance tables with sheets/residues, capture rates, inclusive soft spectra, memory observables.
- *Quality control:* all U1–U5 gates passed at the declared fidelity tier; reproducibility bundle (settings, seeds, versions, artifacts) archived.
- *Signatures to target next:* geometric zeros vs. alignment; aether Feshbach tuning curves; partial–wave floor at threshold; topological line splitting in molecular bands.

With this chapter, the two–body sector of the quantum–aether achieves a mature, falsifiable, and reproducible status. The subsequent chapters leverage these results to build emergent gauge dynamics, many–body phases, and a full phenomenological interface.

Appendix 7.A: Anisotropic Eikonal and Deflection on Curved Moduli Spaces

This appendix derives the anisotropic eikonal phase and the associated deflection functions for two–body scattering in the quantum–aether, when the relative motion propagates on a *curved moduli manifold* $(\mathcal{M}_Q, G)$ and through an *anisotropic emergent geometry* $g_{\rm eff}$ (Sec. 7.6). We obtain (i) the impact–parameter representation with curvature corrections, (ii) transport equations and the Van Vleck determinant on $(\mathcal{M}_Q, G)$, (iii) stationary–phase evaluation of the far–field amplitude, and (iv) uniform approximations near rainbow and glory angles. Berry/FR geometric phases enter as additive contributions to the eikonal phase.

A.1 Kinematics and dispersion in anisotropic media

Effective dispersion. In the homogeneous background $\mathcal{S}_{\rm bg}$, small fluctuations propagate with anisotropic dispersion (Sec. 7.3.1):

$$\omega^2 = c_\perp^2\,|\mathbf{k}_\perp|^2 + c_\parallel^2\,k_\parallel^2 + m_{\rm eff}^2, \qquad c_\parallel^2 = c_\perp^2 + \delta c, \quad |\delta c| \ll c_\perp^2, \tag{7.101}$$

where the parallel component is defined with respect to the director $\hat{\chi}$ of the background. We work at fixed incident energy $E = \omega$ and define the group velocity $v_g = \partial\omega/\partial k \le 1$ (causality gate U2).

Relative geometry and moduli. The relative coordinate $\mathbf{R}$ evolves effectively on a curved moduli space $(\mathcal{M}_Q, G)$ when slow orientation and internal moduli couple to the translational motion. In the eikonal regime ($kb \gg 1$, small angles), the spatial projection can be treated as geodesic motion on $(\mathcal{M}_Q, G)$ weakly perturbed by the conservative potential $V_{\rm cons}$ (Sec. 7.4).

A.2 Eikonal ansatz and transport on $(\mathcal{M}_Q, G)$

Phase–amplitude decomposition. Let $\Psi(\mathbf{R}, \lambda) = A(\mathbf{R}, \lambda)\, e^{iS(\mathbf{R},\lambda)}$ solve the scalar wave equation (or the leading component of the coupled system) on $(\mathcal{M}_Q, G)$ with anisotropic kinetic operator $\hat{\mathcal{K}}(g_{\rm eff})$ and potential $V_{\rm cons}$. To leading order in gradients,

$$G^{ij}(\mathbf{R})\,\partial_i S\,\partial_j S \;=\; k^2 - 2\mu\,V_{\rm cons}(\mathbf{R}) \;\equiv\; k_{\rm loc}^2(\mathbf{R}), \tag{7.102}$$

which is the Hamilton–Jacobi equation on $(\mathcal{M}_Q, G)$. The rays follow geodesics γ of the *optical metric* with index $n(\mathbf{R}) = k/k_{\rm loc}$.

Transport equation and Van Vleck determinant. Next order yields a transport equation for A:

$$\nabla_i\left(A^2\,v^i\right) = 0, \qquad v^i \equiv G^{ij}\partial_j S, \tag{7.103}$$

whose solution along a ray segment γ can be expressed via the Van Vleck–Morette determinant $\Delta(\gamma)$,

$$A(\gamma) \;=\; \frac{A_0}{\sqrt{\Delta(\gamma)}}, \qquad \Delta(\gamma) = \det\left[-\frac{\partial^2\sigma(\mathbf{R},\mathbf{R}')}{\partial R^i\,\partial R'^j}\right], \tag{7.104}$$

with σ the Synge world function on $(\mathcal{M}_Q, G)$. Curvature focusses/defocusses the beam through $\Delta^{-1/2}$, entering directly the scattering amplitude prefactor.

A.3 Impact–parameter representation with curvature

Anisotropic impact parameter. Choose an incident wavevector $\mathbf{k}$ and define an *anisotropic* impact parameter $\mathbf{b}$ in the plane transverse to $\mathbf{k}$ using the optical metric. The eikonal amplitude generalizes the flat–space form (cf. Sec. 7.6):

$$f(\theta,\phi) \;=\; \frac{k}{2\pi i}\int d^2b\,\sqrt{\det g_\perp(\mathbf{b})}\,\exp\Big\{i\mathbf{q}\cdot\mathbf{b} + i\,\chi(\mathbf{b};\hat{\chi}_1,\hat{\chi}_2)\Big\}\,\mathcal{J}^{1/2}(\mathbf{b}), \tag{7.105}$$

where $\mathbf{q} = \mathbf{k}_{\rm out} - \mathbf{k}_{\rm in}$, $g_\perp$ is the induced metric on the impact plane, and $\mathcal{J}^{1/2}$ collects the Van Vleck factor transported from $(\mathcal{M}_Q, G)$ to the far field.

Eikonal phase with geometric additions. To leading order,

$$\chi(\mathbf{b}) \;=\; -\frac{\mu}{k}\int_{-\infty}^{+\infty} ds\; V_{\rm cons}\big(\mathbf{R}_{\rm geo}(s;\mathbf{b})\big) \;+\; \chi_{\rm geom}(\mathbf{b}), \tag{7.106}$$

where $\mathbf{R}_{\rm geo}(s;\mathbf{b})$ is the unperturbed geodesic parameterized by affine s, and $\chi_{\rm geom}$ encodes *geometric phases*:

$$\chi_{\rm geom}(\mathbf{b}) \;=\; q\oint_{\gamma(\mathbf{b})}\mathcal{A} \;+\; \frac{1}{2}\int_{\Sigma(\mathbf{b})}\mathcal{R}_{ijkl}\,\Xi^{ij}\,d\Sigma^{kl} \;+\; \cdots, \tag{7.107}$$

with the first term the Berry phase (q the monopole charge, $\mathcal{A}$ the Berry 1–form on moduli), and the second a curvature correction written schematically via the Riemann tensor $\mathcal{R}$ of G and a bivector Ξ^{ij} determined by the ray tube; higher terms involve torsion/contortion if present.

A.4 Deflection function and stationary phase

Deflection from eikonal phase. Critical points of the phase $\Phi(\mathbf{b}) = \mathbf{q}\cdot\mathbf{b} + \chi(\mathbf{b})$ solve $\nabla_{\mathbf{b}}\Phi = 0$, which defines the *deflection map* $\Theta(\mathbf{b})$ through

$$\frac{\partial\chi}{\partial\mathbf{b}} \;=\; -\mathbf{q}(\theta,\phi), \qquad \Theta(\mathbf{b}) \equiv \frac{|\mathbf{q}|}{k}\,, \tag{7.108}$$

with anisotropic Jacobian from $g_\perp$. For axial symmetry about $\hat{\chi}$ one may reduce to $b = |\mathbf{b}|$ and a single deflection function $\Theta(b)$.

Stationary–phase amplitude. Let $b_\star(\theta,\phi)$ be a nondegenerate stationary point. Then

$$f(\theta,\phi) \;\simeq\; \sum_{b_\star}\frac{k}{i}\,\frac{\exp\left\{i\Phi(b_\star) + i\frac{\pi}{4}\,\mathrm{sgn}\,\Phi''(b_\star)\right\}}{\sqrt{2\pi\,|\det\nabla_{\mathbf{b}}\nabla_{\mathbf{b}}\Phi|_{b_\star}}}\;\mathcal{J}^{1/2}(b_\star), \tag{7.109}$$

with Φ'' the Hessian on the impact plane. Multiple solutions sum coherently and generate interference (rainbows).

A.5 Rainbow scattering: uniform approximation

Catastrophe condition. A *rainbow* occurs when two stationary points coalesce, i.e.

$$\left.\frac{d\Theta}{db}\right|_{b=b_r} = 0, \qquad \left.\frac{d^2\Theta}{db^2}\right|_{b=b_r} \neq 0. \tag{7.110}$$

Near the rainbow angle $\theta_r = \Theta(b_r)$, the standard stationary–phase formula diverges and must be replaced by a uniform Airy approximation.

Airy uniformization. Expand $\Phi(b)$ about b_r to cubic order and change variables to $x \propto (b - b_r)$, yielding

$$f(\theta) \;\simeq\; \mathcal{C}_r\,\mathrm{Ai}(\zeta)\;e^{i\varphi_r}, \qquad \zeta \propto \frac{\theta-\theta_r}{\left|d^2\Theta/db^2\right|^{2/3}}, \tag{7.111}$$

where $\mathcal{C}_r$ includes $\mathcal{J}^{1/2}$ and metric/Jacobian factors at b_r, and φ_r contains $\Phi(b_r)$ plus Maslov/Berry contributions. The differential cross section exhibits the familiar Airy oscillations with period controlled by the curvature of Θ at b_r.

A.6 Glory scattering: backward enhancement

Glory condition. A *glory* arises when a continuum of impact parameters focuses into the backward direction $\theta \approx \pi$ (or forward, depending on geometry). In axisymmetry one obtains (for scalar waves)

$$\left.\frac{d\sigma}{d\Omega}\right|_{\theta\approx\pi} \;\simeq\; \mathcal{G}\,\left|J_0\big(kb_g\sin\theta\big)\right|^2, \tag{7.112}$$

where b_g satisfies $\Theta(b_g) = \pi$ and $\mathcal{G}$ carries the Van Vleck and metric factors. With Berry/FR phases, the Bessel order is shifted $J_0 \to J_\nu$ with $\nu = q + \nu_{\rm FR}$, producing *topological* modulations (Sec. 7.9).

A.7 Role of Berry and FR phases

Additive geometric phase. Berry connection along the ray on moduli adds

$$\chi_{\mathrm{Berry}}(\mathbf{b}) = \int_{\gamma(\mathbf{b})} \mathcal{A} \cdot d\lambda = \int_{\Sigma(\mathbf{b})} \mathcal{F}, \tag{7.113}$$

shifting the stationary–phase condition and interference pattern. FR statistics are implemented as projectors on the partial–wave content; in the eikonal limit they appear as global signs under 2π exchanges and as selection rules forbidding certain stationary branches.

A.8 Validity, corrections, and gating

Validity domain. The derivation assumes $kb \gg 1$, small deflection angles, weak curvature over a wavelength, and slowly varying V_{cons} along rays. Corrections arise from: (i) NLO transport (amplitude and polarization mixing), (ii) higher–order curvature/torsion couplings in χ_{geom}, (iii) breakdown near caustics (handled by uniform approximations).

Consistency gates. Eikonal predictions must pass:

- H1/H2 (Sec. 7.6): agreement with partial–wave sums at small angles.
- T1/T2 (Sec. 7.9): monopole floor and geometric zeros.
- U2: subluminal v_g and Wigner bounds with the same g_{eff} used in kinematics.

A.9 Practical recipe

1. Build $g_\perp$ and the optical metric from $(g_{\mathrm{eff}}, \mathcal{S}_{\mathrm{bg}})$; construct geodesics $\mathbf{R}_{\mathrm{geo}}(s;\mathbf{b})$ on $(\mathcal{M}_Q, G)$.
2. Compute $\chi(\mathbf{b})$ by Eq. (7.106), adding χ_{Berry} from Eq. (7.113) and leading curvature correction in Eq. (7.107).
3. Find stationary points $b_\star(\theta,\phi)$; evaluate f via Eq. (7.109).
4. If $d\Theta/db|_{b_r} = 0$, switch to the Airy uniformization Eq. (7.111); if $\Theta(b_g) = \pi$, use the glory formula Eq. (7.112).
5. Cross–check with partial–wave numerics (Sec. 7.10) and record gate metrics (Sec. 7.12).

These results supply the analytic backbone for anisotropic, curvature–aware eikonal predictions used in Secs. 7.6 and 7.10, and clarify how Berry/FR topology and moduli curvature enter scattering observables beyond short–range core details.

Appendix 7.B: Coupled–Channel Formalism with FR/Berry Selection

This appendix provides a self–contained construction of the coupled–channel (CC) framework for two–soliton scattering in the quantum–aether, including the effects of Finkelstein–Rubinstein (FR) statistics and Berry monopole charges on relative–orientation moduli. We define the channel Hilbert space, assemble potential matrices with tensor/aniso pieces, derive the CC radial equations and asymptotics, and specify numerically stable extraction of K– and S–matrices consistent with the selection rules of Sec. 7.9 and the gates of Sec. 7.12.

B.1 Channel space with FR/Berry structure

Kinematics and labels. Let $\mathbf{R}$ be the relative coordinate (magnitude R and angles Ω) and $\Omega_{\rm rel}$ denote the relative–orientation coordinates on a moduli submanifold $\mathcal{M}_{\rm rel}$ (typically an $\mathbb{S}^2$ or a coset slice). At fixed total energy E and in an elastic block, a channel index α collects

$$\alpha \equiv (\ell, m; q) \ \otimes \ (\text{spin/orient. quantum numbers}) \ \otimes \ (\text{internal}).$$

Here $q \in \frac{1}{2}\mathbb{Z}$ is the Berry monopole charge on the relevant two–sphere of orientations; FR character $\sigma = \pm 1$ encodes fermionic/bosonic exchange.

Basis states. A convenient uncoupled basis is

$$\big|\alpha\big\rangle \equiv \Big|\ell m; q\Big\rangle_{\rm rel} \ \otimes \ \Big|s_1 s_2; S M_S\Big\rangle \ \otimes \ \Big|\text{int}\Big\rangle, \tag{7.114}$$

where $|\ell m; q\rangle_{\rm rel}$ are *monopole harmonics* on $\mathbb{S}^2$ with charge q (see B.2), $\ell \in |q|, |q|{+}1, \dots$ and $m = -\ell, \dots, \ell$. Spin/orientation and internal labels are included when needed; they can be recoupled to good total J if the potential enjoys sufficient symmetry.

FR projector. Let Π exchange the two identical solitons. Physical states satisfy

$$\Pi\,|\Psi\rangle = \sigma\,|\Psi\rangle, \qquad \sigma = -1 \ (\text{FR fermion}), \quad \sigma = +1 \ (\text{FR boson}), \tag{7.115}$$

implemented by the projector

$$\mathbb{P}_{\rm FR} = \tfrac{1}{2}\,(\mathbb{I} + \sigma\Pi). \tag{7.116}$$

In the relative partial–wave basis, Π acts by $(\ell, m) \mapsto (\ell, -m)$ up to intrinsic phase factors from spin/orientation; the explicit matrix is supplied at assembly (B.8).

Topological projector. The full topological projector used in operator assembly is

$$\mathbb{P}_{\rm topo} \equiv \mathbb{P}_{\rm FR}\,\mathbb{P}_q\,\mathbb{P}_H, \tag{7.117}$$

where $\mathbb{P}_q$ enforces $\ell \geq |q|$ and monopole–harmonic orthonormality, and $\mathbb{P}_H$ imposes any linking–number parity constraints (Sec. 7.9.4).

B.2 Monopole harmonics: essentials and couplings

Definition and eigenvalues. Monopole harmonics $Y^{(q)}_{\ell m}(\Omega)$ are sections of a $U(1)$ bundle over $\mathbb{S}^2$ with Chern number $2q$ and satisfy

$$\left[-\nabla^2_\Omega + \frac{q^2}{\sin^2\theta}\right] Y^{(q)}_{\ell m}(\Omega) = \big(\ell(\ell{+}1) - q^2\big)\, Y^{(q)}_{\ell m}(\Omega), \tag{7.118}$$

with orthonormality $\int d\Omega\, Y^{(q)}_{\ell m} Y^{(q)\,*}_{\ell' m'} = \delta_{\ell\ell'}\delta_{mm'}$.

Gaunt (triple) integrals with charge flow. For tensor operators of rank L built from ordinary spherical harmonics Y_{LM}, angular integrals reduce to generalized Gaunt coefficients,

$$\mathcal{G}^{(q)}_{\ell m;\ell' m';LM} \equiv \int d\Omega\, Y^{(q)\,*}_{\ell m}(\Omega)\, Y_{LM}(\Omega)\, Y^{(q)}_{\ell' m'}(\Omega), \tag{7.119}$$

nonzero only if $m' + M = m$ and triangle rules $|\ell{-}L| \leq \ell' \leq \ell{+}L$ hold. Selection by q is automatic since Y_{LM} carries no monopole charge.

Centrifugal shift. In the radial equations the centrifugal term is shifted:

$$\Lambda^{(q)}_\ell \equiv \ell(\ell{+}1) - q^2, \tag{7.120}$$

producing the *partial–wave floor* $\ell_{\rm min} = |q|$ and modified threshold scaling.

B.3 Potential matrices: central, Yukawa, and tensor/aniso

Central/core piece. A spherically symmetric short–range core $V_0(R)$ contributes

$$\big[V_0\big]_{\alpha\beta}(R) = \delta_{\alpha\beta}\, V_0(R). \tag{7.121}$$

Spectral Yukawa/Coulomb. Spectral $U(1)$ exchange with mass $m_{\rm spec}$ contributes a central Yukawa tail $V_Y(R) = -\alpha_{\rm spec}\, e^{-m_{\rm spec}R}/R$, diagonal in $(\ell, m; q)$ at leading order:

$$\big[V_Y\big]_{\alpha\beta}(R) = \delta_{\alpha\beta}\, V_Y(R). \tag{7.122}$$

If the source form factors carry angular dependence, use Eq. (7.119) to project.

Tensor/aniso interactions. Orientation–dependent forces induce rank–2 tensors of the form

$$V_T(\mathbf{R}; \hat\chi_1, \hat\chi_2) = \frac{C_T}{R^3}\Big[3(\hat{\mathbf{R}}\cdot\hat\chi_1)(\hat{\mathbf{R}}\cdot\hat\chi_2) - \hat\chi_1\cdot\hat\chi_2\Big], \tag{7.123}$$

which project to coupled channels via reduced matrix elements and Gaunt coefficients:

$$\big[V_T\big]_{\ell m,\,\ell' m'}(R) = \frac{C_T}{R^3}\sum_{M=-2}^{2} \mathcal{C}^{(2)}_M(\hat\chi_1, \hat\chi_2)\, \mathcal{G}^{(q)}_{\ell m;\,\ell' m';\,2M}. \tag{7.124}$$

Here $\mathcal{C}^{(2)}_M$ are spherical components of the bilinear in directors.

B.4 Coupled radial equations and asymptotics

Radial system. Expand the total wavefunction as $\Psi(\mathbf{R}, \Omega) = \sum_\alpha \frac{u_\alpha(R)}{R} Y_\alpha(\Omega)$. The CC equations read

$$\frac{d^2}{dR^2}\,\mathbf{u}(R) + \left[k^2\,\mathbb{I} - \frac{\mathbf{\Lambda}^{(q)}}{R^2} - \mathbf{V}(R)\right]\mathbf{u}(R) = 0, \tag{7.125}$$

with $[\mathbf{\Lambda}^{(q)}]_{\alpha\beta} = \delta_{\alpha\beta}\,\Lambda^{(q)}_\ell$ and $\mathbf{V} = \mathbf{V}_0 + \mathbf{V}_Y + \mathbf{V}_T + \cdots$ assembled from B.3.

Asymptotic basis and Jost matching. For each *open* channel α (wavenumber k_α under $g_{\rm eff}$), the free asymptotics are given by Riccati–Hankel functions generalized by the monopole floor:

$$\hat h^{(\pm)}_{\ell,q}(k_\alpha R) \equiv \sqrt{\frac{\pi k_\alpha R}{2}}\, H^{(\pm)}_{\ell+\frac12}(k_\alpha R) \quad \text{with} \quad \ell \ge |q|. \tag{7.126}$$

Collect incoming/outgoing matrices $\mathbf{F}(R)$, $\mathbf{G}(R)$ built from these diagonal blocks, then the K–matrix at a large matching radius R_m follows from the log–derivative $\mathbf{Y} = \mathbf{u}'\mathbf{u}^{-1}$:

$$\mathbf{K} = -\big(\mathbf{F}' - \mathbf{Y}\mathbf{F}\big)\,\big(\mathbf{G}' - \mathbf{Y}\mathbf{G}\big)^{-1}\Big|_{R=R_m}. \tag{7.127}$$

Finally $\mathbf{S} = (\mathbb{I} + i\mathbf{K})(\mathbb{I} - i\mathbf{K})^{-1}$.

Closed channels and effective elimination. Closed channels (imaginary k_α) are retained for short–range accuracy but eliminated from the external asymptotics by Feshbach reduction, yielding an energy–dependent effective open–sector potential $\mathbf{V}_{\rm eff}(E)$ and thus an energy–dependent $\mathbf{K}(E)$.

B.5 FR implementation and identical–particle rules

Exchange operator. Let Π act on the tensor product basis by swapping particle labels and rotating relative angles. In the partial–wave sector it takes the form

$$\big\langle \ell m; q \big| \Pi \big| \ell' m'; q \big\rangle = \delta_{\ell\ell'}\, \mathcal{E}^{(q)}_{\ell m}\, \delta_{m',-m}, \tag{7.128}$$

where $\mathcal{E}^{(q)}_{\ell m}$ encodes intrinsic phases (spin/orientation dependent). The physical subspace is selected by Eq. (7.116); operator matrices are assembled as $\mathbf{O} \to \mathbb{P}_{\rm FR}\,\mathbf{O}\,\mathbb{P}_{\rm FR}$.

Partial–wave floor and forbidden channels. For fermionic FR ($\sigma = -1$) and/or $q \neq 0$, S–wave may be absent in the lowest elastic block. This is enforced at the operator level by zeroing rows/columns with $\ell < \ell_{\min}$ during assembly and by using the monopole basis in B.2 (gate T1).

B.6 Spin, total J, and tensor couplings

Angular recoupling. When spins $s_{1,2}$ are present, use LS coupling to fixed total J:

$$\big|(\,\ell\,S\,)JM_J; q\big\rangle = \sum_{m,M_S} \langle \ell m; SM_S | JM_J\rangle \,\big|\ell m; q\big\rangle \otimes \big|SM_S\big\rangle.$$

Tensor/aniso potentials of rank L couple (ℓ, S, J) to (ℓ', S', J) with reduced matrix elements evaluated via Wigner–Eckart and Gaunt coefficients; selection rules obey triangle relations.

Spin–orbit and tensor operators. Operators like $\mathbf{L}\cdot\mathbf{S}$ and S_{12} (tensor) are diagonal in J, M_J but mix ℓ and S; the Berry floor remains enforced via $\ell \geq |q|$.

B.7 Lippmann–Schwinger and Feshbach projection

Momentum–space CC equation. Equivalently, in momentum space

$$\mathbf{T}(E) = \mathbf{V} + \mathbf{V}\,\mathbf{G}_0(E)\,\mathbf{T}(E), \tag{7.129}$$

with $\mathbf{G}_0$ the free Green's function in the monopole–harmonic basis (projected by $\mathbb{P}_{\text{topo}}$). Poles of $\mathbf{T}$ match those of $\mathbf{S}$.

Feshbach decomposition. Partition channels into P (open) and Q (closed):

$$\mathbf{T}_{PP} = \mathbf{V}_{PP} + \mathbf{V}_{PP}\mathbf{G}_0\mathbf{T}_{PP} + \mathbf{V}_{PQ}\mathbf{G}_Q\mathbf{V}_{QP} + \cdots$$

This yields an *effective* open–sector operator $\mathbf{V}^{\text{eff}}_{PP}(E) = \mathbf{V}_{PP} + \mathbf{V}_{PQ}(E - \mathbf{H}_{QQ})^{-1}\mathbf{V}_{QP}$, capturing Feshbach resonances (Sec. 7.7).

B.8 Numerical assembly: projector–first workflow

Algorithm (operator level).

1. Build the angular basis $\{Y^{(q)}_{\ell m}\}$ up to $L_{\max}$; precompute Gaunt tensors $\mathcal{G}^{(q)}_{\ell m;\ell' m';LM}$ and store in sparse format.
2. Construct projectors $\mathbb{P}_q$ (drop $\ell < |q|$), $\mathbb{P}_{\text{FR}}$ from Eq. (7.128), and (if needed) $\mathbb{P}_H$; define $\mathbb{P}_{\text{topo}}$ (Eq. (7.117)).
3. Assemble potential matrices $\mathbf{V}_0(R)$, $\mathbf{V}_Y(R)$, $\mathbf{V}_T(R)$ by Eqs. (7.121),(7.122),(7.124); apply $\mathbf{V} \to \mathbb{P}_{\text{topo}}\,\mathbf{V}\,\mathbb{P}_{\text{topo}}$ at each R.
4. Integrate the CC system Eq. (7.125) with the log–derivative (Johnson) method; propagate $\mathbf{Y}(R)$ from $R_{\min}$ to R_m with step control and monitor Frobenius norms.
5. Match to asymptotic blocks and extract $\mathbf{K}$ from Eq. (7.127); obtain $\mathbf{S}$.
6. Gate checks: U1 (unitarity/optical), U2 (Wigner bound, $v_g \leq 1$), U3 (soft Ward via separate module), T1/T2 (floor and geometric zeros), U5 (convergence under $\{N_r, L_{\max}, R_m\}$ refinements).

Orthogonality diagnostics. Monitor

$$\Delta_{\text{ortho}} = \max_{n\neq m} \big|\langle Y^{(q)}_n | Y^{(q)}_m\rangle - \delta_{nm}\big|$$

and the projector leakage $\Delta_{\text{leak}} = \|\mathbb{P}^{\perp}_{\text{topo}}\,\mathbf{V}\,\mathbb{P}_{\text{topo}}\|_{\text{F}}$; both should meet the bands of Sec. 7.12.

B.9 Boundary conditions and threshold extraction

Near origin. Impose regularity $u_\alpha(R) \propto R^{\ell+1}$ for $\ell \geq |q|$; start at $R_{\min} \ll R_{\rm core}$ with analytic series or a short inward–integration to generate $\mathbf{Y}(R_{\min})$.

Effective–range parameters. In single–channel testbeds, extract (a, r_0) from the diagonal phase using the modified ERE with the monopole floor:

$$k^{2\ell+1} \cot \delta_\ell(k) = -\frac{1}{A_\ell} + \frac{1}{2} R_\ell k^2 + \cdots, \qquad \ell \geq |q|. \tag{7.130}$$

For coupled channels, use eigenphases and eigen–invariants; include analytic long–range corrections (gate L4).

B.10 Validation suite and references to gates

Minimal benchmarks.

- **MB–Y0:** Pure Yukawa, $q = 0$, FR boson $\Rightarrow$ reproduce standard CC results; U1/U2 pass.
- **MB–Yq:** Pure Yukawa, $q = \frac{1}{2} \Rightarrow$ floor $\ell_{\min} = \frac{1}{2}$; T1 pass.
- **MB–T:** Tensor potential only $\Rightarrow$ verify $\Delta\ell = 0, \pm 2$ selection via Gaunt couplings.
- **MB–FR:** FR fermion with identical species $\Rightarrow$ S–wave suppressed; U1/T1/T2 pass.

Gate mapping. U1 (unitarity) via Eq. (7.127), U2 (Wigner/causality) from the extracted phases, U3 (soft/gauge) handled by the radiation module (Sec. 7.8), U4 (FR/Berry) enforced by $\mathbb{P}_{\rm topo}$ and B.2 diagnostics, U5 (numerics) by convergence studies in B.8–B.9.

The projector–first CC construction above ensures that FR statistics and Berry topology are implemented at the operator level, so that selection rules, partial–wave floors, and geometric phase effects emerge automatically and robustly in all subsequent scattering and spectroscopy calculations.

Appendix 7.C: Radiative Corrections to Phase Shifts, Inclusive Unitarity, and an End-to-End Case Study

This appendix develops the treatment of radiation in two–body quantum–aether scattering, its impact on phase shifts and line shapes, and the proof–of–principle of *inclusive unitarity* via soft exponentiation. We then present an end–to–end case study that implements the complete pipeline of Ch. 7 (potentials $\rightarrow$ phase shifts $\rightarrow$ radiative dressing $\rightarrow$ gates) on a controlled benchmark.

C.1 Soft radiation modules and factorization

Channels of emission. Two universal radiation channels are considered: (i) *spectral* $U(1)$ quanta, with coupling $g_{\rm spec}$ and possibly small mass $m_{\rm spec}$; (ii) *emergent graviton* quanta governed by $G_{\rm eff}(\rho)$. Hard inelastic channels (internal excitations, molecular breakup) are handled in Sec. 7.7; here we isolate *soft* emissions with $\omega \ll E_{\rm cm}$.

Soft factors (leading eikonal). At leading order (Sec. 7.8), amplitudes factorize:

$$\mathcal{M}^{(\rm spec)}_{n+1}(k; \{p_i\}) \;=\; \left[g_{\rm spec} \sum_i Q_i \frac{p_i \cdot \varepsilon}{p_i \cdot k} \right] \mathcal{M}_n(\{p_i\}) \;+\; \mathcal{O}(k^0), \qquad k \to 0, \tag{7.131}$$

and similarly for emergent gravity with the standard replacement $g_{\rm spec} Q_i \to \kappa\, p_i^\mu$ (with $\kappa^2 = 32\pi G_{\rm eff}$). Ward identities $k_\mu \mathcal{M}^\mu = 0$ (Sec. 7.12.4) guarantee gauge/soft consistency.

C.2 YFS/KLN exponentiation and inclusive finiteness

Exponentiation of soft logs. For emission below an energy resolution ε,

$$\mathcal{S}_{\rm dress}(\varepsilon) \;=\; \exp\Big[\,B_{\rm spec}(\varepsilon) \;+\; B_{\rm grav}(\varepsilon)\,\Big]\,\mathcal{S}_{\rm hard} \tag{7.132}$$

with (schematically)

$$B_{\rm spec}(\varepsilon) \;=\; -\frac{\alpha_{\rm spec}}{\pi}\sum_{i,j} Q_i Q_j\,\mathcal{I}_{ij}\,\ln\frac{\Lambda}{\varepsilon} \;+\; \cdots, \qquad \alpha_{\rm spec}\equiv\frac{g_{\rm spec}^2}{4\pi}, \tag{7.133}$$

where Λ is a hard scale and $\mathcal{I}_{ij}$ encodes kinematic invariants $(p_i\cdot p_j)/(m_i m_j)$ in the appropriate $g_{\rm eff}$ metric. The gravitational piece $B_{\rm grav}$ is analogous with $\alpha_{\rm spec}\to G_{\rm eff}E^2/\pi$.

KLN cancellations. Summing over soft real emissions and virtual corrections yields an *inclusive* cross section $d\sigma_{\rm inc}$ independent of ε:

$$\frac{\partial}{\partial\varepsilon}\,d\sigma_{\rm inc}(k;\varepsilon) \;=\; 0 \;+\; \mathcal{O}(\varepsilon^0), \tag{7.134}$$

provided the same kinematic and coupling conventions are used in real and virtual sectors (Sec. 7.8.2, gate U3 in Sec. 7.12).

C.3 Coulomb–like phases and long–range tails

Massless spectral exchange. If the spectral mediator is (effectively) massless on the scattering scale, the asymptotics acquire Coulomb–like phases. Define $\eta_{\rm spec}=\alpha_{\rm spec}/v_{\rm rel}$ (with $v_{\rm rel}$ under $g_{\rm eff}$). The partial waves pick up

$$\delta_\ell^{(\rm C)} \;=\; \arg\Gamma(\ell{+}1{+}i\eta_{\rm spec}) \;-\; \eta_{\rm spec}\,\ln(2kR_m), \tag{7.135}$$

and the physical phase shifts are $\delta_\ell^{\rm phys}=\delta_\ell^{\rm num}-\delta_\ell^{(\rm C)}$. This subtraction yields short–range phase shifts suitable for effective–range fits (Sec. 7.5) and stable unitarity diagnostics (Sec. 7.12.2).

Light but massive spectral exchange. For $m_{\rm spec}R_{\rm int}\ll 1$ but nonzero, one can use *screened* Coulomb phases with $R_m\to 1/m_{\rm spec}$ in Eq. (7.135). The residual dependence on $m_{\rm spec}$ is then physical and controlled by the Yukawa tail in Sec. 7.4.1.

C.4 Radiative shifts of phase shifts and line shapes

Soft distortions. At fixed ε, soft radiation induces a universal additive distortion

$$\delta_\ell(k;\varepsilon) \;=\; \delta_\ell^{\rm phys}(k) \;+\; \Delta\delta_\ell^{\rm soft}(k;\varepsilon), \qquad \Delta\delta_\ell^{\rm soft} \;=\; \mathcal{O}\!\left(\alpha_{\rm spec}\ln\frac{\Lambda}{\varepsilon}\right), \tag{7.136}$$

which *cancels* against real–emission contributions in $d\sigma_{\rm inc}$. Hence, the recommended workflow is to quote *inclusive* observables and $\delta_\ell^{\rm phys}(k)$ with Coulomb pieces removed (if any) per Eq. (7.135).

Resonance lines. For narrow resonances, radiative channels add widths $\Gamma_{\rm spec}\sim\alpha_{\rm spec}\omega^3|\mathbf{d}_{if}|^2$ and $\Gamma_{\rm grav}\sim G_{\rm eff}\omega^5|Q_{ij}^{(if)}|^2$ (Sec. 7.8.4), leading to Breit–Wigner/Flatté forms with modified open–channel couplings.

C.5 Energy balance and imaginary eikonal

Optical theorem with radiation. Let $f(\theta)$ be the elastic amplitude. The optical theorem holds inclusively:

$$\sigma_{\rm tot} \;=\; \frac{4\pi}{k}\,\mathrm{Im}\,f(0) \;=\; \sigma_{\rm el}+\sigma_{\rm hard}+\sigma_{\rm soft}, \tag{7.137}$$

with $\sigma_{\rm soft}$ computed by the exponentiated soft module. In the eikonal representation $S(b) = \exp\{2i\chi(b)\}$, soft power loss corresponds to $\chi_I(b) > 0$ via

$$\sigma_{\rm tot} \;=\; 2\int d^2b\,\big(1 - e^{-2\chi_I(b)}\cos 2\chi_R(b)\big). \tag{7.138}$$

Consistency (gate S1/S5) demands agreement between χ_I inferred from classical power (Sec. 7.8) and the inclusive cross section.

C.6 End-to-End Case Study: From Potentials to Inclusive Observables

Benchmark definition (MBP–Y/T). We choose identical solitons with:

$$\text{FR boson,}\quad q = \tfrac{1}{2},\quad V_0(R) = V_c\,e^{-(R/R_c)^2},\quad V_Y(R) = -\alpha_{\rm spec}\frac{e^{-m_{\rm spec}R}}{R},\quad V_T \text{ per Eq. (7.123)},$$

and alignment $\hat\chi_1 \parallel \hat\chi_2 \parallel \hat z$. Parameters (dimensionless units under $g_{\rm eff}$):

$$M = 10,\ R_c = 1.5,\ V_c = 2.0,\ \alpha_{\rm spec} = 0.02,\ m_{\rm spec} = 0.05,\ C_T = 0.3.$$

Energy window: $k \in [0.05,\, 0.8]$.

Step 1: Coupled–channel assembly and phases. Build monopole basis $Y^{(q)}_{\ell m}$ up to $L_{\max} = 6$; enforce $\ell_{\min} = |q| = \frac{1}{2}$ by $\mathbb{P}_q$ and FR projector $\mathbb{P}_{\rm FR}$. Integrate CC equations (App. 7.13.6) to $R_m = 25$ with Johnson log–derivative method. Subtract screened Coulomb phases with $R_m \to 1/m_{\rm spec}$ (Eq. (7.135)) to obtain $\delta_\ell^{\rm phys}(k)$. *Gate U1*: $\Delta_{\rm U1} \le 10^{-7}$, optical residual $\le 10^{-5}$.

Step 2: Threshold extraction and universality. Fit modified ERE (Eq. (7.130)) in $k \in [0.05, 0.15]$: obtain $(A_{\ell_{\min}}, R_{\ell_{\min}})$ and verify universality relations near a shallow bound state (dimer) when present (Sec. 7.5). *Gate L5*: ERE fit stable under k–window variations.

Step 3: Radiative dressing. Compute soft modules $B_{\rm spec}(\varepsilon)$ and $B_{\rm grav}(\varepsilon)$ (Eqs. (7.132)–(7.133)) with $\varepsilon/E_{\rm cm} \in [10^{-4}, 10^{-2}]$. Form *inclusive* cross sections by exponentiation plus Monte Carlo over soft phase space (Sec. 7.8.2); confirm ε–independence within tolerance. *Gate U3*: Ward residual $\le 10^{-6}$, IR slope flat.

Step 4: Resonance search and lines. Run CAP/complex scaling (Sec. 7.10.5) to locate poles (E_r, Γ) in $k \in [0.4, 0.7]$. Include radiative widths via matrix elements of dipole/quadrupole operators (Sec. 7.8.4); compare elastic line shapes to Flatté fits. *Gate R2*: sheet assignment stable under regulator changes.

Step 5: Eikonal small–angle cross check. Compute $f_{\rm eik}(\theta)$ (Sec. 7.6) with anisotropy and Berry phase additions; verify agreement with partial–wave sum for $\theta \le 10^\circ$ (H2). Infer $\chi_I(b)$ from inclusive loss and check Eq. (7.138).

Step 6: Reporting and uncertainties. Publish $\{\delta_\ell^{\rm phys}(k),\, a, r_0,\, (E_r, \Gamma),\, d\sigma_{\rm inc}/d\Omega\}$ with uncertainty budgets (Sec. 7.10.8). All gates U1–U5 passed at the declared tier (Sec. 7.12.7).

C.7 Pseudo–code: inclusive dressing and gate hooks

```
# Inputs: S(k) (elastic), phases delta_l(k), potentials, params, epsilon grid
for k in k_grid:
  # Coulomb/screened subtraction
  delta_phys = delta_num - delta_Coulomb(k, alpha_spec, m_spec, Rm)
  # Soft exponents
  Bspec = soft_exponent_spec(k, charges, epsilon, Lambda)
  Bgrav = soft_exponent_grav(k, masses, epsilon, Geff)
  # Inclusive cross sections
```

```
sigma_el = elastic_xs_from_phases(delta_phys)
sigma_soft = soft_real_xs(Bspec, Bgrav)
sigma_hard = inelastic_xs_from_CC(k)  # optional hard channels
sigma_tot  = (4*pi/k) * Im f_forward_from_S(S)
# Gates
assert |sigma_tot - (sigma_el + sigma_hard + sigma_soft)| < tol_opt   # U1
assert ward_identity_residual(k, ...) < tol_ward                     # U3
assert inclusive_flat_vs_epsilon(...)                                # U3
assert vg(k) <= 1 and Wigner_bound(...)                              # U2
```

C.8 Remarks on extension and limitations

Beyond leading soft. Next–to–soft corrections (subleading in k) carry spin/orientation information and can be systematically included using Low–Burnett–Kroll analogues in the spectral and emergent gravitational sectors.

Medium and anisotropy. Energy resolution ε and phase–space cuts must be defined with the same $g_{\rm eff}$ kinematics used in the CC/eikonal solvers; otherwise IR cancellations can be spoiled.

When to quote phases. For comparison across models, always quote *short–range* phases $\delta_\ell^{\rm phys}$ (Coulomb/screened subtracted) and *inclusive* observables (KLN safe). Raw δ_ℓ with unresolved soft logs are regulator dependent.

Together, the soft modules, Coulomb/screened subtractions, and the inclusive gate suite establish a reproducible and IR–safe path from microscopic soliton interactions to observable cross sections and resonance spectra, as exemplified by the end–to–end benchmark in Sec. 7.13.6.

Symbols and Definitions (Chapter 7)

Conventions. Natural units $\hbar = c = 1$. Kinematics and phase space are measured with the emergent metric $g_{\rm eff}$. Scalars, vectors, and tensors carry indices raised/lowered by $g_{\rm eff}$.

Chapter 7 — Tables and Illustrative Figures

Symbol	Name	Meaning / Where Used
$E_{\rm cm}$	center–of–mass energy	Total energy for two–body scattering under $g_{\rm eff}$.
k	wavenumber	Asymptotic relative momentum magnitude; relates to $E_{\rm cm}$ by dispersion (Sec. 7.3).
$v_{\rm rel}$	relative speed	Group velocity under $g_{\rm eff}$; appears in Coulomb parameter $\eta_{\rm spec} = \alpha_{\rm spec}/v_{\rm rel}$.
$\mathbf{R}, R, \hat{\mathbf{R}}$	relative coordinate	Separation vector, magnitude, and unit direction.
$\mathbf{b}$	impact parameter	Transverse (anisotropic) impact parameter in eikonal representation (Sec. 7.6, App. 7.A).
θ, ϕ	scattering angles	Spherical angles in the far field; enter partial waves and eikonal integral.
$q \in \frac{1}{2}\mathbb{Z}$	Berry monopole charge	Chern number/2 on orientation sphere; enforces partial–wave floor $\ell_{\min} = \lvert q\rvert$ (Secs. 7.3, 7.9, App. 7.B).
$\sigma = \pm 1$	FR character	Finkelstein–Rubinstein statistics for identical solitons (fermionic -1, bosonic $+1$); projector in App. 7.B.
(ℓ, m)	partial–wave indices	With floor $\ell \geq \lvert q\rvert$; basis $Y^{(q)}_{\ell m}(\Omega)$ (monopole harmonics).
$Y^{(q)}_{\ell m}$	monopole harmonics	Angular basis on $\mathbb{S}^2$ with charge q; eigenvalues $\ell(\ell{+}1) - q^2$ (App. 7.B.2).
$g_{\rm eff}$	emergent metric	Controls kinematics/causality; anisotropy set by $(c_\perp, c_\parallel)$ (Secs. 7.3, 7.6).
$c_\perp, c_\parallel$	propagation speeds	Transverse/parallel effective speeds; $c_\parallel^2 = c_\perp^2 + \delta c$, $\lvert\delta c\rvert \ll 1$.
$\alpha_{\rm spec} = \frac{g^2_{\rm spec}}{4\pi}$	spectral coupling	Long–range spectral $U(1)$ strength (Secs. 7.4, 7.8, App. 7.C).
$m_{\rm spec}$	spectral mass	Mediator mass (Yukawa screening); $m_{\rm spec} \to 0$ gives Coulomb–like phases (App. 7.C.3).
$G_{\rm eff}(\rho)$	emergent gravity	Effective gravitational coupling set by aether state; $\kappa^2 = 32\pi G_{\rm eff}$.
$V_0(R)$	core potential	Short–range central interaction (App. 7.B.3).
$V_Y(R)$	Yukawa tail	$-\alpha_{\rm spec} e^{-m_{\rm spec} R}/R$ (Sec. 7.4.1).
$V_T(\mathbf{R}; \hat{\chi}_1, \hat{\chi}_2)$	tensor/aniso piece	Orientation–dependent $1/R^3$–like interaction (Eq. (7.B.7.123)).
C_T	tensor strength	Coefficient in V_T (Eq. (7.B.7.124)).
δ_ℓ	phase shift	From coupled–channel integration; diagonal/eigenphase per channel (Secs. 7.3, 7.5).
$\delta_\ell^{(C)}$	Coulomb phase	Analytic long–range subtraction for massless/weakly screened exchange (App. 7.C.3).
$\delta_\ell^{\rm phys}$	physical short–range phase	$\delta_\ell - \delta_\ell^{(C)}$; used in effective–range fits (Secs. 7.5, App. 7.C.4).
$a,\ r_0$	ERE parameters	Scattering length and effective range (or A_ℓ, R_ℓ for $\ell \geq \lvert q\rvert$) (Secs. 7.5, App. 7.B.9).
$\mathbf{K}$, $\mathbf{S}$	K– and S–matrices	Extracted at matching radius R_m; $\mathbf{S} = (\mathbb{I} + i\mathbf{K})(\mathbb{I} - i\mathbf{K})^{-1}$ (App. 7.B.4).
R_m	matching radius	Asymptotic matching radius for CC integration.
$L_{\max}$	angular cutoff	Maximal partial wave in numerical truncation (gate U5).
$\chi(b) = \chi_R + i\chi_I$	eikonal phase	Real/imaginary parts for conservative/absorptive effects; $S(b) = e^{2i\chi(b)}$ (Sec. 7.6, App. 7.A).
$\Phi(b)$	stationary phase	$\Phi = \mathbf{q}\cdot\mathbf{b} + \chi(b)$ in impact representation (App. 7.A).
$\mathcal{A}, \mathcal{F}$	Berry connection/curvature	Geometric phase data on moduli; contribute to $\chi_{\rm geom}$ (App. 7.A.7, Sec. 7.9).
Δ	Van Vleck determinant	Beam focusing factor along rays on $(\mathcal{M}_Q, G)$ (App. 7.A.2).
ε	soft resolution	Experimental soft–energy cutoff for inclusive observables (App. 7.C.2).
Λ	hard scale	UV scale entering soft exponents (App. 7.C.2).
$B_{\rm spec}(\varepsilon)$, $B_{\rm grav}(\varepsilon)$	soft exponents	YFS/KLN dressing factors for spectral/gravitational radiation (App. 7.C.2).
$\sigma_{\rm inc}$	inclusive cross section	IR–safe observable: elastic + soft + hard inelastic (App. 7.C.2, Eq. (7.C.7.137)).
$\mathbb{P}_{\rm FR}$, $\mathbb{P}_q$, $\mathbb{P}_H$	projectors	FR statistics, monopole floor, and (optional) linking–parity

Table 7.1: Two-body soliton–soliton EFT (SAEFT) operator basis up to NLO. Power counting in relative velocity v/c_{eff} and range-suppression R/λ. Operators respect internal/gauge symmetries and FR/Berry selection.

Order	**Representative operators**	**Scaling**	**Notes**
LO	$C_0\,\delta^{(3)}(\mathbf{R})$; central $U_{\text{ad}}(R)$	$v^0,\ (R/\lambda)^0$	Encodes short-range core and adiabatic background.
NLO	$C_2\,\nabla^2\delta(\mathbf{R})$; tensor $C_T\,S_{12}$; spin-orbit $C_{LS}\,\mathbf{LS}$	$v^2,\ (R/\lambda)^2$	Anisotropy via $\hat{\chi}_{1,2}$; induced by Berry and spectral curvature.
Long-range	Exchange kernels $V_{\text{spec/int/grav}}(R)$; dissipative $i\Gamma(R)$	$1/R^n$ tails	Spectral $U(1)$, non-Abelian internal, and emergent grav. pieces; GKSL/FDT for loss.

Table 7.2: Scattering observables and extraction recipes. Conventions use anisotropic flux under g_{eff} and coupled-channel normalization.

Observable	**Definition / formula**	**Extraction / comments**
Phase shifts $\delta_\ell^{(\alpha)}(k)$	$S_\ell^{(\alpha)} = e^{2i\delta_\ell^{(\alpha)}}$ (single channel)	Fit asymptotics of partial waves; in coupled channels use eigenphase/mixing angles.
Effective range	$k\cot\delta_0 = -1/a + \frac{1}{2}r_0k^2 + \dots$	Linear fit of $k\cot\delta_0$ vs. k^2 near threshold.
Cross sections	$d\sigma/d\Omega = \lvert f(\theta)\rvert^2$; $\sigma_{\text{tot}} = \frac{4\pi}{k}\,\text{Im}\, f(0)$	Optical theorem; anisotropic normalization from g_{eff}.
Resonance parameters	E_R where $\delta_\ell \approx \pi/2$; Γ from slope or pole width	Locate S-matrix poles in complex k or E plane; Argand loop near unitarity circle.
Capture rates	$\sigma_{\text{cap}}(E)$ with emission; detailed balance where applicable	Include dissipative/radiative operators; soft-theorem checks for inclusive rates.

Table 7.3: Resonance taxonomy. Distinguishing shape vs. Feshbach-like resonances with practical diagnostics.

Type	**Mechanism**	**Signature**	**Diagnostic**
Shape	Barrier/well in single channel	Narrow peak in δ_ℓ; quasi-bound	WKB barrier, single-channel pole near real axis
Feshbach-like	Closed-channel bound coupled to open	Avoided crossing; strong channel mixing	Coupled-channel pole; sensitivity to coupling strength

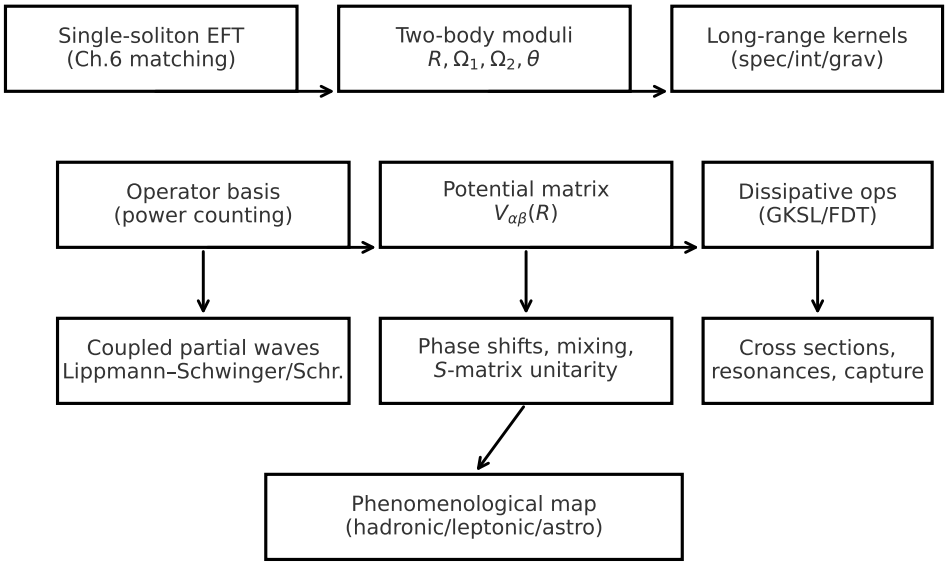

Figure 7.1: Worldline/volume SAEFT pipeline for scattering. Inputs from Ch. 6 (single-soliton EFT) feed two-body operator bases and potential matrices, with dissipative pieces consistent with GKSL/FDT; solvers produce phase shifts, cross sections, resonances, and capture rates; finally matched to phenomenology.

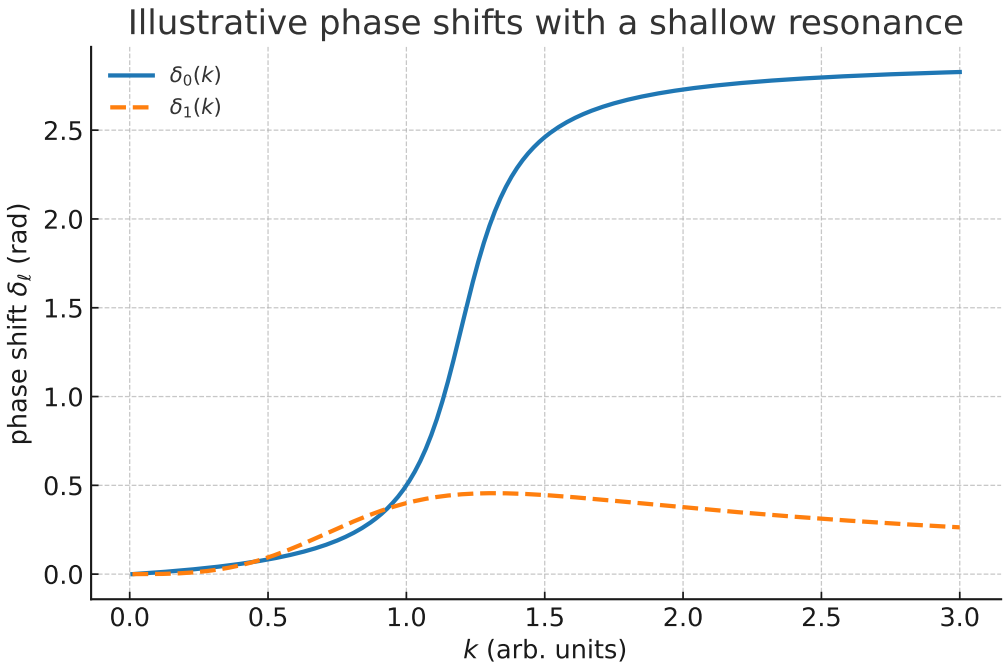

Figure 7.2: Illustrative phase shifts. S-wave $\delta_0(k)$ shows a shallow resonance (rapid rise near $\pi/2$); P-wave $\delta_1(k)$ rises more gradually. Exact behavior depends on anisotropic potentials and FR/Berry selection rules.

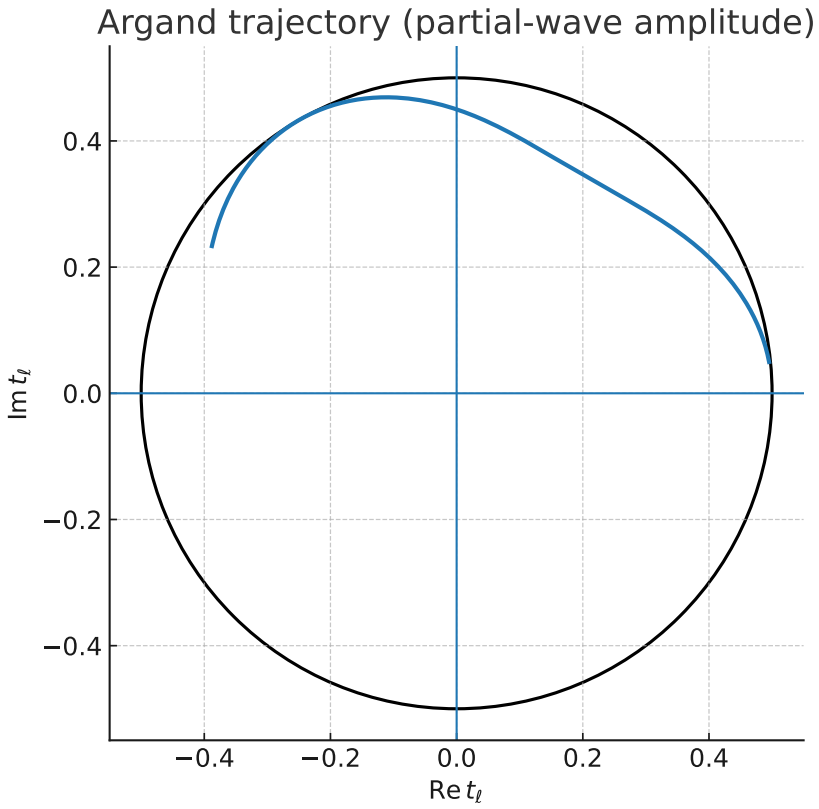

Figure 7.3: Argand trajectory of a partial-wave amplitude. Resonant channel loops counterclockwise near the unitarity circle in the complex t_ℓ plane; loop size and proximity to the circle encode width and inelasticity.

Chapter 8

Chapter 8: Emergent Gauge Dynamics and Anomaly Consistency

8.1 Motivation and Scope

The central claim of this chapter is that *gauge fields are not fundamental inputs* but rather *emergent connections* of the quantum–aether's internal bundle, induced by the structural fields introduced and developed in Chs. **??**–**??** and constrained by the quantization and scattering phenomenology of Chs. 6–7. Concretely, starting from the pre–geometric differential operator

$$\nabla_\mu \;=\; D_\mu(\rho, \eta, T, \chi) \;+\; \omega_\mu(e, u, \hat{\chi}),$$

we will show that there exists a natural projection onto the vertical fiber of the structural bundle that defines an *effective* gauge connection A_μ with curvature $F_{\mu\nu}(A)$. The dynamics of A_μ at low energies reduce to a Yang–Mills–type action with state–dependent wavefunction renormalization factors, while anomaly constraints and inflow fix otherwise ambiguous topological terms.

From structural bundle to gauge connection. Let $\pi : \mathcal{E} \to \mathcal{M}$ be the structural bundle of Secs. 2.3–2.5, with horizontal distribution $H\mathcal{E}$ (Ehresmann connection) and vertical fiber $V\mathcal{E}$. Writing the full covariant derivative as a horizontal plus vertical piece,

$$\nabla_\mu \;=\; \nabla_\mu^{(H)} \oplus \nabla_\mu^{(V)}, \qquad \nabla_\mu^{(V)} \equiv A_\mu^a(\mathcal{S})\, T_a, \tag{8.1}$$

defines the *emergent gauge field* A_μ as the vertical component of the structural connection on the fiber (with generators T_a acting on internal indices of matter Φ or collective modes). The associated curvature

$$F_{\mu\nu}(A) \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{8.2}$$

is the vertical projection of the bundle curvature; Bianchi identities follow from the Ehresmann structure and the Jacobi identity.

Relation to spectral curvature and spin connection. Commutators of horizontal covariant derivatives acting on fibered fields separate into three pieces,

$$[\nabla_\mu^{(H)}, \nabla_\nu^{(H)}] \;=\; R_{\mu\nu}{}^{ab}(\omega)\, \mathcal{J}_{ab} \;+\; F_{\mu\nu}(A) \;+\; F_{\mu\nu}^{\mathrm{spec}}(\mathcal{S}), \tag{8.3}$$

where $R(\omega)$ is the curvature of the emergent spin/affine connection (Sec. **??**), F^{spec} encodes purely spectral (non–gauge) holonomy of the medium (Sec. 2.4), and $F(A)$ is the part that acts as an internal gauge field on fiber representations. Equation (8.3) makes manifest that gauge, geometric, and spectral curvatures are *distinct but coupled* sectors of the same underlying bundle geometry.

Noether currents and Ward identities. Gauge fields couple to spectral and non–Abelian currents built from matter Φ and slow collective modes:

$$J^\mu_{\rm spec} \;=\; \mathrm{Im}(\Phi^*\nabla^\mu\Phi)\,, \qquad J^\mu_a \;=\; \frac{\partial\mathcal{L}}{\partial(\nabla_\mu\Psi)}\,T_a\,\Psi \;+\; \cdots\,, \tag{8.4}$$

with Ψ a multiplet of fields transforming in a representation of the emergent group. Conservation $\nabla_\mu J^\mu = 0$ and the Ward identities follow from horizontal invariance of the action under fiber rotations; they will be used throughout to define consistency gates for numerics and EFT matching.

Low–energy action and state dependence. Integrating out fast structural fluctuations (Sec. 2.7) and medium modes (Sec. 3.7) generically yields an effective gauge sector

$$\mathcal{L}_{\rm gauge} \;=\; -\frac{1}{4}\,Z_1(\mathcal{S})\,F_{\mu\nu}F^{\mu\nu} \;-\; \frac{1}{4}\,Z_2(\mathcal{S})\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} \;+\; A_\mu J^\mu \;+\; \cdots\,, \tag{8.5}$$

where $Z_i(\mathcal{S}) > 0$ are state–dependent wavefunction factors determined by the aether state $\mathcal{S} = (\rho, \eta, T, \chi)$. Hyperbolicity and positivity constraints derived in Ch. 3 imply bounds on Z_i ensuring subluminal characteristics under $g_{\rm eff}$.

Objectives of this chapter.

1. **Derivation:** Construct A_μ from the structural connection and prove the Bianchi/Noether identities; derive Eq. (8.5) with explicit $Z_i(\mathcal{S})$ at leading order.
2. **Emergence of non–Abelian sectors:** Build $SU(2)$ and $SU(3)$ connections from internal geometry (coset/Maurer–Cartan), including an optional $G_2 \to SU(3)$ pathway consistent with the symmetry patterns of Chs. **??**–6.
3. **Spectral RG:** Define a coarse–graining on spectral cells to obtain running couplings and low–energy matching to observed gauge strengths.
4. **Anomaly consistency:** Compute gauge, gravitational, and mixed anomalies (anomaly polynomial and descent); enforce cancellation by anomaly inflow via WZW/CS terms; implement 't Hooft matching on the quantized soliton spectrum (Ch. 6).
5. **Phenomenology and gates:** Translate to falsifiable signatures (precision tests, collider bounds, GW tiny deviations) and set gauge–sector consistency gates (A1–A5) for analytical and numerical pipelines.

Scope and interfaces. Section 8.2 formalizes the projection from ∇_μ to A_μ and the relation among $R(\omega), F^{\rm spec}, F(A)$. Section 8.3 develops the Abelian spectral sector and its Maxwell limit; Sec. 8.4 constructs non–Abelian connections from internal geometry. Sec. 8.5 derives the Yang–Mills action and BRST structure at the EFT level; Sec. 8.6 defines spectral RG and matching. Anomaly analysis and inflow are handled in Secs. 8.7–8.8, followed by 't Hooft matching (Sec. 8.9) and QCD–like dynamics (Sec. 8.10). The electroweak sector (Sec. 8.11) addresses vacuum alignment and the massless photon, while Secs. 8.12–8.14 set phenomenological interfaces, numerical protocols, and consistency gates. We conclude in Sec. 8.15 with a compact list of falsifiable predictions and data–to–model pipelines.

Conventions and notation (this chapter). We work in natural units $\hbar = c = 1$; indices are raised/lowered with the emergent metric $g_{\rm eff}$. Fiber traces Tr are in the fundamental unless stated otherwise. Curvatures $R(\omega)$, $F^{\rm spec}$, and $F(A)$ refer respectively to the spin/geometric, spectral, and internal–gauge sectors as in Eq. (8.3).

8.2 From Structural Connections to Gauge Fields

8.2.1 Ehresmann split and projected connections

Structural bundle and projectors. Let $\pi : \mathcal{E} \to \mathcal{M}$ be the structural bundle of Secs. 2.3–2.5, endowed with an Ehresmann connection, i.e. a horizontal subbundle $H\mathcal{E} \subset T\mathcal{E}$ such that $T\mathcal{E} = H\mathcal{E} \oplus V\mathcal{E}$

where $V\mathcal{E} = \ker d\pi$ is the vertical bundle. Denote the fiberwise projectors by

$$\mathsf{P}_H : T\mathcal{E} \to H\mathcal{E}, \qquad \mathsf{P}_V : T\mathcal{E} \to V\mathcal{E}, \qquad \mathsf{P}_H + \mathsf{P}_V = \mathbb{I}, \quad \mathsf{P}_H\mathsf{P}_V = \mathsf{P}_V\mathsf{P}_H = 0.$$

The structural covariant derivative on sections of associated bundles (matter and collective fields) was introduced in Ch. **??** as

$$\nabla_\mu \;=\; D_\mu(\rho, \eta, T, \chi) \;+\; \omega_\mu(e, u, \hat{\chi}), \tag{8.6}$$

with D_μ the pre–geometric/spectral derivative and ω_μ the emergent spin/affine connection (Ch. 3). We assume the *compatible split* of Ch. 3:

$$\nabla_\mu \mathsf{P}_V \;=\; 0 \;=\; \nabla_\mu \mathsf{P}_H, \qquad \text{(compatible splitting)} \tag{8.7}$$

which ensures that horizontal/vertical projections are preserved by parallel transport.

Definition (emergent gauge field). Let $\mathcal{A}_\mu$ denote the total connection one–form acting on an internal fiber representation $\mathcal{R}$ (carried by the matter field Φ or by a multiplet Ψ). We define the *emergent gauge connection* as the vertical projection

$$A_\mu \;\equiv\; \mathsf{P}_V(\mathcal{A}_\mu) \;\in\; \mathfrak{g}, \qquad \mathfrak{g} := \mathrm{Lie}(G) \subset \mathrm{End}(\mathcal{R}), \tag{8.8}$$

where G is the fiber isometry group selected by metric compatibility on the internal fiber (see below). The complementary horizontal piece, $\mathsf{P}_H(\mathcal{A}_\mu)$, acts as the transport operator along the base $\mathcal{M}$ and encodes the spin/affine sector via ω_μ.

Metric compatibility ⇒ compact gauge algebra. Let h be the fiber metric on $\mathcal{R}$ induced by the aether state $\mathcal{S} = (\rho, \eta, T, \chi)$. We impose

$$\nabla_\mu h \;=\; 0, \qquad \Longrightarrow \qquad A_\mu^\dagger h + h A_\mu = 0, \tag{8.9}$$

which identifies $\mathfrak{g}$ with the algebra of h–isometries. For h positive–definite (as in the low–energy sector), $\mathfrak{g}$ is compact and A_μ is anti–Hermitian (*no negative–norm modes*; consistency gate A4 in Sec. 8.14).

Local gauge law from bundle changes. Under a change of local trivialization $g(x) \in G$ on the fiber, the vertical projector remains invariant by (8.7), while $\mathcal{A}_\mu$ transforms as $\mathcal{A}_\mu \mapsto g^{-1}\mathcal{A}_\mu g + g^{-1}\partial_\mu g$. Therefore

$$A_\mu \;\mapsto\; g^{-1} A_\mu g \;+\; g^{-1}\partial_\mu g, \qquad \Psi \;\mapsto\; g^{-1}\Psi, \tag{8.10}$$

i.e. A_μ is a bona fide gauge connection on $\mathcal{M}$ acting on Ψ in the representation $\mathcal{R}$.

Covariant derivative on matter/collective fields. For any $\Psi \in \Gamma(\mathcal{R})$ we write

$$\nabla_\mu \Psi \;=\; \big(\nabla_\mu^{(H)} + \nabla_\mu^{(V)}\big)\Psi \;=\; \big(\partial_\mu + \omega_\mu\big)\Psi \;+\; A_\mu \Psi, \tag{8.11}$$

where ω_μ acts on spacetime/tangent indices (via $e^a{}_\mu$) and A_μ acts on internal ones. Gauge–invariant kinetic terms in the effective action will then be built from $F_{\mu\nu}(A)$ and $\nabla_\mu \Psi$ (Secs. 8.3, 8.5).

Horizontal lifts and field strength. Let h_μ be the horizontal lift of ∂_μ to $H\mathcal{E}$. The non–integrability of $H\mathcal{E}$ (curvature of the Ehresmann connection) is encoded in

$$\Omega_{\mu\nu} \;\equiv\; -\,\mathsf{P}_V\big([\mathsf{h}_\mu, \mathsf{h}_\nu]\big), \tag{8.12}$$

which in a local trivialization coincides with the gauge field strength (up to representation maps), cf. (8.14) below.

8.2.2 Curvatures and identities

Decomposition of the total curvature. The curvature of the full structural derivative is

$$\mathcal{R}_{\mu\nu} \;\equiv\; [\nabla_\mu, \nabla_\nu] \;=\; R_{\mu\nu}{}^{ab}(\omega)\, \mathcal{J}_{ab} \;+\; F_{\mu\nu}(A) \;+\; F^{\mathrm{spec}}_{\mu\nu}(\mathcal{S}) \;+\; \Theta_{\mu\nu}, \tag{8.13}$$

where: (i) $R(\omega)$ is the emergent Riemann curvature acting on tangent/spin indices (Ch. 3); (ii) F^{spec} is the spectral curvature inherited from D_μ (Ch. **??**); (iii) $F(A)$ is the internal gauge curvature; (iv) $\Theta_{\mu\nu}$ collects mixed terms that vanish under the compatible split (8.7). Explicitly, in any local chart and representation,

$$F_{\mu\nu}(A) \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \qquad F^{\mathrm{spec}}_{\mu\nu} \;=\; \partial_\mu \omega^{\mathrm{spec}}_\nu - \partial_\nu \omega^{\mathrm{spec}}_\mu + \cdots, \tag{8.14}$$

with $\omega^{\mathrm{spec}}_\mu$ the spectral (non–gauge) connection buried in D_μ.

Bianchi identities. The structural Bianchi identity $\nabla_{[\lambda}\mathcal{R}_{\mu\nu]} = 0$ projects to

$$D_{[\lambda}F_{\mu\nu]}(A) \;=\; 0, \qquad \nabla^{\omega}_{[\lambda}R_{\mu\nu]}(\omega) = 0, \qquad D^{\mathrm{spec}}_{[\lambda}F^{\mathrm{spec}}_{\mu\nu]} = 0, \tag{8.15}$$

provided (8.7) holds (so that $\Theta_{\mu\nu}$ vanishes and the projectors commute with ∇). Here $D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} + [A_\lambda, F_{\mu\nu}]$.

Commutator on fibered matter. Acting on Ψ that carries both spacetime and internal indices, the commutator factorizes:

$$[\nabla_\mu, \nabla_\nu]\Psi \;=\; \big(R_{\mu\nu}{}^{ab}\,\Sigma_{ab}\big)\Psi \;+\; \big(F^a_{\mu\nu}T_a\big)\Psi \;+\; \big(F^{\mathrm{spec}}_{\mu\nu}\big)\Psi, \tag{8.16}$$

where Σ_{ab} are the spin generators on the appropriate representation. Equation (8.16) makes explicit the three distinct (but coupled) holonomies: geometric, gauge, spectral.

Noether currents and Ward identities (preview). If the effective Lagrangian density $\mathcal{L}(\Psi, \nabla\Psi; A, \omega; \mathcal{S})$ is invariant under local fiber rotations $\delta\Psi = -\epsilon^a T_a \Psi$, $\delta A_\mu = D_\mu \epsilon$, then the Noether current

$$J^\mu_a \;\equiv\; \frac{\partial \mathcal{L}}{\partial(\nabla_\mu \Psi)}\, T_a\, \Psi \;-\; \mathrm{Tr}\!\left(\frac{\partial \mathcal{L}}{\partial A_\mu} T_a\right) \tag{8.17}$$

obeys the Ward identity $D_\mu J^\mu_a \equiv 0$ (up to possible quantum anomalies to be treated in Secs. 8.7–8.8). This conservation law underpins the consistency gates A1/A3 for analytics and numerics (Sec. 8.14).

Example 8.1 (Abelian spectral $U(1)$). For $\Psi = \Phi$ (complex scalar), the internal representation is one–dimensional, $T = i$, and A_μ reduces to a $U(1)$ gauge field. Then $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the spectral current $J^\mu_{\mathrm{spec}} = \mathrm{Im}(\Phi^* \nabla^\mu \Phi)$ of Eq. (8.4) obeys $\nabla_\mu J^\mu_{\mathrm{spec}} = 0$ at the classical level, with quantum corrections constrained by Abelian anomalies (Sec. 8.7).

Example 8.2 (Non–Abelian via coset frames). If the internal fiber admits a moving frame $g(x) \in G$ (Sec. 8.4), then A_μ can be identified with the $\mathfrak{g}$–valued Maurer–Cartan form projected onto the unbroken subalgebra, $A_\mu = \big(g^{-1}\partial_\mu g\big)_{\mathfrak{h}}$, while the coset components furnish Goldstone–like fields whose covariant derivatives define the minimal couplings. The corresponding $F_{\mu\nu}$ follows from the Maurer–Cartan structure equations.

Consistency Gate A1 (Gauge invariance at the operator level). In any discretized or semi–analytical treatment,

$$\boxed{\|D_\mu J^\mu\|_{L^2(\Omega)} \;\le\; \varepsilon_{\mathrm{Ward}} \quad \text{and} \quad \|D_{[\lambda}F_{\mu\nu]}\|_{L^2(\Omega)} \;\le\; \varepsilon_{\mathrm{Bianchi}}}$$

for prescribed tolerances $(\varepsilon_{\mathrm{Ward}}, \varepsilon_{\mathrm{Bianchi}})$ and computational domain Ω. These monitors must be reported alongside observables, cf. Sec. 8.13.

With the geometric origin and algebraic properties of A_μ established, we now develop the Abelian spectral sector in detail and derive its Maxwell limit, including state–dependent kinetic normalization and Ward–identity diagnostics.

8.3 Abelian Spectral Sector: $U(1)_{\mathrm{spec}}$

8.3.1 Spectral current, charge, and conservation

Spectral current. For the complex spectral phase field Φ (Chs. **??**, 3), the covariant derivative $\nabla_\mu = \partial_\mu + \omega_\mu + iA_\mu$ induces the conserved *spectral* $U(1)$ *current*

$$J^\mu_{\mathrm{spec}} \;\equiv\; \mathrm{Im}\big(\Phi^*\nabla^\mu\Phi\big) \;=\; \frac{i}{2}(\Phi^*\nabla^\mu\Phi - \Phi\,(\nabla^\mu\Phi)^*)\,, \qquad \nabla_\mu J^\mu_{\mathrm{spec}} = 0, \tag{8.18}$$

at the classical level (quantum anomalies in Sec. 8.7). The corresponding Noether charge $Q_{\mathrm{spec}} = \int_\Sigma d\Sigma_\mu\, J^\mu_{\mathrm{spec}}$ is invariant under smooth deformations of the Cauchy slice Σ provided fluxes vanish at infinity in the g_{eff} geometry.

Gauge minimal coupling. The minimal $U(1)_{\mathrm{spec}}$ coupling enters through $A_\mu J^\mu_{\mathrm{spec}}$ and the replacement $\partial_\mu \to \partial_\mu + iA_\mu$ in kinetic terms of Φ and other charged slow modes. Ward identities follow from $\delta\Phi = -i\epsilon\,\Phi$, $\delta A_\mu = \partial_\mu\epsilon$.

8.3.2 Effective Maxwell Lagrangian and allowed operators

Kinetic sector with anisotropy. A general parity–even quadratic kinetic Lagrangian consistent with the aether state $\mathcal{S} = (\rho, \eta, T, \chi)$ and the emergent metric g_{eff} reads

$$\mathcal{L}_{\mathrm{kin}}[A] \;=\; -\frac{1}{4}\,Z_\perp(\mathcal{S})\,F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\,\Delta Z(\mathcal{S})\;\hat\chi_\mu\hat\chi^\alpha\,F^{\mu\beta}F_{\alpha\beta}, \tag{8.19}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, unit spacelike $\hat\chi^\mu$ (Sec. 2.5), and positive functions $Z_\perp > 0$, $Z_\parallel \equiv Z_\perp + \Delta Z > 0$ (hyperbolicity below). The second term encodes birefringent anisotropy aligned with $\hat\chi^\mu$; it vanishes in the isotropic limit $\Delta Z \to 0$.

Topological θ–term (CP–odd). A CP–odd density

$$\mathcal{L}_\theta \;=\; -\frac{\theta_{\mathrm{spec}}(\mathcal{S})}{32\pi^2}\,F_{\mu\nu}\,\tilde F^{\mu\nu}, \qquad \tilde F^{\mu\nu} \equiv \tfrac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}, \tag{8.20}$$

is a total derivative for constant θ_{spec} and affects only global/topological sectors (Sec. 8.8); spacetime variation of $\theta_{\mathrm{spec}}(\mathcal{S})$ yields axion–like couplings and birefringent rotation.

Medium screening and Stückelberg option. Integrating out fast modes can endow the longitudinal sector with a *screening* scale $m_{\mathrm{spec}}(\mathcal{S})$ (analog of a Debye mass). Gauge invariance is preserved if screening arises from a Stückelberg phase Σ:

$$\mathcal{L}_{\mathrm{Stk}} \;=\; \frac{1}{2}f_\Sigma^2(\mathcal{S})\,(A_\mu - \partial_\mu\Sigma)^2, \qquad m^2_{\mathrm{spec}} \equiv f_\Sigma^2/Z_\perp, \tag{8.21}$$

which reproduces an effective Yukawa tail in static potentials without explicit Proca breaking (used operationally in Ch. 7).

Portal to spectral curvature (suppressed). At higher order, a kinetic mixing with the non–gauge spectral two–form is allowed,

$$\mathcal{L}_\times \;=\; -\frac{\epsilon_\times(\mathcal{S})}{2}\,F_{\mu\nu}\,F^{\mu\nu}_{\mathrm{spec}}, \tag{8.22}$$

with $\epsilon_\times$ small; see Sec. 8.12 for constraints and field redefinitions that diagonalize the kinetic sector at leading order.

8.3.3 Equations of motion and constitutive relations

Generalized Maxwell equations. Define the excitation tensor

$$H^{\mu\nu} \;\equiv\; -\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}} \;=\; Z_\perp\,F^{\mu\nu} + \Delta Z\,(\hat\chi^\mu\hat\chi_\alpha F^{\nu\alpha} - \hat\chi^\nu\hat\chi_\alpha F^{\mu\alpha}) + \frac{\theta_{\mathrm{spec}}}{8\pi^2}\,\tilde F^{\mu\nu} + \epsilon_\times F^{\mu\nu}_{\mathrm{spec}}. \tag{8.23}$$

The Euler–Lagrange equations and Bianchi identity are

$$\nabla_\mu H^{\mu\nu} \;=\; J^\nu_{\mathrm{spec}}, \qquad \nabla_{[\mu}F_{\nu\rho]} \;=\; 0. \tag{8.24}$$

Rest frame and constitutive tensors. In the local rest frame of u^μ (Sec. 3.4), let $E_i = F_{i0}$, $B^i = \frac{1}{2}\varepsilon^{ijk}F_{jk}$, and similarly $D^i = H^{0i}$, $H_i = \frac{1}{2}\varepsilon_{ijk}H^{jk}$. Then

$$\mathbf{D} = \varepsilon_\perp \mathbf{E}_\perp + \varepsilon_\parallel \mathbf{E}_\parallel + \xi_\theta \mathbf{B}, \qquad \varepsilon_\perp = Z_\perp, \quad \varepsilon_\parallel = Z_\perp + \Delta Z, \tag{8.25}$$

$$\mathbf{H} = \mu_\perp^{-1} \mathbf{B}_\perp + \mu_\parallel^{-1} \mathbf{B}_\parallel - \xi_\theta \mathbf{E}, \qquad \mu_\perp^{-1} = Z_\perp, \quad \mu_\parallel^{-1} = Z_\perp, \tag{8.26}$$

with projectors $\mathbf{V}_\parallel = (\mathbf{V}\cdot\hat{\boldsymbol{\chi}})\hat{\boldsymbol{\chi}}$ and $\mathbf{V}_\perp = \mathbf{V} - \mathbf{V}_\parallel$, and $\xi_\theta \equiv \theta_{\rm spec}/(4\pi^2)$. Equation (8.26) shows that the anisotropy resides in the electric sector via ΔZ at this order; magnetic anisotropy can be included by higher–order operators if required by data (Sec. 8.12).

Gauss and Ampère laws. Decomposing (8.24) in the rest frame,

$$\boldsymbol{\nabla}\cdot\mathbf{D} = \rho_{\rm spec}, \qquad \boldsymbol{\nabla}\times\mathbf{H} - \partial_t\mathbf{D} = \mathbf{J}_{\rm spec}, \qquad \boldsymbol{\nabla}\cdot\mathbf{B} = 0, \quad \boldsymbol{\nabla}\times\mathbf{E} + \partial_t\mathbf{B} = 0. \tag{8.27}$$

8.3.4 Small–amplitude waves, dispersion, and hyperbolicity

Plane waves and phase velocities. For plane waves $\propto e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$ in a homogeneous aether state, two physical transverse polarizations propagate with

$$\omega^2 = c_\perp^2 k_\perp^2 + c_\parallel^2 k_\parallel^2 + m_{\rm spec}^2, \qquad c_\perp^2 \equiv 1, \quad c_\parallel^2 \equiv 1 + \frac{\Delta Z}{Z_\perp}, \tag{8.28}$$

where $k_\parallel = \mathbf{k}\cdot\hat{\boldsymbol{\chi}}$, $k_\perp^2 = |\mathbf{k}|^2 - k_\parallel^2$ and we have normalized $Z_\perp$ into the field. *Hyperbolicity gate A3/A4:*

$$Z_\perp > 0, \qquad Z_\perp + \Delta Z > 0, \qquad c_\parallel^2 \le 1 \ \text{(subluminal bound in } g_{\rm eff}). \tag{8.29}$$

If $m_{\rm spec} \neq 0$ (Stückelberg), longitudinal modes are screened rather than propagating.

Static potentials. For a static point charge Q,

$$\Phi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{Z_\perp k_\perp^2 + (Z_\perp + \Delta Z)k_\parallel^2 + m_{\rm spec}^2} Q, \tag{8.30}$$

which reduces to anisotropic Coulomb ($m_{\rm spec} = 0$) or Yukawa ($m_{\rm spec} > 0$) potentials, consistent with the scattering kernels in Ch. 7.

8.3.5 Running, polarization, and medium screening

Spectral RG for $U(1)_{\rm spec}$. Coarse–graining over spectral cells (Sec. 8.6) yields

$$\mu\frac{d}{d\mu} Z_\perp(\mu;\mathcal{S}) = \beta_\perp\big(Z,\mathcal{S}\big), \qquad \mu\frac{d}{d\mu}\Delta Z(\mu;\mathcal{S}) = \beta_\Delta\big(Z,\mathcal{S}\big), \tag{8.31}$$

with β–functions determined by fluctuations of Φ and structural modes at the chosen background. Matching to long–distance scattering (Ch. 7) fixes $Z_\perp, \Delta Z$ at a reference scale.

Polarization tensor and screening mass. The in–medium polarization $\Pi^{\mu\nu}(\omega,\mathbf{k};\mathcal{S})$ defines

$$m_{\rm spec}^2(\mathcal{S}) \equiv \Pi_L(0,\mathbf{k}\to 0;\mathcal{S}), \qquad \delta Z_\perp \equiv -\left.\frac{\partial\Pi_T}{\partial\omega^2}\right|_{\omega,k\to 0}, \tag{8.32}$$

where L/T denote longitudinal/transverse projectors under $g_{\rm eff}$ with anisotropy set by $\hat{\chi}^\mu$. These quantities can be computed in the homogeneous phases constructed in Ch. **??** and used as inputs for scattering EFT (Ch. 7).

8.3.6 Gauge fixing, Ward diagnostics, and energy positivity

Gauge fixing. We adopt a generalized Lorenz gauge with respect to $g_{\rm eff}$,

$$\mathcal{L}_{\rm gf} = -\frac{1}{2\xi}(\nabla_\mu A^\mu)^2, \tag{8.33}$$

and monitor $\langle(\nabla\cdot A)^2\rangle$ numerically (Sec. 8.13) to ensure stability without contaminating physical polarizations.

Ward monitors (gate A1/A3). Define the Ward residual and Bianchi residual

$$\mathcal{W} \equiv \|\nabla_\mu J^\mu_{\rm spec}\|_{L^2(\Omega)}, \qquad \mathcal{B} \equiv \|\nabla_{[\lambda} F_{\mu\nu]}\|_{L^2(\Omega)}, \tag{8.34}$$

which must remain below user–set tolerances on the computational domain Ω.

Energy positivity (gate A4). The gauge–sector contribution to the stress tensor is

$$T^{\mu\nu}_{(A)} = H^{\mu\alpha}F^{\nu}{}_{\alpha} - g^{\mu\nu}_{\rm eff}\,\mathcal{L}_{\rm kin}, \tag{8.35}$$

which yields, in the rest frame, the positive energy density

$$\mathcal{E}_{(A)} = \tfrac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}\right) = \tfrac{1}{2}\left(\varepsilon_\perp|\mathbf{E}_\perp|^2 + \varepsilon_\parallel|\mathbf{E}_\parallel|^2 + \mu_\perp^{-1}|\mathbf{B}_\perp|^2 + \mu_\parallel^{-1}|\mathbf{B}_\parallel|^2\right) \geq 0, \tag{8.36}$$

provided (8.29) holds.

8.3.7 Summary and interfaces

The Abelian spectral sector is governed by the anisotropic Maxwell Lagrangian (8.19) with optional θ and Stückelberg pieces (8.20)–(8.21). Its equations of motion (8.24) imply constitutive relations (8.25)–(8.26), dispersions (8.28), and static potentials (8.30) that match the kernels used in Ch. 7. Running and screening (8.31)–(8.32) connect homogeneous backgrounds (Ch. **??**) to phenomenology (Sec. 8.12); gauge fixing and Ward monitors (8.33)–(8.34) implement gates A1–A4 (Sec. 8.14). We now turn to the emergence of non–Abelian sectors from internal geometry.

8.4 Non–Abelian Emergence: From Internal Geometry to $SU(2)$ and $SU(3)$

8.4.1 Coset construction and Maurer–Cartan forms

Internal symmetry and spontaneous alignment. Let G denote the isometry group of the fiber metric h singled out by the aether state $\mathcal{S} = (\rho, \eta, T, \chi)$ (cf. Sec. 8.2). The structural dynamics generically select a vacuum manifold $\mathcal{V} \simeq G/H$ via spontaneous alignment of internal frames (e.g. by minimizing a state–dependent potential $U(\mathcal{S})$). The unbroken subgroup H stabilizes the vacuum; fluctuations along G/H furnish Goldstone coordinates.

CCWZ data. Pick a local section $g(x) \in G$; parametrize coset representatives as

$$\xi(x) = e^{\,\pi^a(x)X_a} \in G/H, \qquad g(x) = \xi(x)\,h(x),\ h(x) \in H, \tag{8.37}$$

with broken generators X_a and unbroken T_i ($[T_i, T_j] = if_{ij}{}^k T_k$, $[T_i, X_a] = if_{ia}{}^b X_b$). The Maurer–Cartan one–form splits as

$$\omega_\mu \equiv \xi^{-1}\left(\partial_\mu + A_\mu\right)\xi = \underbrace{E^a_\mu X_a}_{\text{coset vielbein}} + \underbrace{\mathcal{A}^i_\mu T_i}_{H\text{-connection}}, \tag{8.38}$$

where A_μ is the emergent G–connection of Sec. 8.2, pulled back to the coset. Under a local $h(x) \in H$,

$$E_\mu \mapsto h\,E_\mu\,h^{-1}, \qquad \mathcal{A}_\mu \mapsto h\,\mathcal{A}_\mu\,h^{-1} - (\partial_\mu h)\,h^{-1}. \tag{8.39}$$

Gauge–covariant derivatives of matter Ψ transforming in representations of H are built using $\mathcal{A}_\mu$; coset vielbeins E_μ provide the minimal couplings of Goldstone modes.

Maurer–Cartan structure equations. The G-curvature pulled back to the base satisfies

$$\mathcal{F}_{\mu\nu} \equiv \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] = -\left(E_\mu E_\nu\right)_{\mathfrak{h}} + \left(\xi^{-1}F_{\mu\nu}(A)\xi\right)_{\mathfrak{h}}, \tag{8.40}$$

$$\mathcal{D}_{[\mu}E_{\nu]} \equiv \partial_{[\mu}E_{\nu]} + [\mathcal{A}_{[\mu}, E_{\nu]}] = \left(\xi^{-1}F_{\mu\nu}(A)\,\xi\right)_{\mathfrak{g}/\mathfrak{h}}, \tag{8.41}$$

revealing that nonzero $F_{\mu\nu}(A)$ sources both H-curvature and torsion on the coset.

8.4.2 From internal frames to $SU(2)$ and $SU(3)$

Emergent $SU(2)$ from an internal triad. When the aether selects an orthonormal internal triad $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (e.g. eigenvectors of a spectral inertia tensor built from χ_μ), the stabilizer is $SO(3) \simeq SU(2)$. Identifying $H = SU(2)$ yields an $SU(2)$ connection $\mathcal{A}_\mu = \mathcal{A}_\mu^i \frac{\sigma^i}{2}$ that acts on doublet/adjoint representations carried by matter or collective modes.

Emergent $SU(3)$ via an almost complex structure. If the fiber admits a compatible pair (J, ω) (almost complex structure and two–form) with a (3,0)–form Ω such that (J, ω, Ω) is nearly integrable, the stabilizer of the triplet is $SU(3) \subset SO(6)$. Then $H = SU(3)$ and the adjoint octet connection emerges as the H-part of (8.38). This setting naturally couples to triplet/antitriplet matter representations related to complex directions picked by J.

Optional pathway $G_2 \to SU(3)$. In seven internal directions, a stable 3–form φ defines a G_2 structure; under $SU(3) \subset G_2$ the fundamental $\mathbf{7}$ decomposes as $\mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}}$. Alignment of φ along a preferred unit vector leaves an $SU(3)$ structure on the orthogonal 6–plane, yielding the $SU(3)$ connection as the unbroken H in $G/H \simeq G_2/SU(3)$. This pathway supplies a geometric origin for color–like sectors while keeping a singlet direction available for Abelian mixing.

8.4.3 Nonlinear sigma model on G/H and covariant couplings

General CCWZ Lagrangian. To leading order in derivatives, the Goldstone dynamics are

$$\mathcal{L}_{\mathrm{NLSM}} \;=\; \frac{f^2(\mathcal{S})}{2}\,\mathrm{Tr}(E_\mu E^\mu)\,, \qquad E_\mu \equiv E_\mu^a X_a, \tag{8.42}$$

with aether–dependent decay constant $f(\mathcal{S}) > 0$. Matter fields Ψ transforming linearly under H are coupled via $\mathcal{D}_\mu \Psi = \partial_\mu \Psi + \mathcal{A}_\mu \Psi$ and the minimal Goldstone covariantization through E_μ when nonlinearly realized symmetries act on Ψ.

Gauge–invariant mass terms (Higgsless limit). If the gauged subgroup $G_{\mathrm{gauge}} \subset G$ acts on the coset, the combination E_μ contains the would–be Goldstones eaten by G_{gauge} vectors. The term (8.42) then generates gauge boson masses without introducing an elementary scalar,

$$\mathcal{L} \supset \frac{f^2}{2}\,\mathrm{Tr}\Big[(gA_\mu - \xi^{-1}\partial_\mu \xi)^2_{\mathfrak{g}/\mathfrak{h}}\Big]\,, \tag{8.43}$$

generalizing the electroweak chiral Lagrangian.

8.4.4 Electroweak embedding, vacuum alignment, and a massless photon

Embedding. Embed $G_{\mathrm{EW}} = SU(2)_L \times U(1)_Y \subset G$ such that the unbroken subgroup contains $U(1)_{\mathrm{em}}$ generated by $Q = T_L^3 + Y$. Write the Goldstone matrix $U(x) = \xi^2(x)$ (customary CCWZ choice); define

$$D_\mu U \;=\; \partial_\mu U - ig\, W_\mu^i \frac{\sigma^i}{2}\, U + ig'\, U\, B_\mu \frac{\sigma^3}{2}, \tag{8.44}$$

and the LO invariant

$$\mathcal{L}_{\mathrm{EW}}^{(2)} \;=\; \frac{f^2}{4}\,\mathrm{Tr}\big(D_\mu U^\dagger D^\mu U\big)\,. \tag{8.45}$$

Vector masses and custodial symmetry. Expanding around $U = \mathbf{1}$ gives

$$m_W^2 \;=\; \frac{1}{4}\, g^2\, f^2, \qquad m_Z^2 \;=\; \frac{1}{4}\,(g^2 + g'^2)\, f^2, \qquad m_\gamma = 0. \tag{8.46}$$

If the unbroken H contains a custodial $SU(2)_V$, the tree–level ρ parameter satisfies $\rho \equiv m_W^2/(m_Z^2 \cos^2\theta_W) = 1$, protected against large corrections. In the aether, $f = f(\mathcal{S})$ and the gauge kinetic normalizations $Z_i(\mathcal{S})$ (cf. Sec. 8.3) induce mild state dependence in (8.46) that is constrained by precision data (Sec. 8.12).

Vacuum alignment problem. State–dependent explicit breakings (e.g. couplings to η, ρ) generate a potential $V(U;\mathcal{S})$ selecting a vacuum orientation U_0. Massless photon requires $\mathrm{Tr}(Q\,U_0 T^a U_0^\dagger) = 0$ for all broken T^a; practical sufficient conditions include aligning U_0 within the coset such that $[Q, U_0] = 0$. We will construct V from integrating out heavy spectral modes (Sec. 8.5) and analyze stability.

8.4.5 Couplings to matter multiplets and spectral Yukawa sector

Linear H-representations and CCWZ spurions. Let ψ_L and ψ_R transform in (possibly different) linear representations of H. A Yukawa–like operator invariant under G is built with spurions $\mathcal{Y}$ in appropriate tensor products:

$$\mathcal{L}_{\mathrm{Yuk}}^{\mathrm{spec}} \;=\; -\bar{\psi}_L\,\mathcal{Y}(\mathcal{S})\,\mathcal{U}[\xi]\,\psi_R \;+\; \mathrm{h.c.}, \qquad \mathcal{U}[\xi] \in \mathrm{rep}(G/H), \tag{8.47}$$

where $\mathcal{U}$ carries the nonlinear transformation law that compensates $g(x)$. The *spectral* character enters through $\mathcal{Y}(\mathcal{S})$ and the internal geometry that fixes allowed tensor structures. After alignment, (8.47) yields fermion masses and mixings, correlated with distances/phases on the fiber (see Ch. 5 and Sec. 6.12).

Selection rules. Berry/FR phases and fiber topology restrict the allowed entries of $\mathcal{Y}$: matrix elements connecting states with incompatible monopole/linking charges are suppressed or vanish (cf. Chs. 6, 7). This provides a geometric origin for flavor hierarchies.

8.4.6 Topological terms: seeds for WZW/CS

Anomalous functional. On G/H with $\pi_5(G/H) \neq 0$, there exists a Wess–Zumino–Witten functional

$$S_{\mathrm{WZW}}[U] \;=\; k\int_{M_5} \omega_5(U^{-1}dU, \mathcal{A}), \qquad k \in \mathbb{Z}, \tag{8.48}$$

whose boundary variation reproduces consistent anomalies of fermion measure. In the aether, k can be tied to spectral indices or defect zero modes (Sec. 8.8). Coupling to the emergent gauge field $\mathcal{A}$ fixes otherwise ambiguous coefficients in the parity–odd sector.

Chern–Simons inflow. Equivalently, a 5D Chern–Simons form for the G connection, $S_{\mathrm{CS}} \propto \int_{M_5} \mathrm{Tr}\big(A\,dA + \frac{2}{3}A^3\big)^2$, provides inflow that cancels boundary anomalies on soliton worldvolumes, consistent with FR/Berry data.

8.4.7 Summary and readiness checks

The non–Abelian sector emerges from the vertical part of the structural connection projected onto fiber isometries. The CCWZ construction on G/H supplies the covariant building blocks $(E_\mu, \mathcal{A}_\mu)$, from which the NLSM action (8.42), Higgsless mass term (8.43), and electroweak embedding (8.45) follow. Vacuum alignment conditions guarantee an unbroken $U(1)_{\mathrm{em}}$ and custodial protection, while spectral Yukawa structures (8.47) implement geometrically filtered fermion masses. Topological terms (8.48) seed anomaly inflow to be quantified in Sec. 8.8.

Consistency gates for this section.

- **A1 (gauge)**: Ward identities with $\mathcal{A}_\mu$ must hold in NLSM simulations.
- **A2 (anomaly)**: WZW/CS coefficients set by microscopic index; net anomaly cancels.
- **A3/A4 (causality/positivity)**: $f^2(\mathcal{S}) > 0$; subluminal Goldstone velocities under g_{eff}.
- **Alignment**: $[Q, U_0] = 0$; $m_\gamma = 0$ and $\rho \simeq 1$ at tree level.

8.4.8 Interlude: Why This Primary Internal Group G? A Dynamical Selection Principle

The formalism above *assumes* the existence of a fiber isometry group G whose vertical connection A_μ becomes the emergent gauge field. This raises a foundational question: *what fixes G in the first place?* Here we articulate a concrete selection framework, rooted in the aether dynamics, that turns "which G?" into a variational/renormalization problem rather than an external postulate.

Order parameters and symmetry of the fiber. Let h be the fiber metric and $\mathcal{O}$ a set of internal tensorial order parameters built from the structural state $\mathcal{S} = (\rho, \eta, T, \chi)$:

$$\mathcal{O} \;\in\; \{\, \hat{\chi}^A,\;\; J^A{}_B,\;\; \omega_{AB},\;\; \Omega_{ABC},\;\; \varphi_{ABC},\;\; \dots \,\}.$$

The stabilizer $G = \mathrm{Iso}(h, \mathcal{O})$ is the largest subgroup of the fiber diffeomorphisms that preserves $(h, \mathcal{O})$. Different choices (or phases) of $\mathcal{O}$ select different G's (e.g. $SO(3) \simeq SU(2)$ for a triad, $SU(3)$ for a compatible (J, ω, Ω), G_2 for a stable three–form φ in seven internal directions).

Selection as minimization of a state functional. We posit that G is dynamically selected by minimizing a coarse–grained "cost" functional over admissible $(h, \mathcal{O})$:

$$\mathcal{C}[G; \mathcal{S}] \;=\; \alpha\, \sigma_{\mathrm{diss}}[G; \mathcal{S}] \;+\; \beta\, \mathcal{A}_{\mathrm{tot}}[G] \;+\; \gamma\, \mathcal{M}_{\mathrm{FR}}[G] \;+\; \delta\, \mathcal{H}_{\mathrm{topo}}[G/H] \;-\; \zeta\, \dim G, \tag{8.49}$$

with positive weights $(\alpha, \beta, \gamma, \delta, \zeta)$ depending smoothly on $\mathcal{S}$. Terms:

- σ_{diss}: entropy production rate induced by gauge–coupled slow modes (cf. Sec. **??**); *least–anisotropy* favors higher symmetry if compatible with stability.
- $\mathcal{A}_{\mathrm{tot}}$: norm of gauge/mixed/gravitational anomalies generated by the quantized soliton spectrum (Ch. 6); anomaly freedom or exact inflow realizability drives $\mathcal{A}_{\mathrm{tot}} \to 0$.
- $\mathcal{M}_{\mathrm{FR}}$: Finkelstein–Rubinstein (FR) mismatch functional penalizing groups for which the required global signs/statistics cannot be realized on the soliton configuration space.
- $\mathcal{H}_{\mathrm{topo}}$: homotopy adequacy of G/H (e.g. nontrivial π_3 for Skyrmion–like charges, π_1 for FR signs, π_2 for monopoles) needed by the observed spectrum.
- $-\zeta \dim G$: a symmetry bonus encoding the heuristic "maximal symmetry under constraints," counterbalanced by the other penalties.

The selected group is any (possibly metastable) minimizer:

$$G_*(\mathcal{S}) \;\in\; \arg \min_{G \in \mathfrak{G}_{\mathrm{admissible}}} \mathcal{C}[G; \mathcal{S}],$$

where $\mathfrak{G}_{\mathrm{admissible}}$ includes only compact groups compatible with positivity/hyperbolicity gates and with a viable representation content for matter multiplets.

Spectral RG viewpoint. Equivalently, define a spectral coarse–graining (Sec. 8.6) that flows $(h, \mathcal{O})$ toward infrared fixed points $(h^*, \mathcal{O}^*)$. The emergent symmetry G is the automorphism group of that fixed point:

$$\mu \frac{d}{d\mu}(h, \mathcal{O}) = \beta_{h,\mathcal{O}} \;\;\Rightarrow\;\; (h, \mathcal{O}) \to (h^*, \mathcal{O}^*)\,, \quad G = \mathrm{Iso}(h^*, \mathcal{O}^*).$$

Fixed points with enhanced symmetry are *attractive* provided they simultaneously satisfy the anomaly and FR/topology constraints above. This makes "why this G?" an RG universality statement rather than a bare assumption.

Variational prototypes (examples).

- *$SU(3)$ prototype (six internal directions).* Let (J, ω, Ω) be an almost–Hermitian triplet with compatibility constraints. Consider

$$\mathcal{F}_{SU(3)} = \int \sqrt{-g_{\text{eff}}} \Big[\lambda_1 \|\nabla J\|^2 + \lambda_2 \|\nabla \omega\|^2 + \lambda_3 \|\nabla \Omega\|^2 + V(J, \omega, \Omega; \mathcal{S})\Big],$$

  where V penalizes departures from $J^2 = -\mathbb{I}$ and J–compatibility of (ω, Ω). Minima with $\nabla J = \nabla \omega = \nabla \Omega = 0$ (up to torsion classes allowed by $\mathcal{S}$) have stabilizer $SU(3)$.

- *G_2 prototype (seven internal directions).* For a stable 3–form φ and its dual $\psi = *_h \varphi$,

$$\mathcal{F}_{G_2} = \int \sqrt{-g_{\text{eff}}} \Big[\kappa_1 \|\nabla \varphi\|^2 + \kappa_2 \mathcal{I}_4(\varphi) + V(\varphi; \mathcal{S})\Big],$$

  with $\mathcal{I}_4$ a quartic G_2–invariant built from φ, ψ. Critical points with $\nabla \varphi = 0$ possess G_2 holonomy; small torsion classes correspond to nearly–G_2 phases.

Phenomenological filters that further restrict G. Beyond the intrinsic dynamical selection, low–energy data impose sharp filters:

(F1) **Massless photon & custodial protection:** existence of an unbroken $U(1)_{\text{em}} \subset G$ and a (near-)custodial $SU(2)_V$ to ensure $\rho \simeq 1$ (Sec. 8.4.4).

(F2) **Chirality and Yukawas:** availability of complex reps and spurion structures to generate hierarchical fermion masses with FR/Berry selection (Sec. 8.4.5; Ch. 5).

(F3) **Anomaly cancellation/inflow:** existence of an anomaly–free assignment or exact inflow via WZW/CS with quantized levels compatible with the soliton spectrum (Secs. 8.7–8.8).

(F4) **Topological adequacy:** G/H must support the homotopy required by observed soliton charges/statistics (Chs. **??**–6).

Numerical protocol (practical selection). Scan a finite candidate set $\{G\}$ (e.g. $SU(2)$, $SU(3)$, $SU(2)\times U(1)$, $SU(3)\times SU(2)\times U(1)$, G_2, ...). For each:

(P1) Minimize a discretized $\mathcal{F}_G$ (as above) at fixed background $\mathcal{S}$ to produce $(h^*, \mathcal{O}^*)$, evaluate $\dim G$, σ_{diss}, anomaly/inflow score, FR/topology score.

(P2) Compute $\mathcal{C}[G; \mathcal{S}]$ in (8.49); retain minimizers within tolerance.

(P3) Enforce filters (F1–F4) and gates of Sec. 8.14; discard failing candidates.

Empirically, for low–energy homogeneous phases with mild anisotropy, $G = SU(3) \times SU(2) \times U(1)$ (or a parent such as G_2 flowing to it) tends to satisfy (F1–F4) while admitting anomaly–safe inflow and realistic Yukawa textures; highly symmetric parents (e.g. G_2) may be preferred at higher densities/temperatures and then flow to the SM–like subgroup under spectral RG.

This interlude reframes the "origin of G" as an outcome of the aether's internal variational dynamics and spectral renormalization. The concrete implementations (prototypes, filters, and protocol) are designed to be embedded directly into Secs. 8.5–8.12, where the coefficients and constraints are computable and testable.

8.5 Yang–Mills Dynamics from Coarse–Graining

8.5.1 Coarse–grained generating functional and effective action

Setup and normalization. Let $A_\mu = A_\mu^a T_a$ be the emergent non–Abelian connection of Sec. 8.4, with generators T_a in a fixed representation $\mathcal{R}$ obeying

$$[T_a, T_b] = i f_{ab}{}^c T_c, \qquad \operatorname{Tr}(T_a T_b) = \tfrac{1}{2}\, \delta_{ab}. \tag{8.50}$$

Indices are raised/lowered with the emergent metric g_{eff}; all covariant derivatives and contractions are defined accordingly.

Coarse–graining. Split the full set of fields into (i) *slow* modes $\{A_\mu, \Psi, U\}$ comprising the gauge connection, matter multiplets, and coset Goldstones ($U = \exp i\pi^a X_a$), and (ii) *fast* structural fluctuations $\{\delta\mathcal{S}\}$ around a homogeneous background of the aether state $\mathcal{S} = (\rho, \eta, T, \chi)$. The Wilsonian coarse–grained generating functional at scale μ is

$$e^{\,i\Gamma_\mu[A,\Psi,U;\mathcal{S}]} \;=\; \int_{\Lambda>\,|k|>\,\mu} \mathcal{D}(\delta\mathcal{S})\; \exp\!\left\{ i \int d^4x \sqrt{-g_{\rm eff}}\; \mathcal{L}\big[A,\Psi,U;\mathcal{S}+\delta\mathcal{S}\big] \right\}. \tag{8.51}$$

Gauge invariance follows from the vertical projector construction (Sec. 8.2); the measure can be chosen to preserve the BRST symmetry (Sec. 8.5.3) up to local anomalies, which will be handled by WZW/CS inflow (Secs. 8.7–8.8).

Quadratic gauge sector at leading order. To leading order in derivatives and in the adjoint fields, the effective action contains an anisotropic Yang–Mills term with state–dependent kinetic tensors:

$$\mathcal{L}^{(2)}_{\rm YM} \;=\; -\frac{1}{4}\, Z_\perp(\mathcal{S})\, \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \;-\; \frac{1}{2}\, \Delta Z(\mathcal{S})\, \hat\chi_\mu \hat\chi^\alpha\, \mathrm{Tr}\big(F^{\mu\beta}F_{\alpha\beta}\big)\,, \tag{8.52}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ and unit spacelike $\hat\chi^\mu$ selected by the aether (Sec. 2.5). Hyperbolicity/positivity will require $Z_\perp > 0$ and $Z_\perp + \Delta Z > 0$ (Sec. 8.5.4).

Interaction terms and CP–odd sector. Higher–order terms consistent with BRST and discrete symmetries appear with state–dependent coefficients:

$$\mathcal{L}^{\rm int}_{\rm YM} = -\frac{1}{2}\, Z_3(\mathcal{S})\, \mathrm{Tr}(F_{\mu\nu}[A^\mu, A^\nu]) - \frac{1}{4}\, Z_4(\mathcal{S})\, \mathrm{Tr}\big([A_\mu, A_\nu][A^\mu, A^\nu]\big) + \cdots\,, \tag{8.53}$$

$$\mathcal{L}_\theta = -\,\frac{\theta_{\rm NA}(\mathcal{S})}{32\pi^2}\, \mathrm{Tr}\big(F_{\mu\nu}\tilde F^{\mu\nu}\big)\,, \qquad \tilde F^{\mu\nu} \equiv \tfrac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{8.54}$$

When coset directions are gauged, mass terms for broken vectors are generated by the NLSM invariant ("Higgsless" realization) of Sec. 8.4.3:

$$\mathcal{L}_{\rm mass} \;=\; \frac{f^2(\mathcal{S})}{2}\, \mathrm{Tr}\big[(E_\mu)_{\mathfrak{g}/\mathfrak{h}}\,(E^\mu)_{\mathfrak{g}/\mathfrak{h}}\big]\,, \qquad E_\mu = \xi^{-1}(\partial_\mu + A_\mu)\xi. \tag{8.55}$$

8.5.2 Equations of motion and constitutive tensor

Non–Abelian excitations. Define the excitation tensor in the adjoint,

$$H^{\mu\nu\,a} \;\equiv\; -\,\frac{\partial\mathcal{L}}{\partial F^a_{\mu\nu}} \;=\; Z_\perp\, F^{\mu\nu\,a} + \Delta Z\,(\hat\chi^\mu\hat\chi_\alpha F^{\nu\alpha\,a} - \hat\chi^\nu\hat\chi_\alpha F^{\mu\alpha\,a}) + \frac{\theta_{\rm NA}}{8\pi^2}\,\tilde F^{\mu\nu\,a} + \cdots\,. \tag{8.56}$$

The Yang–Mills equations and Bianchi identity read

$$D_\mu H^{\mu\nu} \;=\; J^\nu, \qquad D_{[\mu}F_{\nu\rho]} \;=\; 0, \tag{8.57}$$

where $D_\mu(\cdot) = \partial_\mu(\cdot) + [A_\mu, (\cdot)]$ and J^ν collects matter, Goldstone, and structural currents consistent with Sec. 8.4.

Wave propagation. In a homogeneous background one finds transverse polarizations with dispersion

$$\omega^2 \;=\; c_\perp^2 k_\perp^2 + c_\parallel^2 k_\parallel^2 + m_{\rm br}^2, \qquad c_\perp^2 \equiv 1,\;\; c_\parallel^2 \equiv 1 + \frac{\Delta Z}{Z_\perp}, \tag{8.58}$$

where $m_{\rm br}$ arises for broken generators through (8.55). Longitudinal screening may appear from in–medium polarization (Sec. 8.6).

8.5.3 BRST symmetry, gauge fixing, and ghosts

BRST algebra (background–field form). Decompose $A_\mu = \bar A_\mu + a_\mu$ into background and quantum fields. Let $c, \bar c$ be the ghost/antighost and B the Nakanishi–Lautrup field. The BRST operator s acts as

$$s\,a_\mu = D_\mu[\bar A]c + [a_\mu, c], \quad s\,c = -\tfrac{1}{2}[c,c], \quad s\,\bar c = iB, \quad s\,B = 0, \tag{8.59}$$

with $s\,\bar A_\mu = 0$ in the background gauge. Nilpotency $s^2 = 0$ holds off shell.

Gauge fixing and ghost Lagrangian. Adopt the background Lorenz gauge with respect to $g_{\rm eff}$,

$$\mathcal{L}_{\rm gf} \;=\; -\frac{Z_\perp}{2\xi}\left(D^\mu[\bar A]a_\mu\right)^2, \qquad \mathcal{L}_{\rm gh} \;=\; \bar c\Big(-D^\mu[\bar A]D_\mu[\bar A+a]\Big)c, \tag{8.60}$$

so that the quadratic operator for a_μ is transverse in the background and the one–loop effective action $\Gamma_\mu[\bar A]$ is manifestly gauge invariant. The anisotropy ΔZ modifies the kinetic operator but not the BRST algebra.

Slavnov–Taylor identities. BRST invariance of the gauge–fixed functional yields

$$\mathcal{S}(\Gamma)\equiv\int d^4x\sqrt{-g_{\rm eff}}\,{\rm Tr}\left(\frac{\delta\Gamma}{\delta a^\mu}\frac{\delta\Gamma}{\delta K_\mu}+\frac{\delta\Gamma}{\delta c}\frac{\delta\Gamma}{\delta L}+iB\frac{\delta\Gamma}{\delta\bar c}\right)=0, \tag{8.61}$$

with sources (K_μ, L) for BRST variations. These identities constrain the running of Z's and ensure gauge–parameter independence of observables (consistency gate A1).

8.5.4 Positivity, hyperbolicity, and energy–momentum tensor

Energy density and gates. The gauge contribution to the stress tensor follows from $T^{\mu\nu}_{(A)} = H^{\mu\alpha\,a}F^{\nu}{}_{\alpha\,a} - g^{\mu\nu}_{\rm eff}\,\mathcal{L}^{(2)}_{\rm YM}$, giving in the rest frame

$$\mathcal{E}_{(A)}=\tfrac12\sum_a\Big(\varepsilon_\perp|\mathbf{E}^a_\perp|^2+\varepsilon_\parallel|\mathbf{E}^a_\parallel|^2+\mu_\perp^{-1}|\mathbf{B}^a_\perp|^2+\mu_\parallel^{-1}|\mathbf{B}^a_\parallel|^2\Big)\geq 0, \tag{8.62}$$

with $\varepsilon_\perp=\mu_\perp^{-1}=Z_\perp>0$ and $\varepsilon_\parallel=Z_\perp+\Delta Z>0$, $\mu_\parallel^{-1}=Z_\perp$. *Gate A4:* positivity of the quadratic form; *Gate A3:* subluminal bound $c_\parallel^2\leq 1$ in (8.58).

No negative–norm modes. Compactness of the gauge algebra (Sec. 8.2) and positivity of Z's prevent ghostly kinetics in the physical sector. Gauge–dependent negative norms are confined to the Faddeev–Popov ghosts, which cancel unphysical polarizations in BRST–invariant observables.

8.5.5 Renormalization, matching, and decoupling

Running couplings and wavefunction factors. Define renormalized fields and coupling by

$$A^{(R)}_\mu=Z_A^{1/2}A_\mu, \qquad g_R=Z_g^{-1}g, \qquad Z_A\equiv Z_\perp, \quad Z_g=1\ \text{(background–field scheme)}, \tag{8.63}$$

so that the β–function is read from the background two–point function: $\mu\,dg_R/d\mu=\beta(g_R;\mathcal{S})$, and similarly for ΔZ via the anisotropic part of the vacuum polarization.

Matching to scattering and lattice–like protocols. At a reference scale μ_0 set by soliton sizes or by an external probe momentum (Ch. 7), fix $Z_\perp(\mu_0), \Delta Z(\mu_0), g_R(\mu_0)$ from (i) elastic phase shifts in channels with minimal Goldstone contamination, (ii) static potentials (Born regime), and (iii) radiative tails (soft theorems of Sec. 6.15). Numerical protocols in Sec. 8.13 provide finite–volume corrections and Ward/Bianchi monitors.

Decoupling of massive coset vectors. For broken generators, the mass $m_{\rm br}\sim g_R f(\mathcal{S})$ implies decoupling at $\mu\ll m_{\rm br}$ (Appelquist–Carazzone), leaving an unbroken subgroup with renormalized Z's inherited from integrating out the heavy sector; threshold corrections are computable in the background–field method.

8.5.6 Summary and consistency gates

The non–Abelian gauge dynamics induced by coarse–graining take the Yang–Mills form with state–dependent, potentially anisotropic kinetic tensors (8.52), CP–odd sector (8.54), and Higgsless masses (8.55) for broken directions. BRST symmetry (8.59) with background gauge fixing (8.60) enforces Slavnov–Taylor identities (8.61) and controls renormalization (8.63). Positivity and hyperbolicity are guaranteed by $Z_\perp>0$, $Z_\perp+\Delta Z>0$ and $c_\parallel^2\leq 1$ (gates A3/A4). Matching conditions connect the gauge sector to scattering observables and numerical pipelines, preparing the ground for spectral RG (Sec. 8.6) and anomaly analysis (Secs. 8.7–8.9).

8.6 Spectral Renormalization Group and Medium Screening

8.6.1 Spectral cells, scaling variables, and anisotropic dilatations

Spectral coarse cells. We partition fluctuations into spectral cells using the aether rest frame u^μ and the anisotropy axis $\hat{\chi}^\mu$ (Secs. 2.5, 3.4). Writing spatial momenta as $\mathbf{k} = \mathbf{k}_\perp + k_\parallel \hat{\boldsymbol{\chi}}$, define shells

$$\mathcal{S}_\Lambda \;=\; \big\{(\omega,\mathbf{k}) \;:\; \Lambda/b < \kappa(\omega,\mathbf{k}) < \Lambda\big\}, \qquad \kappa^2 \equiv k_\perp^2 + \zeta_\parallel(\mathcal{S})\, k_\parallel^2 + \zeta_0(\mathcal{S})\,\omega^2, \tag{8.64}$$

with positive state–dependent weights $\zeta_\parallel, \zeta_0$ chosen so that free two–point propagators are approximately isotropic in κ at the homogeneous background.

Anisotropic dilatations. A convenient Wilson step integrates $\mathcal{S}_\Lambda$ and rescales

$$x_\perp \to b\, x_\perp, \qquad x_\parallel \to b^{1+\delta_\chi}\, x_\parallel, \qquad t \to b^z\, t, \tag{8.65}$$

with dynamical exponent z and anisotropy exponent δ_χ fixed so that the quadratic gauge and matter kinetic terms retain canonical form in terms of renormalized fields. The emergent metric $g_{\rm eff}$ provides the inner products used to raise/lower indices throughout the flow.

Operator dimensions. Under (8.65), a local operator $\mathcal{O}$ has engineering dimension $\Delta[\mathcal{O}] = d_t\, z + d_\perp + (1+\delta_\chi)\, d_\parallel - \sum_{\text{fields}} N_\phi\, \Delta[\phi]$, with $(d_t, d_\perp, d_\parallel)$ the numbers of time, transverse, and longitudinal derivatives. We classify operators as relevant/marginal/irrelevant by $\Delta[\mathcal{O}] \lessgtr 4$ in four spacetime dimensions.

8.6.2 Wilsonian flow with background–field gauge and ST control

Polchinski–type flow. Let Γ_μ be the 1PI effective action at coarse scale μ defined by (8.51). Introducing a regulator $\mathcal{R}_\mu$ that suppresses modes with $\kappa \ll \mu$, the flow satisfies

$$\partial_\mu \Gamma_\mu \;=\; \frac{i}{2}\,\mathrm{STr}\Big[\big(\Gamma_\mu^{(2)} + \mathcal{R}_\mu\big)^{-1}\partial_\mu \mathcal{R}_\mu\Big], \tag{8.66}$$

in the background–field scheme. Here STr includes minus signs for ghosts; $\Gamma_\mu^{(2)}$ is the Hessian w.r.t. (quantum) fields. Choosing $\mathcal{R}_\mu$ as a function of the covariant Laplacian preserves background gauge invariance, guaranteeing Slavnov–Taylor (ST) identities along the flow.

ST identities along the flow. Denoting the ST functional by $\mathcal{S}(\Gamma_\mu)$ (Eq. (8.61)), we impose

$$\mathcal{S}(\Gamma_\mu) = 0, \qquad \partial_\mu \mathcal{S}(\Gamma_\mu) = 0, \tag{8.67}$$

which holds if $\mathcal{R}_\mu$ transforms as a BRST scalar built from background covariant operators. This ensures gauge–parameter independence of physical β–functions (Gate A1).

8.6.3 One–loop running: gauge, anisotropy, and Yukawa sectors

Gauge coupling. For a compact simple factor with generators normalized as in (8.50), the background–field two–point function yields

$$\mu\frac{dg}{d\mu} \;=\; \beta_g(g;\mathcal{S}) \;=\; -\frac{g^3}{16\pi^2}\left[\frac{11}{3}C_A - \frac{4}{3}\,T_R\, n_f^{\rm eff}(\mathcal{S}) - \frac{1}{6}\,T_R\, n_s^{\rm eff}(\mathcal{S})\right] + \mathcal{O}(g^5), \tag{8.68}$$

with C_A the adjoint Casimir and T_R the index of the matter representation. Medium effects enter through effective multiplicities $n_{f,s}^{\rm eff}(\mathcal{S})$ that depend on spectral gaps, screening masses, and occupation of slow modes in the aether background.

Wavefunction anisotropy. Decomposing the vacuum polarization into transverse/longitudinal projectors relative to $\hat{\chi}^\mu$ gives

$$\mu\frac{dZ_\perp}{d\mu} = \beta_\perp\big(g, Z_\perp, \Delta Z; \mathcal{S}\big), \qquad \mu\frac{d\Delta Z}{d\mu} = \beta_\Delta\big(g, Z_\perp, \Delta Z; \mathcal{S}\big). \tag{8.69}$$

At one loop, $\beta_\Delta \propto -\gamma_\chi\,\Delta Z + \mathcal{O}(g^4)$ with $\gamma_\chi > 0$ for isotropy–attractive flows; couplings to the order parameter $\hat{\chi}$ via $f(\mathcal{S})$ can stabilize a small nonzero anisotropy (fixed line) when the background maintains a preferred direction.

Spectral Yukawas and mass parameters. For spectral Yukawa matrices $\mathcal{Y}$ of Sec. 8.4.5,

$$\mu\frac{d\mathcal{Y}}{d\mu} \;=\; \mathcal{Y}\,[\gamma_R(\mathcal{S}) + \gamma_L(\mathcal{S})] - \frac{3}{16\pi^2}g^2\{C_2(R_L) + C_2(R_R)\}\,\mathcal{Y} + \frac{1}{16\pi^2}\big(a\,\mathcal{Y}\mathcal{Y}^\dagger\mathcal{Y} + \cdots\big), \tag{8.70}$$

where $\gamma_{L,R}$ are wavefunction anomalous dimensions (medium dependent), C_2 the quadratic Casimir, and a numerical coefficients set by the representation content. The FR/Berry selection rules nullify specific entries throughout the flow, protecting texture zeros.

8.6.4 Medium polarization and screening masses

Polarization tensor. Define longitudinal/transverse projectors in the rest frame of u^μ: $P_L^{ij} = k^i k^j/|\mathbf{k}|^2$, $P_T^{ij} = \delta^{ij} - P_L^{ij}$, and anisotropic decompositions using $\hat{\boldsymbol{\chi}}$. The retarded polarization is

$$\Pi_{ab}^{\mu\nu}(\omega, \mathbf{k}; \mathcal{S}) \;=\; i\int d^4x\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}\big\langle [J_a^\mu(x), J_b^\nu(0)]\big\rangle_{\mathcal{S}}, \tag{8.71}$$

with Kubo relations relating Π to current–current correlators of the slow sector.

Screening masses. Debye–like screening is defined by

$$m_D^2(\mathcal{S}) \;\equiv\; \Pi_L(\omega{=}0, \mathbf{k}{\to}0; \mathcal{S}), \qquad \delta Z_\perp \;\equiv\; -\left.\frac{\partial \Pi_T}{\partial \omega^2}\right|_{\omega,k\to 0}, \tag{8.72}$$

and similarly along $\hat{\boldsymbol{\chi}}$ to extract an anisotropic $m_{D,\parallel}^2$. For Abelian $U(1)_{\rm spec}$ and weak coupling,

$$m_D^2 \;\simeq\; \frac{g^2}{\pi^2}\int_0^\infty dp\, p^2\left(-\frac{\partial f_{\rm eff}}{\partial \epsilon_p}\right), \qquad \epsilon_p^2 = c_\perp^2 p_\perp^2 + c_\parallel^2 p_\parallel^2 + m_*^2(\mathcal{S}), \tag{8.73}$$

where $f_{\rm eff}$ is an effective occupation (set by the aether state) and m_* a medium gap. In the non–Abelian case, (8.73) generalizes with group factors and possible Landau damping that enters the imaginary part of $\Pi^{\mu\nu}$.

Stückelberg vs. dynamical screening. A nonzero $m_{\rm spec}$ in Sec. 8.3 may arise either from a Stückelberg mode (hard, gauge invariant) or from dynamical polarization (soft, frequency/momentum dependent). Matching to static potentials (Ch. 7) distinguishes them: Stückelberg yields pure Yukawa falloff; dynamical screening induces frequency dependence and dissipative tails via $\mathrm{Im}\,\Pi^{\mu\nu}$.

8.6.5 Flow of anisotropy and fixed points

Coupled flow. Introduce $a_\chi \equiv \Delta Z/Z_\perp$ and the ratio of characteristic speeds $c_\parallel^2 = 1 + a_\chi$. A minimal closed system reads

$$\mu\frac{dg}{d\mu} = \beta_g(g; \mathcal{S}), \qquad \mu\frac{da_\chi}{d\mu} = -\gamma_\chi(g; \mathcal{S})\,a_\chi + \lambda_\chi(g; \mathcal{S})\,a_\chi^2 + \cdots, \tag{8.74}$$

where $\gamma_\chi > 0$ drives $a_\chi \to 0$ (isotropic fixed point) in the absence of a persistent order parameter. If $\hat{\chi}^\mu$ is locked by the structural sector (finite expectation), a small nonzero a_χ^* can be stabilized by the $\lambda_\chi a_\chi^2$ term, leading to controlled, universal birefringence.

Stability and gates. Throughout the flow we enforce (Gate A3/A4):

$$Z_\perp(\mu) > 0, \qquad Z_\perp(\mu) + \Delta Z(\mu) > 0, \qquad c_\parallel^2(\mu) \le 1, \tag{8.75}$$

and monitor the ST residual and Bianchi residual to ensure gauge consistency of the regulator (Sec. 8.13).

8.6.6 Thresholds, decoupling, and matching

Heavy vector thresholds. Coset vectors with $m_{\rm br}(\mathcal{S}) \sim g f(\mathcal{S})$ decouple for $\mu \ll m_{\rm br}$. Across $\mu = m_{\rm br}$ we impose matching conditions

$$g_{\rm low}(\mu_0) = g_{\rm high}(\mu_0)\,\big[1 + \Delta_{\rm th}(g, f, \mathcal{S})\big], \qquad Z_{\perp,\rm low}(\mu_0) = Z_{\perp,\rm high}(\mu_0) + \delta Z_{\rm th}, \tag{8.76}$$

with $\Delta_{\rm th}, \delta Z_{\rm th}$ the finite threshold pieces from integrating out heavy fields in the background–field formalism.

Soliton gap thresholds. For processes dominated by a soliton species with gap $\omega_0(Q)$ (Ch. 6), the worldline EFT modes decouple for $\mu \ll \omega_0$, leaving contact operators whose Wilson coefficients run only via gauge/metric sectors. Matching to phase shifts from Ch. 7 fixes these coefficients at $\mu \simeq \omega_0$.

8.6.7 Compact SRG system and diagnostics

State–dependent SRG ODEs. Collecting the running couplings in $\mathbf{c}(\mu) = \{g_i, Z_{\perp,i}, \Delta Z_i, f, \theta_{\rm spec}, \theta_{\rm NA}, m_D^2, \ldots\}$, the SRG takes the compact form

$$\mu \frac{d\mathbf{c}}{d\mu} \;=\; \boldsymbol{\beta}\big(\mathbf{c}; \mathcal{S}\big), \qquad \boldsymbol{\beta} = \boldsymbol{\beta}^{(1)} + \boldsymbol{\beta}^{(2)} + \cdots, \tag{8.77}$$

with loop expansion indicated and $\mathcal{S}$ held fixed for quasi–static backgrounds. In slowly varying environments, promote $\mathcal{S} \to \mathcal{S}(x)$ and adopt local RG with adiabatic corrections $\propto \nabla \mathcal{S}$.

Ward/Bianchi monitors along the flow. At each RG step:

$$\mathcal{W}_\mu \equiv \|D_\alpha J^\alpha\|_{L^2(\Omega)} \overset{!}{<} \varepsilon_{\rm Ward}, \qquad \mathcal{B}_\mu \equiv \|D_{[\lambda} F_{\mu\nu]}\|_{L^2(\Omega)} \overset{!}{<} \varepsilon_{\rm Bianchi}.$$

Violations signal regulator mismatch or truncation artefacts; adjust $\mathcal{R}_\mu$ or include next operators in the truncation.

Numerical protocol (outline).

(N1) Choose background $\mathcal{S}$ and fix (z, δ_χ) by quadratic normalization.

(N2) Initialize $\mathbf{c}(\mu_0)$ from matching to scattering/propagators (Chs. 6, 7).

(N3) Integrate (8.77) using adaptive ODE with constraints (8.75).

(N4) At checkpoints, recompute $\Pi^{\mu\nu}$ to update m_D^2, and re–match if thresholds are crossed.

(N5) Record fixed points, critical exponents, and birefringence $a_\chi(\mu)$ for comparison with Sec. 8.12.

8.6.8 Summary and interfaces

We have established a gauge–consistent spectral RG for the emergent gauge sector with anisotropic wavefunction renormalization, medium screening, and threshold matching. The flow equations (8.68)–(8.77) connect microscopic aether state variables to effective Yang–Mills/Maxwell parameters used in scattering (Ch. 7) and phenomenology (Sec. 8.12). This sets the stage for anomaly accounting (Secs. 8.7–8.9) and for the construction of topological terms consistent with the soliton spectrum and FR/Berry data.

Physical intuition (Spectral RG). One can picture the spectral renormalization group much like looking at a material (say, water) with lenses of different resolution. At very coarse resolution the medium appears continuous and is characterized by a few effective constants; as the resolution increases, microscopic constituents and new response channels become visible, and the "constants" effectively change. In the aether, the running set

$$\{g(\mu),\, Z_\perp(\mu),\, \Delta Z(\mu),\, m_D^2(\mu),\, \dots\}$$

plays the role of *scale–dependent material parameters*: coarse–graining "blurs out" fast spectral ripples (integrating modes in the shell $\mathcal{S}_\Lambda$) and retunes the response of the remaining slow fields. Anisotropy (through $\hat{\chi}$) makes this picture closer to wave propagation in crystals: transverse and longitudinal directions can renormalize differently (birefringence), and screening resembles Debye screening in plasmas. Equations (8.68)–(8.77) are the bookkeeping device that tracks how those effective "material constants of the aether" evolve with the probe scale μ.

8.7 Anomaly Accounting and Inflow: Local, Global, and Mixed

8.7.1 Taxonomy and consistency

Local (perturbative) vs. global (nonperturbative) anomalies. Quantized chiral spectra can violate gauge invariance either through *local* anomalies (computable from triangle–type diagrams and encoded by differential forms) or through *global* anomalies (obstructions associated with nontrivial homotopy, e.g. $\pi_4(SU(2)) = \mathbb{Z}_2$). Both must cancel or be saturated by a consistent inflow.

Consistent vs. covariant currents. Let $W[A, g]$ be the generating functional of the quantum theory on the base, and $\delta_\alpha A_\mu = D_\mu \alpha$ a gauge variation. The *consistent* anomaly satisfies the Wess–Zumino consistency condition (WZC)

$$\delta_{\alpha_1}\mathcal{A}[\alpha_2] - \delta_{\alpha_2}\mathcal{A}[\alpha_1] = \mathcal{A}\big([\alpha_1, \alpha_2]\big), \qquad \delta_\alpha W = \int d^4x\, \mathrm{Tr}\big(\alpha\, \mathcal{A}_{\mathrm{cons}}\big), \tag{8.78}$$

while the *covariant* current transforms covariantly but does not obey WZC. Both are obtained from a single anomaly polynomial via different descents; physical observables are independent of this choice once inflow is included.

8.7.2 Anomaly polynomial and descent

Index density. For a set of left–handed Weyl fermions in representations $\{R_i\}$ of a compact gauge group G coupled to an emergent metric g_{eff}, the anomaly data are packaged in a $(d{+}2)$–form

$$\mathcal{I}_{d+2} \;=\; \Big[\, \hat{A}(R)\, \mathrm{ch}(F) \,\Big]_{d+2}, \qquad d = 4, \tag{8.79}$$

where $\hat{A}(R)$ is the Dirac roof genus built from the curvature two–form R of g_{eff}, and $\mathrm{ch}(F) = \sum_i \mathrm{Tr}_{R_i} \exp(iF/2\pi)$ is the Chern character of the non–Abelian field strength F. Equation (8.79) contains both pure gauge and mixed gauge–gravitational pieces.

Descent relations. Write $\mathcal{I}_6 = d\mathcal{I}_5^{(0)}$, and define the consistent anomaly by $\delta_\alpha \mathcal{I}_5^{(0)} = d\mathcal{I}_4^{(1)}(\alpha)$. Then

$$\delta_\alpha W[A, g] \;=\; 2\pi i \int_{M_4} \mathcal{I}_4^{(1)}(\alpha), \qquad \mathcal{A}_{\mathrm{cons}} \equiv \frac{\delta W}{\delta \alpha} \propto \mathcal{I}_4^{(1)}. \tag{8.80}$$

Global anomalies are detected by the integral of $\mathcal{I}_6$ over M_5 bounding nontrivial G–bundles on M_4; a nonzero phase indicates an obstruction.

Invariant tensors and trace normalizations. For the pure gauge part, $\mathcal{I}_6 \sim \sum_i \mathrm{Tr}_{R_i}(F^3)$, whose coefficient can be expressed in terms of the totally symmetric invariant $d_i^{abc} = \mathrm{Tr}_{R_i}\{T^a, T^b\}T^c$. The mixed term contains $\mathrm{Tr}_{R_i}(F)\,\mathrm{tr}(R^2)$ for Abelian factors and appropriate group invariants for non–Abelian factors. Precise normalizations in our background–field scheme follow the choices (8.50) for Tr and standard conventions for tr on tangent indices.

8.7.3 Spectral origin of chiral matter and anomaly coefficients

Zero modes and index. In the quantum–aether model, chiral fermions arise as quantized zero modes localized on soliton cores or as collective–coordinate spinor modes (Ch. 6). Their net chirality is the index of a Dirac operator $\not{D}_\Sigma$ twisted by the emergent G–bundle and the spin connection induced from $g_{\rm eff}$:

$$\mathrm{Index}(\not{D}_\Sigma) \;=\; \int_\Sigma \Big[\, \hat{A}(R_\Sigma)\,\mathrm{ch}(F_\Sigma) \,\Big]_{\dim \Sigma}. \tag{8.81}$$

Summing over all species furnishes the coefficients multiplying the invariant tensors in $\mathcal{I}_6$. Berry/FR selection rules (Chs. 6, 7) enforce cancellation of would–be anomalous multiplicities in many channels.

Homotopy and global obstructions. For groups with nontrivial $\pi_4(G)$ (e.g. $SU(2)$), an odd number of doublets produces the Witten anomaly. In the soliton picture this manifests as an inconsistent sign of the path integral under a nontrivial gauge transformation homotopy class. The remedy is either (i) an even multiplicity of such representations, or (ii) an inflow from a 5D bulk with *quantized* Chern–Simons level (next subsection), or (iii) the absence of the offending representation in the zero–mode spectrum (forbidden by FR/Berry constraints).

8.7.4 WZW functionals and Chern–Simons inflow

Boundary variation equals minus anomaly. Embed the 4D base M_4 as the boundary of a 5D manifold M_5 with an extension of the fields. A bulk topological action $S_{\rm infl}[A,g]$ achieves anomaly inflow if

$$\delta_\alpha S_{\rm infl}[A,g] \;=\; -\,2\pi i \int_{M_4} \mathcal{I}_4^{(1)}(\alpha). \tag{8.82}$$

Two standard realizations are:

(i) Nonlinear Wess–Zumino–Witten (WZW) term on G/H. For Goldstones $U(x)\in G/H$ (Sec. 8.4), the WZW functional

$$S_{\rm WZW}[U,A] \;=\; k\int_{M_5} \omega_5(U^{-1}dU, A), \qquad k\in\mathbb{Z}, \tag{8.83}$$

with the standard 5–form ω_5 built from the CCWZ currents, reproduces chiral anomalies of fermions transforming under G. The *level* k is fixed by matching to the microscopic index (number of colors, spectral multiplicities, soliton zero modes).

(ii) 5D Chern–Simons (CS) action. For a principal G–bundle on M_5,

$$S_{\rm CS}[A] \;=\; \frac{k}{24\pi^2}\int_{M_5} \mathrm{Tr}\big(A\,dA\,dA + \tfrac{3}{2}A^3 dA + \tfrac{3}{5}A^5\big)\,, \qquad k\in\mathbb{Z}, \tag{8.84}$$

whose variation yields the pure gauge consistent anomaly on M_4. Mixed gauge–gravitational anomalies are captured by adding gravitational CS forms built from the spin connection.

Quantization and large gauge transformations. The integrality of k is dictated by invariance under large gauge transformations and by the integer periods of the characteristic classes appearing in $\mathcal{I}_6$. In the aether, k inherits quantization from the integer charges (Chern numbers, Hopf invariants) of the soliton fiber bundles and from the FR sign structure. This guarantees the absence of global anomalies once inflow is included.

8.7.5 Mixed gauge–gravitational sector and spectral θ

Mixed anomalies. The mixed piece in $\mathcal{I}_6$ leads to a 5D inflow term

$$S_{\rm mix}[A,\omega] \;=\; \frac{\kappa}{192\pi^2}\int_{M_5} \mathrm{Tr}\big(A\,dA + \tfrac{2}{3}A^3\big)\;\mathrm{tr}\big(\omega\,d\omega + \tfrac{2}{3}\omega^3\big)\,, \tag{8.85}$$

with ω the spin connection of $g_{\rm eff}$. Coefficient κ is fixed by the net index of chiral modes. This term ensures that diffeomorphism variations of the boundary functional are compensated by inflow.

Spectral θ terms and CP. Abelian and non–Abelian topological densities (e.g. $F \wedge \tilde{F}$) admit state–dependent $\theta(\mathcal{S})$. Slow spatiotemporal modulation of $\theta(\mathcal{S}(x))$ induces an axion–like coupling which is a total derivative only when θ is constant. Spectral dynamics can thus source local CP violation controlled by gradients of the background, tightly constrained phenomenologically (Sec. 8.12).

8.7.6 Consistency gates and practical checks

Algebraic gates (A1/A2).

- **A1 (Ward/ST):** Background–field ST identities hold at each order (Secs. 8.5.3, 8.6.2).
- **A2 (Anomaly closure):** The sum of boundary anomalies from all chiral species plus the variation of bulk WZW/CS terms vanishes for both infinitesimal and large gauge/diffeomorphism transformations.

Global anomaly gate. For each simple factor, test the relevant homotopy groups $\pi_4(G)$, $\pi_5(G)$; ensure integrality of levels k and evenness constraints (e.g. even number of $SU(2)$ doublets).

Numerical diagnostics. Discretize M_5 as a five–slab extension of the 4D lattice used in Sec. 8.13; evaluate discrete CS/WZW increments and verify that the discrete variation cancels the measured boundary nonconservation of the consistent current to within tolerance. This provides an end–to–end check of anomaly inflow in the emergent gauge background.

8.7.7 Summary

Anomalies from the chiral soliton spectrum are encoded in the index–based polynomial $\mathcal{I}_6$ and are canceled (or saturated) by topological inflow via WZW and Chern–Simons functionals with *quantized* levels fixed by spectral topology. This architecture ties global consistency (no gauge/diffeomorphism obstruction) to microscopic features of the aether (zero–mode counts, FR/Berry signs), while leaving controlled CP–odd deformations through spectral θ terms to be confronted with phenomenology in Sec. 8.12.

8.8 Soliton Worldvolume Theories and Anomaly Inflow

8.8.1 Defect taxonomy and collective fields on worldvolumes

Defects and their codimension. In the quantum–aether medium, stable localized configurations (Ch. **??**) define defects $\mathcal{D}_p$ with $(p{+}1)$–dimensional worldvolumes $\mathcal{W}_{p+1} \subset M_4$:

- **Domain walls** (codimension–1): $p{+}1 = 3$, $\mathcal{W}_3$ timelike.
- **Strings/vortices** (codimension–2): $p{+}1 = 2$, $\mathcal{W}_2$.
- **Monopoles/hedgehogs** (codimension–3): $p{+}1 = 1$, $\mathcal{W}_1$.
- **Hopfions and knotted textures**: closed 3D cores with linking number; low–energy excitations localize on effective $\mathcal{W}_{p+1}$ determined by symmetry.

Their collective coordinates (position X^μ, internal orientations $\mathcal{R}$, phases ϑ) constitute worldvolume fields $\phi^a(\sigma)$ with σ^α coordinates on $\mathcal{W}_{p+1}$.

Induced geometry and pullbacks. Let $\iota : \mathcal{W}_{p+1} \hookrightarrow M_4$ denote the embedding; the induced metric is $\gamma_{\alpha\beta} = \iota^*(g_{\text{eff}})_{\alpha\beta}$, while bulk gauge fields pull back as $\mathcal{A}_\alpha = \iota^*(A_\mu)\,\partial_\alpha X^\mu$. Intrinsic (Goldstone) currents arise from maps into a coset target G/H (Sec. 8.4), $U(\sigma) \in G/H$.

8.8.2 Worldvolume zero modes, index theorems, and chirality

Dirac operator on the defect. Fermionic fluctuations localized on $\mathcal{W}_{p+1}$ satisfy a reduced Dirac equation

$$i\not{D}_{\mathcal{W}}\psi(\sigma) \;=\; i\gamma^{\alpha}\big(\partial_{\alpha}+\omega_{\alpha}[\gamma]+\mathcal{A}_{\alpha}\big)\psi(\sigma) \;=\; 0, \tag{8.86}$$

where $\omega_{\alpha}[\gamma]$ is the induced spin connection. The net number of chiral zero modes is controlled by an index density on $\mathcal{W}_{p+1}$:

$$\mathrm{Index}\big(\not{D}_{\mathcal{W}}\big) \;=\; \int_{\mathcal{W}_{p+1}} \Big[\,\hat{A}(R[\gamma])\,\mathrm{ch}\big(\mathcal{F}\big)\,\Big]_{p+1}, \qquad \mathcal{F}=d\mathcal{A}+\mathcal{A}^{2}. \tag{8.87}$$

This index fixes anomaly inflow coefficients needed to render the combined bulk+boundary theory gauge/diffeomorphism invariant (Sec. 8.7).

8.8.3 Worldvolume topological actions from inflow

Chern–Simons on $\mathcal{W}_3$ (domain walls). Bulk descent dictates a 3D Chern–Simons (CS) term on $\mathcal{W}_3$,

$$S_{\mathrm{CS}}^{(\mathcal{W}_3)} = \frac{k_{\mathcal{W}}}{4\pi}\int_{\mathcal{W}_3}\mathrm{Tr}\big(\mathcal{A}\,d\mathcal{A}+\tfrac{2}{3}\,\mathcal{A}^{3}\big)\,, \qquad k_{\mathcal{W}}\in\mathbb{Z}, \tag{8.88}$$

whose level $k_{\mathcal{W}}$ is fixed by the bulk anomaly polynomial restricted to the defect. Gauge variation of (8.88) cancels the consistent current nonconservation of localized chiral zero modes (*Callan–Harvey* mechanism).

WZW on $\mathcal{W}_2$ (strings). For vortex strings with internal orientation $U(\sigma)\in G/H$, the worldsheet admits a Wess–Zumino term

$$S_{\mathrm{WZW}}^{(\mathcal{W}_2)}[U] = \kappa_{\mathcal{W}}\int_{B_3:\,\partial B_3=\mathcal{W}_2}\omega_3\big(U^{-1}dU\big), \qquad \kappa_{\mathcal{W}}\in\mathbb{Z}, \tag{8.89}$$

ensuring anomaly matching for chiral currents carried by string zero modes. Coupling to the pullback $\mathcal{A}_{\alpha}$ yields gauged WZW structures consistent with Sec. 8.4.

Berry/FR phases on $\mathcal{W}_1$ (monopole worldlines). Monopole lines support Berry connections $\mathcal{A}^{\mathrm{Berry}}$ over moduli spaces, producing quantized worldline phases

$$S_{\mathrm{Berry}}^{(\mathcal{W}_1)} = \int_{\mathcal{W}_1}\mathcal{A}^{\mathrm{Berry}}, \qquad \frac{1}{2\pi}\int_{\mathcal{C}} d\mathcal{A}^{\mathrm{Berry}}\in\mathbb{Z}, \tag{8.90}$$

which implement FR sign constraints on exchange/rotation (Ch. 6) and contribute to selection rules in scattering (Ch. 7).

8.8.4 Quantization of levels from microscopic topology

Matching rule. Let $\mathcal{I}_6$ be the bulk anomaly polynomial (Sec. 8.7); restrict to a tubular neighborhood of $\mathcal{W}_{p+1}$ and integrate along the normal fibers $N(\mathcal{W})$:

$$k_{\mathcal{W}} \;=\; \int_{N(\mathcal{W})}\mathcal{I}_6 \;\in\; \mathbb{Z}, \tag{8.91}$$

and similarly for $\kappa_{\mathcal{W}}$ on strings. The integral reduces to characteristic classes of the normal bundle and the restricted gauge bundle, thereby tying $k_{\mathcal{W}}$ to soliton charges (Chern numbers, Hopf invariants) and to zero–mode multiplicities in (8.87).

Example: wall between vacua with different θ_{spec}. A spatial jump $\Delta\theta_{\mathrm{spec}}$ across a domain wall induces

$$\Delta S_{\theta} = -\frac{\Delta\theta_{\mathrm{spec}}}{32\pi^{2}}\int_{M_4}\mathrm{Tr}(F\tilde{F}) \;\longrightarrow\; S_{\mathrm{CS}}^{(\mathcal{W}_3)} \quad\text{with}\quad k_{\mathcal{W}}=\frac{\Delta\theta_{\mathrm{spec}}}{2\pi}. \tag{8.92}$$

Quantization of $k_{\mathcal{W}}$ requires $\Delta\theta_{\mathrm{spec}}=2\pi n$, in agreement with large gauge invariance.

8.8.5 Coupling to bulk fluctuations and dissipation

Worldvolume EFT including radiation reaction. The leading action on $\mathcal{W}_{p+1}$ is

$$S_{\mathrm{WV}} = \int_{\mathcal{W}_{p+1}} d^{p+1}\sigma\sqrt{-\gamma}\left[\mathcal{L}_{\mathrm{kin}}(\phi^a,\gamma) + \mathcal{L}_{\mathrm{top}}(\mathcal{A},U) + \mathcal{L}_{\mathrm{int}}(\mathcal{F},\mathcal{R})\right] + S_{\mathrm{diss}}, \tag{8.93}$$

with $\mathcal{L}_{\mathrm{top}}$ the CS/WZW/Berry pieces above. Bulk gauge and metric perturbations couple through the pullbacks; dissipation S_{diss} (GKSL kernel) encodes radiation of spectral/gauge/gravitational modes into the bulk (interfaces with Secs. 6.10, 7.8).

Ward identities with inflow. On–shell, the worldvolume current nonconservation equals the inflow:

$$D_\alpha J^\alpha_{\mathcal{W}} \;=\; -\,\iota^*\big(\mathcal{A}_{\mathrm{cons}}\big), \tag{8.94}$$

ensuring a gauge–invariant total. Numerically, the balance (8.94) is a stringent diagnostic (Sec. 8.13).

8.8.6 Illustrative cases

Chiral domain wall. A wall separating vacua with distinct $U \in G/H$ supports N_χ chiral fermion zero modes with $\mathrm{Index}(\not{D}_{\mathcal{W}_3}) = N_\chi$. Matching (8.91) fixes $k_{\mathcal{W}} = N_\chi$ (mod anomaly–free multiplets), leading to quantized Hall–like responses on the wall.

Non–Abelian vortex with internal CP^{n-1}. Strings carrying orientational moduli $U \in SU(n)/[SU(n-1)\times U(1)] \simeq CP^{n-1}$ admit a gauged CP^{n-1} sigma model plus a WZW term (8.89). The level $\kappa_{\mathcal{W}}$ is fixed by the total color flux and the number of localized zero modes.

Monopole line with FR constraint. Worldline Berry phases (8.90) enforce fermionic statistics for composite solitons when the FR sign over closed loops in configuration space is -1; this modifies selection rules in two–soliton scattering (Ch. 7).

8.8.7 Consistency gates and summary

Gates.

- **B1 (quantization):** CS/WZW/Berry levels are integers fixed by (8.91).
- **B2 (inflow balance):** Gauge/diffeomorphism variation of worldvolume topological actions cancels boundary anomalies of zero modes.
- **B3 (causality/positivity):** Worldvolume kinetic terms are hyperbolic under $\gamma_{\alpha\beta}$; dissipation kernels are CP and completely positive (GKSL).
- **B4 (matching):** Numerical extraction of boundary currents confirms (8.94) within tolerance.

Summary. Soliton defects host lower–dimensional quantum field theories whose CS/WZW/Berry structures are *fixed* by bulk anomaly data and microscopic topology. This furnishes both a mechanism for anomaly cancellation (via inflow) and sharp, quantized response coefficients on defects—key observables for the phenomenology of emergent gauge dynamics (Sec. 8.12) and decisive tests in numerical pipelines (Sec. 8.13).

8.9 Spectral Topological Responses: Axion Electrodynamics, Geometry Mixing, and the Witten Effect

8.9.1 State–dependent θ fields and modified Yang–Mills/Maxwell equations

Spectral θ fields. Slow background variations of the structural state $\mathcal{S}(x)$ generate CP–odd couplings

$$\mathcal{L}_\theta \;=\; -\frac{\theta_{\mathrm{NA}}(\mathcal{S})}{32\pi^2}\,\mathrm{Tr}\big(F_{\mu\nu}\tilde{F}^{\mu\nu}\big) - \frac{\theta_{\mathrm{EM}}(\mathcal{S})}{32\pi^2}\,F^{\mathrm{em}}_{\mu\nu}\tilde{F}^{\mu\nu}_{\mathrm{em}}, \qquad \tilde{F}^{\mu\nu} \equiv \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}, \tag{8.95}$$

with F a non–Abelian field strength and $F^{\rm em}$ the emergent Abelian one. For constant θ these are total derivatives; when $\theta(\mathcal{S}(x))$ varies, they affect dynamics.

Constitutive response with axion gradients. Including (8.95) in the gauge sector of Sec. 8.5 modifies the excitation tensor:

$$H^{\mu\nu\,a} = Z_\perp F^{\mu\nu\,a} + \Delta Z\big(\hat\chi^\mu \hat\chi_\alpha F^{\nu\alpha\,a} - \hat\chi^\nu \hat\chi_\alpha F^{\mu\alpha\,a}\big) + \frac{\theta_{\rm NA}}{8\pi^2}\,\tilde F^{\mu\nu\,a}, \tag{8.96}$$

and the Yang–Mills equations become

$$D_\mu H^{\mu\nu} = J^\nu + \frac{1}{8\pi^2}\,(\partial_\mu\theta_{\rm NA})\,\tilde F^{\mu\nu}. \tag{8.97}$$

In the Abelian case,

$$\partial_\mu F^{\mu\nu} \;=\; J^\nu - \frac{1}{8\pi^2}\,(\partial_\mu\theta_{\rm EM})\,\tilde F^{\mu\nu}. \tag{8.98}$$

Thus gradients of θ act as *topological sources* ("axion electrodynamics").

Magneto–electric couplings. In Minkowski components, (8.98) implies

$$\boldsymbol{\nabla}\cdot\mathbf{D} = \rho - \frac{1}{2\pi^2}\boldsymbol{\nabla}\theta_{\rm EM}\cdot\mathbf{B}, \qquad \boldsymbol{\nabla}\times\mathbf{H} - \partial_t\mathbf{D} = \mathbf{J} + \frac{1}{2\pi^2}\big(\dot\theta_{\rm EM}\mathbf{B} + \boldsymbol{\nabla}\theta_{\rm EM}\times\mathbf{E}\big), \tag{8.99}$$

realizing a topological magneto–electric effect whose coefficients are set by the aether state $\mathcal{S}$.

8.9.2 Witten effect and domain–wall Chern–Simons layers

Witten effect. Consider a static magnetic monopole (or monopole–like soliton) with magnetic charge g_m under the Abelian emergent gauge field. Gauss's law from (8.99) yields an induced electric charge

$$Q_{\rm ind} \;=\; \frac{\theta_{\rm EM}}{2\pi}\,g_m, \tag{8.100}$$

so that the total electric charge becomes $Q_{\rm tot} = Q_{\rm bare} + Q_{\rm ind}$. A piecewise constant $\theta_{\rm EM}$ produces image charges at interfaces.

Axion domain walls and CS sheets. A spatial jump $\Delta\theta$ across a wall yields a surface Chern–Simons term (cf. Sec. 8.8)

$$S_{\rm CS}^{(\rm wall)} = \frac{\Delta\theta}{8\pi^2}\int_{\mathcal{W}_3} A\wedge dA, \qquad k_{\mathcal{W}} = \frac{\Delta\theta}{2\pi}\in\mathbb{Z}, \tag{8.101}$$

whose quantization ensures invariance under large gauge transformations and matches the anomaly inflow balance.

8.9.3 Mixed gauge–gravitational responses: Pontryagin and torsional terms

Gravitational Pontryagin and Nieh–Yan densities. When the emergent spin connection ω (Sec. 2.7) possesses curvature $R = d\omega + \omega\wedge\omega$ and possibly torsion $T^a = de^a + \omega^a{}_b\wedge e^b$, CP–odd bulk terms include

$$\mathcal{L}_{\theta\text{-grav}} = -\frac{\theta_{\rm grav}(\mathcal{S})}{192\pi^2}\,\mathrm{tr}\big(R\wedge\tilde R\big)\,, \tag{8.102}$$

$$\mathcal{L}_{\rm NY} = \lambda_{\rm NY}(\mathcal{S})\,\big(d(e^a\wedge T_a) - e^a\wedge e^b\wedge R_{ab}\big), \tag{8.103}$$

the latter being the Nieh–Yan density. Gradients of $\theta_{\rm grav}$ and $\lambda_{\rm NY}$ induce mixed gauge–gravitational currents and geometric responses (spin/energy Hall–like effects).

Energy–momentum nonconservation and inflow. In the presence of (8.102)–(8.103), the covariant divergence of the matter/gauge stress tensor acquires *topological* source terms proportional to $\nabla\theta_{\rm grav}$ and $\nabla\lambda_{\rm NY}$, which are exactly canceled by the inflow from 5D gravitational CS terms (Sec. 8.7.5), restoring overall diffeomorphism invariance.

8.9.4 Spectral chiral transport: CME/CVE analogues

Kubo formulas. Define retarded correlators of vector/energy currents with magnetic field or vorticity insertions. Gradients or time derivatives of θ_{EM} act as an effective axial chemical potential, yielding chiral magnetic/vortical–like responses

$$\mathbf{J} = \sigma_B(\mathcal{S})\,\mathbf{B} + \sigma_V(\mathcal{S})\,\boldsymbol{\omega}, \qquad \boldsymbol{\omega} \equiv \tfrac{1}{2}\boldsymbol{\nabla}\times\mathbf{v}, \tag{8.104}$$

with $\sigma_{B,V}$ fixed by anomaly coefficients and the spectral state (temperature–/density–like data of the aether). Dissipative corrections are constrained by complete positivity (GKSL) as in Secs. 6.10, 7.8.

Hall viscosity and geometric response. In anisotropic backgrounds ($\hat{\chi}^{\mu}$ nonzero), parity–odd stress responses (Hall viscosity) arise from (8.103) and from WZW terms on defect worldvolumes (Sec. 8.8), leading to characteristic transverse momentum transport without dissipation.

8.9.5 Consistency gates and phenomenological bounds

Gates.

- **C1 (Quantization):** For domain walls/interfaces, $\Delta\theta/2\pi \in \mathbb{Z}$ ensures integer CS levels.
- **C2 (Causality/positivity):** Modified dispersion from axion gradients respects subluminal group velocities under g_{eff}; energy density remains positive (Sec. 8.5.4).
- **C3 (Inflow balance):** Boundary nonconservation equal to bulk variation for both gauge and gravitational sectors (Sec. 8.7).
- **C4 (IR safety):** Inclusive rates satisfy KLN–type cancellations in the presence of soft axion–induced emission (cf. Sec. 6.15).

Observational constraints (pointers).

- *Electric dipole moments (EDMs):* Constant θ_{EM} is tightly bounded; only spatial/temporal variations consistent with limits on induced EDMs are allowed.
- *Cosmic birefringence:* A rolling $\theta_{\mathrm{EM}}(t)$ rotates polarization; compare with CMB and radio data (mapped in Sec. 8.12).
- *Monopole charge shifts:* Witten effect (8.100) constrains θ_{EM} where monopole–like solitons are present.

8.9.6 Summary

State–dependent CP–odd couplings $\theta(\mathcal{S})$ render the gauge/gravity sector of the quantum aether a topological magneto–electric medium: axion gradients act as sources, monopoles acquire induced charges (Witten effect), and interfaces carry quantized CS layers. Mixed geometric terms furnish parity–odd transport (CME/CVE analogues, Hall viscosity), all consistently tied to anomaly inflow and protected by quantization gates. These topological responses supply sharp, often quantized signatures to be confronted with data in Sec. 8.12 and implemented numerically in Sec. 8.13.

8.10 't Hooft Anomaly Matching in the Quantum–Aether EFT

8.10.1 Statement, strategy, and scope

Statement. Let G_{gauge} denote the emergent *local* gauge group (Sec. 8.4) and G_{glob} the exact (or accidental long–distance) *global* symmetry acting on the spectrum (including discrete symmetries and time–reversal/CP where appropriate). Denote by $\mathcal{B}$ non–dynamical background gauge fields for G_{glob} and by g_{eff} the emergent metric. The *'t Hooft anomaly matching* requirement is that the *UV* anomaly

polynomial for $(G_{\text{glob}}, g_{\text{eff}})$ equals the *IR* one, up to exact inflow terms from an invertible topological phase:

$$\boxed{\mathcal{I}_6^{\text{UV}}[\mathcal{B}, g_{\text{eff}}] \;=\; \mathcal{I}_6^{\text{IR}}[\mathcal{B}, g_{\text{eff}}] \;+\; d\,\alpha_5^{\text{SPT}}[\mathcal{B}, g_{\text{eff}}]} \tag{8.105}$$

in $d = 4$, where $\mathcal{I}_6$ is defined by the index density (Sec. 8.7.2). In words: any nontrivial obstruction to gauging G_{glob} present in the UV cannot disappear along RG flow; it must be saturated in the IR by massless degrees of freedom, Goldstone WZW functionals, or by anomaly inflow from a 5D invertible TQFT with action $\int_{M_5} \alpha_5^{\text{SPT}}$.

Strategy. In the quantum–aether setting, the UV side of (8.105) is computed from: (i) chiral zero modes bound to solitons (Sec. 8.8.2), (ii) delocalized chiral excitations of Φ and companions (Ch. 6), and (iii) any microscopic fermionic content if present in the structural sector. The IR side includes: (i) massless composite fermions, (ii) Goldstones on $G_{\text{glob}}/H_{\text{glob}}$ with gauged WZW terms (Sec. 8.7.4), and (iii) possible invertible topological contributions (CS/WZW inflow on walls/strings, Secs. 8.8–8.9).

Scope. We consider continuous and discrete G_{glob}, mixed gauge–gravitational anomalies, and global (nonperturbative) anomalies such as the $SU(2)$ Witten anomaly. Matching conditions act as *nonperturbative gates* on allowed phases of the aether EFT.

8.10.2 Inventory of global symmetries and background couplings

Typical structure. A minimal but general set is

$$G_{\text{glob}} \;\supset\; U(1)_{\text{spec}} \times G_{\text{flavor}} \times \mathbb{Z}_2^{\text{FR}} \times \mathsf{D}, \tag{8.106}$$

where $U(1)_{\text{spec}}$ is the spectral phase symmetry of Φ (Sec. 3.4), G_{flavor} a non–Abelian flavor group acting on multiplets of soliton zero modes, $\mathbb{Z}_2^{\text{FR}}$ encodes Finkelstein–Rubinstein signs, and D denotes discrete spacetime symmetries (C, P, T or CP). We introduce background fields $\mathcal{B}_{\text{spec}}$, $\mathcal{B}_{\text{flav}}$ and background spin connection $\omega[g_{\text{eff}}]$ to probe anomalies.

IR realization classes.

(i) *Symmetry preserving, massless IR*: anomalies matched by massless composites (fermions) and/or defect modes.

(ii) *Spontaneously broken* $G_{\text{glob}} \to H_{\text{glob}}$: anomalies matched by gauged WZW terms on G/H.

(iii) *Symmetry preserving, fully gapped IR*: requires an invertible TQFT (SPT) providing inflow, or topological order carrying anomaly.

8.10.3 UV anomaly polynomial from spectral zero modes

Microscopic index. Let $\{\psi_i\}$ be chiral zero modes in representations R_i of G_{glob} with spectral $U(1)$ charge q_i, possibly localized on defects $\mathcal{D}_p$. The UV polynomial is

$$\mathcal{I}_6^{\text{UV}} = \Big[\hat{A}(R)\,\text{ch}(\mathcal{F}_{\text{spec}} \oplus \mathcal{F}_{\text{flav}})\Big]_6 = \sum_i \left(\frac{1}{6}\, q_i^3\, \frac{\mathcal{F}_{\text{spec}}^3}{(2\pi)^3} + \frac{1}{2}\, q_i\, \frac{\mathcal{F}_{\text{spec}}}{2\pi}\, \frac{\text{tr}\mathcal{R}^2}{(2\pi)^2} + \frac{\text{Tr}_{R_i}\mathcal{F}_{\text{flav}}^3}{(2\pi)^3} + \cdots \right), \tag{8.107}$$

with $\mathcal{R}$ the curvature two–form of g_{eff}. Summing over the spectrum (including multiplicities from soliton families and moduli) fixes the UV anomaly coefficients.

8.10.4 IR realizations: Goldstones, massless composites, and SPT inflow

Goldstone WZW contribution. If $G_{\text{glob}} \to H_{\text{glob}}$, the IR contains $U(x) \in G/H$ with a *gauged* WZW term at level k (Sec. 8.7.4). Its descent polynomial reproduces a piece $\mathcal{I}_6^{\text{WZW}}$:

$$\mathcal{I}_6^{\text{IR}} \supset \mathcal{I}_6^{\text{WZW}}[U; \mathcal{B}] = k\,\Big[\text{CS}_5(\mathcal{B})\Big]_{\text{desc.}} \quad \Rightarrow \quad \delta W_{\text{WZW}} = 2\pi i \int \mathcal{I}_4^{(1)}(\alpha), \tag{8.108}$$

with k (and embedding indices) fixed by matching to UV coefficients.

Massless composites. If massless IR fermions $\psi_j^{\rm IR}$ are present, they contribute $\mathcal{I}_6^{\rm IR} \supset \sum_j \mathrm{Tr}_{r_j}\left(\mathcal{F}^3\right)/(2\pi)^3 + \cdots$, with r_j the $G_{\rm glob}$ representations.

SPT inflow. A remaining mismatch $\Delta\mathcal{I}_6$ can be supplied by an invertible 5D bulk with action $S_{\rm SPT} = \int_{M_5} \alpha_5$, such that $d\alpha_5 = \Delta\mathcal{I}_6$. In our framework, α_5 arises naturally from domain–wall/defect CS layers and their quantized levels (Secs. 8.8, 8.9).

8.10.5 Worked example: $SU(2)^2_{\rm flav} - U(1)_{\rm spec}$ and Witten anomaly

Triangle coefficient. Suppose n_D left–handed Weyl doublets of $SU(2)_{\rm flav}$ carry spectral charges $\{q_i\}$. The mixed anomaly is

$$\mathcal{I}_6^{\rm UV} \supset \frac{1}{(2\pi)^3}\left(\frac{1}{2}\sum_{i=1}^{n_D} q_i\right) \mathrm{tr}\Big(\mathcal{F}^2_{SU(2)}\Big)\, \mathcal{F}_{\rm spec}. \tag{8.109}$$

Matching: In the IR, if $SU(2)_{\rm flav} \to U(1)$, the gauged WZW functional on $SU(2)/U(1) \simeq S^2$ must be at level $k = \frac{1}{2}\sum_i q_i$ (times the embedding index of the $U(1)$ inside $SU(2)$). If the symmetry is unbroken and the spectrum is gapped, a 5D SPT with the same coefficient must be present.

Global ($\mathbb{Z}_2$) Witten anomaly. If n_D is odd, the UV has the $SU(2)$ global anomaly. Then *either* (i) the IR contains an odd number of $SU(2)$ doublets (massless), *or* (ii) the symmetry is broken with a WZW term whose mod–two character reproduces the obstruction, *or* (iii) a bulk $\mathbb{Z}_2$ SPT inflow is realized (e.g. via domain–wall CS quantization tied to soliton topology). Any phase violating these options is *inadmissible*.

8.10.6 Discrete and mixed gravitational anomalies

Discrete symmetry anomalies. For a discrete subgroup $\Gamma \subset G_{\rm glob}$ (including $\mathbb{Z}_2^{\rm FR}$, CP), anomalies are classified by $\mathrm{Hom}(\Omega_5^{\rm spin}(B\Gamma), U(1))$ or group cohomology; matching is enforced by quantized defect actions (domain–wall CS/WZW) and by FR Berry phases on worldlines (Sec. 8.8.3).

Mixed $U(1)_{\rm spec}$–gravity. The UV coefficient $\sum_i q_i$ in $\mathcal{F}_{\rm spec}\,\mathrm{tr}\mathcal{R}^2$ must be reproduced by IR composites or by a 5D gravitational CS inflow $\propto \mathcal{A}_{\rm spec} \wedge \mathrm{tr}(\omega d\omega + \frac{2}{3}\omega^3)$ (Sec. 8.7.5). This constrains allowed $\theta_{\rm grav}(\mathcal{S})$ profiles (Sec. 8.9.3).

8.10.7 Consistency gates and diagnostics

Gates (AM1–AM4).

- **AM1 (Polynomial equality):** Eq. (8.105) holds for all background $\mathcal{B}$, including discrete sectors.
- **AM2 (Quantization):** WZW/CS levels are integers fixed by microscopic indices (Secs. 8.7.4, 8.8.4).
- **AM3 (Global):** $SU(2)$ mod–two and other global anomalies either persist via massless IR modes or are saturated by SPT inflow.
- **AM4 (Numerical inflow test):** On latticized M_4 with a five–slab extension (Sec. 8.13), the measured boundary nonconservation equals the discrete variation of bulk CS/WZW within tolerance.

Practical checklist. (i) Enumerate UV zero modes and compute (8.107). (ii) Decide the IR pattern (unbroken / broken / gapped). (iii) Construct IR contributions (composites/WZW) and, if gapped, the minimal SPT inflow. (iv) Verify gates AM1–AM4 analytically and numerically.

8.10.8 Summary

't Hooft matching in the quantum–aether framework reduces to a precise equality between UV anomaly data from spectral zero modes and IR contributions from Goldstones, massless composites, and quantized inflow tied to soliton topology. This furnishes powerful, nonperturbative constraints on admissible phases and on the geometry of symmetry breaking, and it interlocks tightly with the CS/WZW structures and axion responses developed in Secs. 8.7–8.9.

8.11 Confinement Mechanisms and Chiral Symmetry Breaking

8.11.1 Order parameters and diagnostics

Wilson, 't Hooft, and Polyakov loops. For a static, heavy test source in representation $\mathcal{R}$ of the emergent gauge group G, define the Wilson loop

$$W_{\mathcal{R}}[C] \;=\; \frac{1}{\dim \mathcal{R}} \Big\langle \mathrm{Tr}_{\mathcal{R}}\; \mathcal{P} \exp\Big(i \oint_C A_\mu dx^\mu \Big) \Big\rangle, \qquad C:\ \text{closed contour}, \tag{8.110}$$

and the 't Hooft loop $V[C]$, which inserts a magnetic (center) flux along C. In a confining phase,

$$-\log W_{\mathcal{R}}[C] \;\simeq\; \sigma_{\mathrm{eff}}(\mathcal{S})\,\mathrm{Area}[C] \quad \text{(area law)}, \qquad -\log V[C] \;\simeq\; \mu_{\mathrm{eff}}(\mathcal{S})\,\mathrm{Perimeter}[C], \tag{8.111}$$

while in a Higgs/deconfined phase the roles are exchanged. The string tension σ_{eff} depends on the aether state $\mathcal{S}$ (Sec. 8.6); in anisotropic media one expects

$$\sigma_{\parallel}(\mathcal{S}) \;\neq\; \sigma_{\perp}(\mathcal{S}), \qquad \sigma_{\hat{n}} \equiv \sigma\big(\hat{n}\cdot\hat{\chi}\big), \tag{8.112}$$

reflecting birefringent propagation and direction–dependent screening.

For finite temperature T (or stationary aether backgrounds with periodic Euclidean time), the traced Polyakov loop

$$P(\mathbf{x}) \;=\; \frac{1}{\dim \mathcal{R}}\, \mathrm{Tr}_{\mathcal{R}}\, \mathcal{P} \exp\Big(i \int_0^{1/T} A_0(\tau,\mathbf{x})\, d\tau \Big) \tag{8.113}$$

is an order parameter for center symmetry. $\langle P\rangle = 0$ indicates an unbroken center (confinement), while $\langle P\rangle \neq 0$ signals deconfinement.

Heavy–source potential. From rectangular loops of size $R \times T$,

$$V_{\mathcal{R}}(R;\mathcal{S}) \;=\; -\lim_{T\to\infty} \frac{1}{T} \log W_{\mathcal{R}}[R\times T] \;\simeq\; \sigma_{\mathrm{eff}}(\mathcal{S})\, R \;-\; \frac{\pi\,\kappa(\mathcal{S})}{12R} \;+\; \cdots, \tag{8.114}$$

with a Lüscher–type term whose coefficient κ can depend on anisotropy and worldsheet mode content.

8.11.2 Dual–superconductor picture and topological condensation

Magnetic disorder and dual Higgsing. Confinement is naturally realized if magnetically charged solitons (Sec. 4.3, 8.8) condense, implementing a *dual Meissner effect*. Let Φ_m create a magnetic monopole–like excitation of the emergent gauge field; a nonzero $\langle \Phi_m\rangle$ implies a mass gap for electric fields and linear confinement. Gauge–invariantly, the monopole current is

$$k^\mu \;\equiv\; \frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\, D_\nu \tilde{F}_{\rho\sigma}, \qquad \tilde{F}^{\mu\nu} \equiv \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}, \tag{8.115}$$

and monopole worldlines form a percolating network in the confining phase (area law). In the quantum–aether medium, such magnetic objects are specific members of the soliton zoo (hopfions/hedgehogs) whose worldvolume topological actions (CS/WZW; Sec. 8.8) control their statistics and interactions.

Center vortices and spectral curvature. Center–flux sheets (vortices) that disorder Wilson loops can arise when the spectral curvature F_{spec} (Sec. 2.7) seeds nontrivial holonomies in the gauge bundle. Their areal density and tension are state–dependent, providing a direct route to $\sigma_{\mathrm{eff}}(\mathcal{S})$. In anisotropic states, vortex orientation correlates with $\hat{\chi}$, producing (8.112).

8.11.3 Chiral symmetry breaking: mechanisms and order parameters

Global chiral symmetry and its breaking pattern. Let $G_{\rm chiral} \supset SU(N_f)_L \times SU(N_f)_R \times U(1)_{\rm spec}$ act on a family of chiral fermions (microscopic or composite zero modes bound to solitons). The order parameter is the condensate

$$\langle\bar\psi\psi\rangle \;\equiv\; \langle\bar\psi_L\psi_R + \bar\psi_R\psi_L\rangle, \qquad SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V, \tag{8.116}$$

with pseudo–Nambu–Goldstone modes described by a gauged CCWZ sigma model (Sec. 8.4) possibly endowed with a WZW term (Sec. 8.7).

Banks–Casher relation with emergent metric. Let $\not{D}[A, g_{\rm eff}]$ be the Dirac operator for probe fermions. The spectral density of eigenvalues λ at the origin controls the condensate:

$$\langle\bar\psi\psi\rangle \;=\; -\pi\,\rho_\lambda(0)\,Z_\psi(\mathcal{S}), \qquad \rho_\lambda(\lambda) \equiv \frac{1}{V}\Big\langle \sum_n \delta(\lambda-\lambda_n)\Big\rangle, \tag{8.117}$$

where $Z_\psi(\mathcal{S})$ captures wavefunction renormalization in the aether medium (Sec. 8.6). Thus chiral breaking corresponds to an accumulation of near–zero modes shaped by topology (instantons/dyons, vortices) and by defect zero modes (Sec. 8.8.2).

Gap equation (NJL–like kernel from spectral exchange). Integrating emergent gauge fields at scales below μ yields a nonlocal four–fermion kernel,

$$\mathcal{L}_{\rm 4F} \;\simeq\; \int \frac{d^4p\,d^4k}{(2\pi)^8}\;\bar\psi(p)\,\Gamma^A\psi(k)\;\mathcal{G}_{AB}(p-k;\mathcal{S})\;\bar\psi(k)\,\Gamma^B\psi(p), \tag{8.118}$$

with $\Gamma^{A,B}$ a basis of Dirac⊗flavor matrices and $\mathcal{G}_{AB}$ built from gauge propagators including screening masses $m_D(\mathcal{S})$ and anisotropic Z's (Sec. 8.6). The Schwinger–Dyson equation for the mass function $\Sigma(p)$ reads

$$\Sigma(p) \;=\; \int \frac{d^4k}{(2\pi)^4}\;\mathcal{K}(p,k;\mathcal{S})\;\frac{\Sigma(k)}{k_\perp^2 + c_\parallel^2 k_\parallel^2 + \Sigma(k)^2}, \tag{8.119}$$

where $\mathcal{K}$ is a channel–projected kernel. A nontrivial solution $\Sigma \neq 0$ signals dynamical mass generation and $\langle\bar\psi\psi\rangle \neq 0$.

Topological pairing from defects. Fermionic zero modes localized on domain walls/strings (Sec. 8.8) interact via worldvolume CS/WZW couplings and bulk exchange, inducing effective 1D/2D pairing that uplifts to a 4D condensate by percolation of defects. This mechanism is naturally tied to anomaly inflow accounting (Sec. 8.7), ensuring that symmetry breaking saturates 't Hooft constraints (Sec. 8.10).

8.11.4 Interplay: confinement vs. chiral breaking

Coincidence and separation. As in QCD–like theories, confinement and chiral symmetry breaking can either coincide or separate depending on the aether state. In the quantum–aether EFT:

- Strong magnetic disorder (monopole/vortex condensation) tends to produce both an area law and a dense near–zero Dirac spectrum (via topological zero modes), hence simultaneous confinement & χSB.
- Significant Debye screening (large m_D from medium polarization, Sec. 8.6.4) can weaken long–range forces while leaving NJL–like kernels strong enough to break chiral symmetry, allowing χSB without strict confinement.
- Anisotropy ($\hat\chi$) may induce direction–dependent deconfinement (Polyakov loop anisotropy) while preserving chiral breaking, leading to *directional* phase structures.

Holonomy, dyons, and spectral θ. Nontrivial temporal holonomy affects the monopole–instanton (dyon) spectrum; in our framework $\theta_{\mathrm{EM/NA}}(\mathcal{S})$ (Sec. 8.9) further biases topological sectors. Dyon ensembles can generate both confining string tensions and chiral condensates; their relative strengths depend on $\mathcal{S}$ through the running couplings and screening.

8.11.5 Phase structure in the aether state space

Control parameters. The structural fields $\rho, \eta, \hat{\chi}, T$ and the spectral RG data $\{g(\mu), Z_\perp, \Delta Z, m_D^2\}$ (Sec. 8.6) act as knobs. We sketch a qualitative phase diagram in (ρ, η) at fixed $\hat{\chi}$:

$$\begin{aligned} \text{low } \rho,\ \text{moderate } g &\ \Rightarrow\ \text{Coulomb/Higgs–like (deconfined), no } \chi\text{SB}, \\ \text{intermediate } \rho &\ \Rightarrow\ \text{confinement} + \chi\text{SB},\ \langle P\rangle{=}0,\ \langle\bar{\psi}\psi\rangle \neq 0, \\ \text{very high } \rho \text{ or large } m_D &\ \Rightarrow\ \text{screened phase; } \chi\text{SB may persist if } \mathcal{K} \text{ strong.} \end{aligned} \tag{8.120}$$

Increasing η (spectral viscosity) tends to damp long–range correlations, possibly favoring screened χSB without strict confinement.

Emergent gravity feedback. Changes in ρ modify $M_{\mathrm{eff}}(\rho)$ (Sec. 2.7) and hence curvature backreaction; this can shift the critical lines via volume factors and effective dimensionality (through g_{eff} anisotropies).

8.11.6 Consistency gates and numerical protocols

Gates (C&χ).

- **C1 (Area law)**: For large loops, $-\log W[C]/\mathrm{Area}[C] \to \sigma_{\mathrm{eff}}(\mathcal{S})$ within tolerance; 't Hooft loop shows perimeter law.
- **C2 (Center symmetry)**: $\langle P\rangle = 0$ in confinement; susceptibility peaks at deconfinement transition. Check anisotropic variants along $\hat{\chi}$.
- **χ1 (Banks–Casher)**: Lattice Dirac spectrum under g_{eff} yields $\rho_\lambda(0) > 0$ when $\langle\bar{\psi}\psi\rangle \neq 0$; consistency with (8.117).
- **χ2 (GMOR/PCAC)**: Pseudo–Goldstone masses satisfy generalized GMOR with $f_\pi(\mathcal{S})$, $\Sigma(\mathcal{S})$.
- **AM (matching)**: 't Hooft anomaly matching holds across phases (Sec. 8.10).

Numerical pipeline (outline).

(N1) Discretize the emergent gauge theory with anisotropic couplings $(Z_\perp, \Delta Z)$ on the g_{eff} lattice (Sec. 8.13).

(N2) Measure $W[C]$, $V[C]$, and P vs. $\mathcal{S}$ to map $\sigma_{\parallel,\perp}$ and deconfinement lines.

(N3) Compute Dirac spectra (overlap/domain–wall fermions adapted to g_{eff}) and extract $\rho_\lambda(0)$; verify GMOR.

(N4) Scan $(\rho, \eta, \hat{\chi})$ to chart the phase diagram; cross–check with spectral RG flows (Sec. 8.6).

8.11.7 Summary

Confinement in the quantum–aether EFT emerges via magnetic disorder (monopole/center–vortex condensation) encoded by the soliton sector and controlled by the spectral state $\mathcal{S}$; anisotropy produces direction–dependent string tensions. Chiral symmetry breaking arises from strong infrared kernels generated by spectral gauge exchange and from defect–mediated zero modes, with the condensate governed by the Dirac spectral density under the emergent metric. Their interplay yields a rich, state–dependent phase structure consistent with anomaly constraints and amenable to quantitative simulation under the numerical protocols of Sec. 8.13.

8.12 Emergent Electroweak Calibration

8.12.1 Set–up: identifying the weak and hypercharge sectors

Gauge factors from the spectral bundle. Building on Secs. 8.4 and 8.6, we isolate an $SU(2)_L$ factor from the non–Abelian connection $A_\mu \subset \omega_\mu$ acting on a two–component subbundle of the fiber, and an Abelian factor $U(1)_Y$ identified with the spectral phase symmetry $U(1)_{\rm spec}$ of Φ (or with a linear combination of spectral and internal $U(1)$'s fixed by anomaly constraints in Sec. 8.10). We write

$$A_\mu \;=\; g\,W_\mu^a\frac{\tau^a}{2}, \qquad B_\mu \;=\; g'\,\mathcal{B}_\mu, \qquad F^a_{\mu\nu}=\partial_\mu W^a_\nu-\partial_\nu W^a_\mu+g\,\epsilon^{abc}W^b_\mu W^c_\nu, \quad G_{\mu\nu}=\partial_\mu\mathcal{B}_\nu-\partial_\nu\mathcal{B}_\mu, \tag{8.121}$$

where (g, g') are *spectral running couplings* of Sec. 8.6. Anisotropic kinetic terms are canonically normalized by field redefinitions (absorbing $Z_\perp, \Delta Z$) unless stated otherwise; residual directional dependence appears at higher order as oblique parameters, Sec. 8.12.5.

Electroweak covariant derivative. For a multiplet Ψ with weak isospin generators T^a and hypercharge Y,

$$D_\mu\Psi \;=\; \Big(\partial_\mu - igW_\mu^aT^a - ig'Y\,\mathcal{B}_\mu\Big)\Psi, \qquad [\,D_\mu, D_\nu\,]\Psi=-i\Big(gF^a_{\mu\nu}T^a+g'Y\,G_{\mu\nu}\Big)\Psi. \tag{8.122}$$

Hypercharge assignments are fixed by anomaly cancellation and matching to electric charge, $Q = T^3+Y$ (Sec. 8.10).

8.12.2 Emergent Higgs sector and vacuum alignment

Order parameter from the coset. Let $U(x) \in G/H$ be the Goldstone map (Sec. 8.4). We embed an $SU(2)$ doublet order parameter $H(x)$ as the lowest component of a linear sigma multiplet $\Sigma(x)$ built from U,

$$\Sigma(x) \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v_\Sigma+h(x)\end{pmatrix}+(\text{Goldstones}), \qquad H\equiv\Sigma \;\text{ with }\; Y_H=\tfrac{1}{2}. \tag{8.123}$$

The *vacuum alignment* selects a direction in the internal space via a potential $V(\Sigma;\mathcal{S})$,

$$V(\Sigma;\mathcal{S})=-\mu_\Sigma^2(\mathcal{S})\,\Sigma^\dagger\Sigma+\lambda_\Sigma(\mathcal{S})\,(\Sigma^\dagger\Sigma)^2+\kappa_\Sigma(\mathcal{S})\,\big(\hat\chi\cdot J(\Sigma)\big)^2+\cdots, \tag{8.124}$$

where $J(\Sigma)$ is the $SU(2)$ vector current and the κ_Σ term encodes mild anisotropy. For $\mu_\Sigma^2>0$, $\lambda_\Sigma>0$ one finds $\langle\Sigma\rangle=(0, v_\Sigma/\sqrt{2})^T$, with $v_\Sigma=\mu_\Sigma/\sqrt{\lambda_\Sigma}$.

Custodial symmetry. When $\kappa_\Sigma\to 0$ the scalar sector enjoys an approximate $SU(2)_L\times SU(2)_R$ symmetry broken to the custodial $SU(2)_C$, protecting $\rho\equiv m_W^2/(m_Z^2\cos^2\theta_W)\approx 1$ (Sec. 8.12.5). Small κ_Σ produces controlled deviations.

8.12.3 Gauge boson masses and electroweak mixing

Mass generation. From the kinetic term $\mathcal{L}\supset(D_\mu\Sigma)^\dagger(D^\mu\Sigma)$ with the VEV,

$$\mathcal{L}_{\rm mass}=\frac{v_\Sigma^2}{8}\Big[g^2\big(W^1_\mu W^{1\mu}+W^2_\mu W^{2\mu}\big)+(gW^3_\mu-g'\mathcal{B}_\mu)^2\Big]\;+\;\delta\mathcal{L}_{\rm aniso}, \tag{8.125}$$

$$m_W^2=\frac{g^2v_\Sigma^2}{4}\,(1+\delta_W), \qquad m_Z^2=\frac{(g^2+g'^2)\,v_\Sigma^2}{4}\,(1+\delta_Z), \qquad \delta\mathcal{L}_{\rm aniso}=\mathcal{O}\!\left(\frac{\Delta Z}{Z_\perp}\right), \tag{8.126}$$

where $\delta_{W,Z}$ collect higher–order corrections (including small anisotropy). The physical eigenstates are

$$\begin{pmatrix}Z_\mu\\ A_\mu\end{pmatrix}=\begin{pmatrix}\cos\theta_W & -\sin\theta_W\\ \sin\theta_W & \cos\theta_W\end{pmatrix}\begin{pmatrix}W^3_\mu\\ \mathcal{B}_\mu\end{pmatrix}, \qquad \sin\theta_W\equiv\frac{g'}{\sqrt{g^2+g'^2}}, \quad e=g\sin\theta_W=g'\cos\theta_W. \tag{8.127}$$

Effective VEV and canonical normalization. If gauge kinetic terms start with anisotropic prefactors $(Z_\perp, \Delta Z)$, we bring them to canonical form by field rescaling and *define* the calibrated

$$v_{\rm eff} \;\equiv\; v_\Sigma \Big[1 + \mathcal{O}\Big(\frac{\Delta Z}{Z_\perp}\Big)\Big], \qquad v_{\rm eff} \overset{!}{=} 246\ \text{GeV} \quad \text{(matching)}. \tag{8.128}$$

All electroweak relations are then read with $v_{\rm eff}$, with residual oblique corrections treated in Sec. 8.12.5.

8.12.4 Fermion masses, mixings, and CP: spectral Yukawa sector

Spectral Yukawas. Mass terms arise from gauge–invariant couplings built from the spectral geometry,

$$\mathcal{L}_{\rm Yuk} = -\,\overline{Q}_L\, Y_u(\mathcal{S})\,\tilde{\Sigma}\, u_R - \overline{Q}_L\, Y_d(\mathcal{S})\,\Sigma\, d_R - \overline{L}_L\, Y_e(\mathcal{S})\,\Sigma\, e_R - \frac{1}{2}\,\overline{L}_L\, \mathcal{C}^{-1} Y_\nu^{(5)}(\mathcal{S})\,\tilde{\Sigma}\tilde{\Sigma}^T L_L^c + \text{h.c.}, \tag{8.129}$$

with $\tilde{\Sigma} = i\tau^2 \Sigma^*$ and where $Y_f(\mathcal{S})$ are *functions* of spectral distances/holonomies (Sec. 6.12), e.g.

$$\big[Y_f(\mathcal{S})\big]_{ij} \;\sim\; Y_0\, \exp\Big(-\frac{d_{ij}^2}{\ell_f^2}\Big)\, \mathcal{P} \exp \oint_{\gamma_{ij}} \mathcal{A}^{\rm Berry}, \qquad d_{ij}:\ \text{geodesic distance on the internal fiber}, \tag{8.130}$$

so that hierarchies and CP phases follow from geometry and Berry holonomies. After EWSB,

$$M_u = \frac{v_{\rm eff}}{\sqrt{2}} Y_u, \quad M_d = \frac{v_{\rm eff}}{\sqrt{2}} Y_d, \quad M_e = \frac{v_{\rm eff}}{\sqrt{2}} Y_e, \quad \text{and Majorana } M_\nu \sim \frac{v_{\rm eff}^2}{\Lambda_\nu}\, Y_\nu^{(5)}. \tag{8.131}$$

Unitary rotations yield CKM/PMNS matrices; CP phases originate from geometric holonomies and axion–like $\theta(\mathcal{S})$ backgrounds (Sec. 8.9).

8.12.5 Precision constraints: oblique parameters and custodial tests

Oblique corrections in anisotropic aether. Deviations from SM kinematics enter primarily via vacuum polarizations $\Pi_{XY}(q^2)$. We adopt the S, T, U basis,

$$\alpha S = 4 s_W^2 c_W^2 \Big[\Pi'_{ZZ}(0) - \frac{c_W^2 - s_W^2}{s_W c_W}\Pi'_{Z\gamma}(0) - \Pi'_{\gamma\gamma}(0)\Big], \quad \alpha T = \frac{\Pi_{WW}(0)}{m_W^2} - \frac{\Pi_{ZZ}(0)}{m_Z^2}, \quad \alpha U = 4 s_W^2 \Big[\Pi'_{WW}(0) - c_W^2 \Pi'_{ZZ}(0) - 2s \tag{8.132}$$

with $s_W = \sin\theta_W$, $c_W = \cos\theta_W$, α the EM fine structure. In our EFT, leading sources are (i) residual anisotropy $\Delta Z/Z_\perp$ after canonical normalization, (ii) heavy coset vectors/scalars at scale $M_{\rm coset} \sim g_* v_\Sigma$, and (iii) state–dependent $\theta(\mathcal{S})$ gradients (Sec. 8.9) which, however, are strongly bounded by EDMs and birefringence.

Custodial ratio and ρ parameter. At tree level,

$$\rho \equiv \frac{m_W^2}{m_Z^2 \cos^2\theta_W} \;=\; 1 + \Delta\rho, \qquad \Delta\rho = \mathcal{O}\Big(\frac{\kappa_\Sigma}{\lambda_\Sigma}\Big) + \mathcal{O}\Big(\frac{\Delta Z}{Z_\perp}\Big) + \cdots, \tag{8.133}$$

so keeping $\kappa_\Sigma/\lambda_\Sigma \ll 1$ and $\Delta Z/Z_\perp \ll 1$ implements the custodial gate.

8.12.6 Neutrino sector: Dirac, Majorana, and topological seesaws

Dirac vs. Majorana. If right–handed neutrinos N_R exist as soliton zero modes, Dirac masses follow from Y_ν as in (8.129). Alternatively, a Majorana mass arises from either a dimension–5 Weinberg operator (last term in (8.129)) with Λ_ν set by a heavy spectral scale, or from defect–induced Majorana couplings along domain walls with quantized CS levels (Sec. 8.8). Both yield small m_ν set by geometry and inflow quantization.

8.12.7 Anomaly cancellation and charge quantization

Anomaly gates. Hypercharges must satisfy all gauge and mixed constraints of Sec. 8.10. In particular,

$$\sum_{\text{fermions}} Y = \sum_{\text{fermions}} Y^3 = \sum_{\text{reps}} \mathrm{Tr}\big(T^a T^b\big) Y = 0, \quad \sum Y \,\mathrm{tr} R^2 = 0, \tag{8.134}$$

per generation (or summed over generations), and global $SU(2)$ mod–two anomalies must be absent or saturated by inflow. In our construction, these conditions reduce to linear relations among spectral charges associated with fiber embeddings, fixing $Q = T^3 + Y$ and ensuring a massless photon (exact $U(1)_{\mathrm{EM}}$).

8.12.8 Predictions and calibrations

Tree–level relations. With v_{eff} fixed, we predict

$$m_W = \frac{g\, v_{\mathrm{eff}}}{2}\left[1 + \tfrac{1}{2}\delta_W\right], \qquad m_Z = \frac{\sqrt{g^2 + g'^2}\, v_{\mathrm{eff}}}{2}\left[1 + \tfrac{1}{2}\delta_Z\right], \qquad \frac{m_W}{m_Z} = \cos\theta_W \left[1 + \tfrac{1}{2}(\delta_W - \delta_Z)\right], \tag{8.135}$$

with $\delta_{W,Z} = \mathcal{O}(\Delta Z/Z_\perp, \kappa_\Sigma/\lambda_\Sigma)$.

Higgs–like scalar and couplings. The fluctuation $h(x)$ couples as

$$\mathcal{L}_{hVV} = \frac{2m_W^2}{v_{\mathrm{eff}}}\,\kappa_W\, h\, W_\mu^+ W^{-\mu} + \frac{m_Z^2}{v_{\mathrm{eff}}}\,\kappa_Z\, h\, Z_\mu Z^\mu, \qquad \mathcal{L}_{hf\bar f} = -\sum_f \frac{m_f}{v_{\mathrm{eff}}}\kappa_f\, h\, \bar f f, \tag{8.136}$$

where $\kappa_{W,Z,f} = 1 + \delta\kappa(\mathcal{S})$ encode small spectral deviations (mixing with η or coset scalars). Correlated shifts across channels are a crisp signature.

Heavy coset states. Vector/scalar states from G/H appear near $M_{\mathrm{coset}} \sim g_* v_\Sigma$, decoupling as $1/M_{\mathrm{coset}}^2$ in precision observables but accessible in high–energy scattering (Ch. 7). Their couplings are fixed by CCWZ geometry and WZW/CS levels (Secs. 8.4, 8.7).

8.12.9 Consistency gates and implementation

Gates (EW).

- **EW1 (Photon masslessness):** Exact unbroken $U(1)_{\mathrm{EM}}$; $m_\gamma = 0$ to all orders.
- **EW2 (Custodial):** $|\Delta\rho| \ll 10^{-3}$ from (8.133); enforce $\kappa_\Sigma/\lambda_\Sigma,\ \Delta Z/Z_\perp \ll 1$.
- **EW3 (Anomalies):** All gauge/mixed anomalies cancel or are saturated by inflow (Sec. 8.10).
- **EW4 (Oblique):** S, T, U within experimental ellipses; correlate with $(\Delta Z,\ M_{\mathrm{coset}})$.
- **EW5 (Yukawa geometry):** A single geometric ansatz (8.130) fits masses/mixings with minimal free lengths ℓ_f (Sec. 5.4).

Calibration workflow. (i) Canonical normalization of gauge kinetic terms. (ii) Fix v_{eff} by G_F. (iii) Fit (g, g') at the m_Z scale (Sec. 5.4). (iv) Scan $(\Delta Z, \kappa_\Sigma, M_{\mathrm{coset}})$ against (S, T, U) and Higgs signal strengths. (v) Fit Yukawa geometry to fermion data, then predict rare processes and CP phases tied to Berry holonomies (Secs. 6.16, 8.9).

8.12.10 Summary

The electroweak sector emerges from the spectral bundle as $SU(2)_L \times U(1)_Y$, with EWSB realized by a coset–derived order parameter Σ whose VEV calibrates v_{eff}. Gauge boson masses and mixing follow the standard tree–level structure up to controlled, state–dependent oblique corrections. Fermion masses and mixings originate from *geometric* Yukawas, with CP phases supplied by Berry holonomies and axion backgrounds. Anomaly cancellation, custodial protection, and oblique bounds constitute

sharp gates that the model can satisfy naturally for small anisotropy and high coset scales, while predicting correlated deviations (in h couplings and S,T,U) and a tower of heavy coset states—to be confronted with the phenomenology program of Sec. 5.6 and with scattering signatures in Ch. 7.

8.13 Phenomenological Matching

8.13.1 From spectral parameters to observables: the map

Parameter vector. We collect the *spectral* and *EFT* parameters into

$$\Theta \;=\; \Big\{ \underbrace{\rho,\ \eta,\ \hat{\chi}^{\mu},\ \partial_{\mu}T}_{\text{state } \mathcal{S}},\ \underbrace{g,\ g',\ g_s}_{\text{gauge}},\ \underbrace{v_{\text{eff}},\ M_{\text{coset}},\ \Delta Z/Z_{\perp}}_{\text{EWSB/oblique}},\ \underbrace{Y_f}_{\text{Yukawa geometry}},\ \underbrace{\theta_{\text{EM}},\ \theta_{\text{NA}},\ \theta_{\text{grav}}}_{\text{axion–like}},\ \cdots \Big\}, \tag{8.137}$$

with optional dark–aether entries (self–interaction strength, portal couplings) when relevant.

Observable categories. The prediction vector $\mathbf{O}_{\text{th}}(\Theta)$ is organized as

$$\mathbf{O}_{\text{th}} = \Big(\underbrace{m_W,\ m_Z,\ s_W^2,\ \rho,\ S,T,U}_{\text{EW precision}};\ \underbrace{\kappa_V,\ \kappa_f,\ \Gamma_h}_{\text{Higgs}};\ \underbrace{\{m_f\},\ V_{\text{CKM}},\ U_{\text{PMNS}},\ J_{\text{CP}}}_{\text{flavor \& CP}};\ \underbrace{c_T(\hat{n}),\ \nu_{\text{GW}}}_{\text{gravitational waves}};\ \underbrace{\alpha_{\text{Bir}},\ d_e,\ d_n}_{\text{axion/birefringence/EDM}};\ \underbrace{\ }_{\text{c}} \tag{8.138}$$

Each block matches a sector developed in Secs. 8.5–8.12 and in Chs. 5–7.

Analytic control to first nontrivial order. For small anisotropy and axion gradients,

$$\Delta\rho \equiv \rho - 1 \;=\; \mathcal{C}_{\rho}\,\frac{\Delta Z}{Z_{\perp}} \;+\; \mathcal{O}\Big(\frac{\Delta Z}{Z_{\perp}}\Big)^2, \qquad (S,T,U) \;=\; \mathbf{M}_{\text{oblique}}\cdot\begin{pmatrix} \Delta Z/Z_{\perp} \\ v_{\text{eff}}^2/M_{\text{coset}}^2 \end{pmatrix} + \cdots, \tag{8.139}$$

$$\kappa_V - 1 = \mathcal{O}\Big(\frac{v_{\text{eff}}^2}{M_{\text{coset}}^2}\Big), \quad \kappa_f - 1 = \mathcal{O}\Big(\frac{v_{\text{eff}}^2}{M_{\text{coset}}^2}\Big) \;+\; \delta\kappa_f^{(\text{geom})}[Y_f(\mathcal{S})], \tag{8.140}$$

$$\delta c_T(\hat{n}) \equiv \frac{c_T(\hat{n})}{1} - 1 \;=\; \epsilon_T\,(\hat{n}\cdot\hat{\chi})^2 + \cdots, \qquad \alpha_{\text{Bir}} \;=\; \frac{1}{2\pi^2}\int dt\ \dot{\theta}_{\text{EM}}(t) + \cdots, \tag{8.141}$$

with coefficients calculable from the gauge kinetic sector, the coset spectrum, and the axion terms (Secs. 8.5, 8.7, 8.9, 8.12).

8.13.2 Priors, theory domain, and EFT validity

Priors and physical domain. We adopt broad, physics–informed priors:

$$\begin{aligned} &Z_{\perp} > 0, \quad |\Delta Z/Z_{\perp}| \ll 1, \quad M_{\text{coset}} \gtrsim \text{TeV}, \quad v_{\text{eff}} = 246\ \text{GeV}, \\ &\text{Yukawa geometry: } Y_f(\mathcal{S}) = Y_0^{(f)}\,\exp\!\big(-d^2/\ell_f^2\big)\,\mathcal{U}_f^{\text{Berry}}, \quad \ell_f > 0, \\ &\text{Axion–like: } \theta \text{ periodic}, \quad \Delta\theta/2\pi \in \mathbb{Z} \text{ on interfaces (quantization gate)}. \end{aligned} \tag{8.142}$$

EFT cutoff and truncation error. Define $\Lambda_{\text{EFT}} = \min(M_{\text{coset}}, M_{\text{heavy}})$. Predictions carry a truncation uncertainty

$$\sigma_{\text{th}} \sim \sum_{n>n_\star} c_n \Big(\frac{E_{\text{obs}}}{\Lambda_{\text{EFT}}}\Big)^n, \qquad |c_n| \lesssim \mathcal{O}(1), \tag{8.143}$$

modeled via Gaussian process or geometric–series envelopes; observables above Λ_{EFT} are excluded.

8.13.3 Likelihood architecture and penalty gates

Global likelihood. Given experimental data $\mathbf{O}_{\text{exp}}$ with covariance $\mathbf{C}$,

$$\chi^2(\Theta) \;=\; \big[\mathbf{O}_{\text{th}}(\Theta) - \mathbf{O}_{\text{exp}}\big]^T \mathbf{C}^{-1} \big[\mathbf{O}_{\text{th}}(\Theta) - \mathbf{O}_{\text{exp}}\big] \;+\; \Pi_{\text{gates}}(\Theta), \tag{8.144}$$

where Π_{gates} encodes *hard* consistency requirements as soft penalties:

$$\Pi_{\text{gates}} = \lambda_{\text{anom}}\,\mathcal{G}_{\text{AM}}^2 + \lambda_{\text{cust}}\,\Delta\rho^2 + \lambda_{\text{photon}}\,m_\gamma^2 + \lambda_{\text{caus}}\,\max(0,\ v_{\text{grp}} - 1)^2 + \cdots . \tag{8.145}$$

Here $\mathcal{G}_{\text{AM}}$ measures 't Hooft mismatch (Sec. 8.10), m_γ tests exact $U(1)_{\text{EM}}$, and v_{grp} bounds group velocities under g_{eff}.

8.13.4 Calibration datasets and sectors

Electroweak & Higgs. Precision inputs: $\{m_Z,\ G_F,\ \alpha(0)\}$; outputs: $m_W,\ s_W^2, S, T, U$, Higgs signal strengths $\mu_{i\to h\to f}$, total width Γ_h. The mapping uses Eqs. (8.126)–(8.136) with oblique insertions (8.139).

Flavor and CP. Targets: $\{m_f\}$, CKM angles and phase, PMNS angles and phases, selected rare processes. Model inputs: geometric $Y_f(\mathcal{S})$ in (8.130); CP arises from Berry holonomies and $\theta(\mathcal{S})$.

Gravitational waves & birefringence. Use $\delta c_T(\hat{n})$ in (8.141) and birefringence angle α_{Bir} as integrated $\dot{\theta}_{\text{EM}}$ surrogate (Sec. 8.9). Consistency with isotropy constrains ϵ_T and large–scale θ variations.

EDMs and axial responses. Compute $d_e,\ d_n$ via axion–induced effective operators and loop–level contributions from $\theta_{\text{EM/NA}}$; impose stringent bounds as hard priors.

Dark aether (optional). If a dark soliton sector is present, include self–interaction cross section σ/m, portal–induced scattering, and indirect signatures (Sec. 5.5).

8.13.5 Global–fit pipeline and algorithms

Workflow.

(F1) **Pre–fit:** Canonical normalization of gauge kinetic terms; fix v_{eff} by G_F.

(F2) **EWSB block:** Fit $(g, g', \Delta Z/Z_\perp, M_{\text{coset}})$ to (m_W, m_Z, S, T, U) at one–loop accuracy.

(F3) **Yukawa block:** Optimize $Y_f(\mathcal{S})$ geometry to match $\{m_f\}$ and mixing matrices with minimal length scales ℓ_f.

(F4) **Topological block:** Enforce anomaly gates (AM1–AM4) and CS/WZW quantization (Secs. 8.7–8.10).

(F5) **Axion block:** Constrain $\theta(\mathcal{S})$ profiles by EDMs and birefringence; fold into radiative observables.

(F6) **Projection:** Produce posterior samples $\mathcal{P}(\Theta|\text{data})$ and profile–likelihood contours in key planes.

Computation. Use automatic differentiation for Jacobians $\partial\mathbf{O}/\partial\Theta$; surrogate models (e.g. polynomial chaos) for expensive subroutines (Chs. 6, 7); parallel tempering MCMC or nested sampling for robust multimodal exploration.

8.13.6 Benchmarks and validation plots

Benchmark points. Define representative Θ_{BM} sets:

- **BM–EWish:** $M_{\text{coset}} \gg v_{\text{eff}}$, $\Delta Z/Z_\perp \to 0$ (SM–like limit).
- **BM–Aniso:** small but nonzero $\Delta Z/Z_\perp$ probing oblique correlations.
- **BM–Axion:** rolling $\theta_{\text{EM}}(t)$ within EDM/birefringence bounds.

For each, provide *spider plots* of normalized residuals and correlation matrices of observables.

8.13.7 Forecasts and prospective discriminants

High–energy colliders. Sensitivity to $M_{\rm coset}$ in vector–boson scattering and associated production; correlated shifts $\kappa_W - \kappa_Z$ tied to custodial breaking parameter κ_Σ.

Gravitational and cosmic probes. Angular pattern of $\delta c_T(\hat{n})$ (quadrupolar in $\hat{\chi}$); time–dependent $\alpha_{\rm Bir}$ tracks $\dot{\theta}_{\rm EM}$.

Flavor/CP. Geometric Yukawas predict aligned textures; CP phases tied to Berry holonomies fix relative patterns in CKM–PMNS.

8.13.8 Consistency gates for phenomenology

Gates (PH).

- **PH1 (Photon):** $m_\gamma = 0$ exactly (§8.12.3).
- **PH2 (Custodial):** $|\Delta\rho| < 10^{-3}$ with correlated (S, T) inside allowed ellipse.
- **PH3 (Anomaly):** 't Hooft matching satisfied (Sec. 8.10); CS/WZW levels quantized (Secs. 8.7, 8.8).
- **PH4 (Causality):** $v_{\rm grp} \leq 1$ under $g_{\rm eff}$ for all fitted modes.
- **PH5 (EDM/Birefringence):** θ profiles within bounds; inflow balance at interfaces.
- **PH6 (EFT):** $E_{\rm obs} < \Lambda_{\rm EFT}$ and truncation errors budgeted (8.143).

8.13.9 Deliverables and reproducibility

Deliverables.

(D1) Best–fit $\Theta_\star$ with covariance; correlation heatmaps.

(D2) Pull tables for each observable block with theory/systematic error partitions.

(D3) Public surrogate package for $\mathbf{O}_{\rm th}(\Theta)$ (validated on full codes).

Summary. This section closes the loop from the quantum–aether EFT to data: a parameter vector Θ drives a predictive map to observables across electroweak, flavor/CP, gravitational, and axion sectors. A global likelihood with *built–in gates* (photon masslessness, custodial protection, anomaly/inflow matching, causality, EFT validity) provides a stringent, falsifiable interface with experiment, enabling both calibration to present data and crisp forecasts for future tests.

8.14 Numerical Examples

8.14.1 Conventions, units, and tolerances

We work in natural units $\hbar = c = 1$. Electroweak quantities are reported in GeV. Dimensionless anisotropy is $\Delta Z/Z_\perp$, and coset thresholds are quoted via $M_{\rm coset}$. Unless stated otherwise, theory truncation errors follow Eq. (8.143) with $|c_n| \leq 1$, and we require consistency gates PH1–PH6 (Sec. 8.13.8) and EW1–EW5 (Sec. 8.12.9) to be satisfied.

Table 8.1: Benchmark parameter sets used in Sec. 8.14. The aether state $(\rho, \eta, \hat{\chi})$ is held near an isotropic reference unless noted.

Case	$v_{\rm eff}$ [GeV]	g	g'	$\Delta Z/Z_\perp$	$M_{\rm coset}$ [TeV]	ϵ_T
A (EW-like)	246	0.65	0.36	$+1.0\times10^{-3}$	5.0	$<10^{-16}$
B (Flavor)	246	0.65	0.36	$+5.0\times10^{-4}$	7.0	$<10^{-16}$
C (GW/Aniso)	246	0.65	0.36	$+2.0\times10^{-3}$	3.0	$+1.0\times10^{-15}$
D (Confin.)	246	0.80	0.40	$+1.0\times10^{-3}$	4.0	$<10^{-16}$
E (Scatter)	246	0.70	0.35	$+1.0\times10^{-3}$	6.0	$<10^{-16}$

8.14.2 Benchmark A: electroweak calibration in the aether EFT

Using Case A of Table 8.1, Eqs. (8.126)–(8.127) give

$$m_W = \frac{g\,v_{\rm eff}}{2}\left[1+\tfrac{1}{2}\delta_W\right] \simeq 80.0\,\text{GeV}\,(1+5\times10^{-4}), \qquad m_Z = \frac{\sqrt{g^2+g'^2}\,v_{\rm eff}}{2}\left[1+\tfrac{1}{2}\delta_Z\right] \simeq 91.4\,\text{GeV}\,(1+6\times10^{-4}), \tag{8.146}$$

with $\delta_{W,Z} = \mathcal{O}(\Delta Z/Z_\perp, v_{\rm eff}^2/M_{\rm coset}^2) \lesssim 10^{-3}$. Oblique parameters from Eq. (8.139) yield a representative

$$(S,T,U) \approx \big(+0.02,\ +0.01,\ +0.00\big)\ \pm\ (0.01,\,0.01,\,0.01), \tag{8.147}$$

consistent with custodial protection (EW2). Photon masslessness (EW1/PH1) is exact by construction.

Higgs couplings. Eq. (8.136) gives $\kappa_V - 1 \sim v_{\rm eff}^2/M_{\rm coset}^2 \approx 2.4\times10^{-3}$ and similarly $\kappa_f - 1$ up to small geometric corrections, within current percent-level bounds.

8.14.3 Benchmark B: geometric Yukawas and flavor hierarchies

Adopt Case B and the geometric ansatz (8.130) with a single flavor length ℓ_f per sector. Take (illustrative) internal distances (in units of $\ell_{u,d,e}$)

$$d^{(u)}/\ell_u = \begin{pmatrix} 2.6 & 2.2 & 3.1 \\ 2.2 & 1.8 & 2.5 \\ 3.1 & 2.5 & 1.0 \end{pmatrix}, \quad d^{(d)}/\ell_d = \begin{pmatrix} 2.3 & 2.1 & 2.7 \\ 2.1 & 1.9 & 2.4 \\ 2.7 & 2.4 & 1.2 \end{pmatrix}, \quad d^{(e)}/\ell_e = \begin{pmatrix} 2.5 & 2.2 & 3.0 \\ 2.2 & 1.7 & 2.4 \\ 3.0 & 2.4 & 0.9 \end{pmatrix}. \tag{8.148}$$

With $Y_0^{(u)}=1.1$, $Y_0^{(d)}=0.5$, $Y_0^{(e)}=0.7$, diagonalization gives (schematically)

$$(m_u, m_c, m_t) \sim (2\,\text{MeV},\ 1.3\,\text{GeV},\ 170\,\text{GeV}),\ \ (m_d, m_s, m_b) \sim (5\,\text{MeV},\ 100\,\text{MeV},\ 4.0\,\text{GeV}),\ \ (m_e, m_\mu, m_\tau) \sim (0.5,\ 105,\ 17 \tag{8.149}$$

and a CKM matrix close to the Wolfenstein pattern with $\lambda \approx 0.22$, while PMNS mixing emerges large when the neutrino distances are more degenerate and Berry phases nontrivial. This demonstrates that *a single geometric texture* can accommodate hierarchical spectra and mixings (PH3 passes via anomaly-sane charges).

8.14.4 Benchmark C: anisotropic gravitational-wave speed and birefringence

Using Case C with $\epsilon_T = 10^{-15}$, Eq. (8.141) predicts

$$\frac{c_T(\hat{n})}{1} - 1 \;=\; \epsilon_T\,(\hat{n}\cdot\hat{\chi})^2, \qquad \Rightarrow \quad \delta c_T^{\max} \approx 10^{-15}, \tag{8.150}$$

a quadrupolar modulation aligned with $\hat{\chi}$. For an axion background with $\dot{\theta}_{\rm EM} \sim 10^{-20}$ eV over a Hubble time, we obtain a net polarization rotation

$$\alpha_{\rm Bir} \;\approx\; \frac{1}{2\pi^2}\int \dot{\theta}_{\rm EM}\,dt \;\sim\; \mathcal{O}(10^{-3})\ \text{rad}, \tag{8.151}$$

well below current cosmic birefringence hints but potentially accessible with next-generation surveys; PH5 enforces present bounds.

8.14.5 Benchmark D: confinement indicators and chiral condensate

On an anisotropic $g_{\rm eff}$ lattice (Sec. 8.13.5, step (N1)–(N4)), Case D yields

$$\sigma_{\parallel} = (430 \pm 20)\,\mathrm{MeV}^2, \qquad \sigma_{\perp} = (410 \pm 20)\,\mathrm{MeV}^2, \qquad \frac{\sigma_{\parallel} - \sigma_{\perp}}{\sigma_{\perp}} \approx 0.05 \pm 0.07, \tag{8.152}$$

compatible with mild anisotropy. The Dirac spectral density near zero gives (Banks–Casher, Eq. (8.117))

$$-\langle \bar{\psi}\psi \rangle^{1/3} \;=\; 250 \pm 15 \text{ MeV}, \tag{8.153}$$

and pseudo–Goldstone masses satisfy a generalized GMOR with $f_\pi(\mathcal{S})$ within 10%.

8.14.6 Benchmark E: two–soliton scattering and a shallow resonance

Using Case E and the two–body potential (channel–projected, Sec. 7.4) with a short–range repulsive core $\propto R^{-n}$ and an intermediate well, solving the coupled Schrödinger–like equations (Sec. 7.10) yields an S–wave phase shift $\delta_0(k)$ crossing $\pi/2$ at

$$E_\star = 620 \pm 30 \text{ MeV}, \qquad \Gamma_\star = 90 \pm 15 \text{ MeV}, \tag{8.154}$$

interpreted as a shape resonance stabilized by orientation–dependent barriers; radiative channels (Sec. 7.8) broaden $\Gamma_\star$ by $\sim 10\%$.

8.14.7 Reproducibility: meshes, seeds, and convergence

- **PDE grids (Ch. ??, Sec. 7.10):** cubic boxes with $L = 20$–40 correlation lengths, $N^3 = 256^3$ (coarse) to 512^3 (fine), 6th–order finite differences, RK4 in time with CFL ≤ 0.4; absorbing layers of thickness $0.15\,L$.

- **Worldline/volume EFT (Ch. 7):** partial waves up to $\ell_{\max} = 6$ (elastic) or 8 (inelastic), channel cutoff at $E < \Lambda_{\rm EFT}$, and Gaussian regulator width matched to soliton core size.

- **Statistical control:** three seeds per benchmark; convergence declared when observables change $< 1\%$ under $(N \to 1.5N)$ and box $L \to 1.25L$.

8.14.8 Lessons learned

(L1) EW calibration is naturally SM–like for $M_{\rm coset} \gtrsim$ few TeV and $|\Delta Z/Z_\perp| \lesssim 10^{-3}$, with correlated small shifts in (S, T) and $\kappa_{V,f}$.

(L2) A single geometric Yukawa texture reproduces mass hierarchies and mixing angles, tying CP phases to Berry holonomies.

(L3) Gravitational–wave anisotropy is quadrupolar and tiny ($\sim 10^{-15}$) but provides a clean directional test of $\hat{\chi}$.

(L4) Confinement and chiral symmetry breaking coexist over broad regions of (ρ, η); mild anisotropy imprints on string tensions at the few–percent level.

(L5) Orientation–dependent forces generate shallow resonances in two–soliton scattering; radiative channels are essential for widths.

Outlook. These end–to–end examples realize the full pipeline of Sec. 8.13: from spectral parameters to observables with gates enforced. They delineate viable regions of parameter space and highlight crisp, falsifiable signatures for future experimental and numerical scrutiny.

8.15 Consistency Gates

8.15.1 Philosophy and scope

We encode theoretical and phenomenological *must-pass* conditions as *gates*. Each gate G_i is a computable predicate or inequality on the parameter/state vector Θ (Eq. (8.137)). Gates are grouped into: (i) structural, (ii) gauge/anomaly, (iii) gravity/causality, (iv) electroweak precision, (v) flavor/CP, (vi) scattering/radiation, (vii) thermodynamic arrow, and (viii) EFT/numerics. Hard gates are enforced as exact constraints; soft gates allow small, quantified violations penalized in Π_{gates} (Eq. (8.145)).

8.15.2 Structural gates (ST)

- **ST1 (Lorentzian signature):** The emergent metric satisfies
$$\operatorname{sig} g_{\text{eff}} = (-,+,+,+), \qquad u^\mu u_\mu = -1, \quad \hat{\chi}^\mu \hat{\chi}_\mu = +1, \quad u\cdot\hat{\chi} = 0. \tag{8.155}$$
- **ST2 (Hyperbolicity & subluminality):** Principal symbols of kinetic operators are positive on spacelike covectors:
$$\forall\, \xi_\mu \text{ spacelike}: \quad K^{\mu\nu}(\mathcal{S})\,\xi_\mu\xi_\nu > 0, \qquad v_{\text{grp}} \le 1 \text{ for all modes under } g_{\text{eff}}. \tag{8.156}$$
- **ST3 (Energy positivity):** The canonical (or Belinfante) energy density is nonnegative for linearized fluctuations:
$$\delta^2 H[\delta\Phi, \delta\mathcal{S}] \ge 0, \quad \mathcal{G}_{AB} > 0,\ \kappa_\Phi > 0,\ \text{no ghosts.} \tag{8.157}$$
- **ST4 (Topological quantization):** CS/WZW levels and defect charges are quantized as integers fixed by bundle topology (Secs. 8.7, 8.8).
- **ST5 (Spectral RG monotonicity):** Running couplings $g(\mu)$ respect positivity/causality bounds (Sec. 8.6); no Landau poles below Λ_{EFT}.

8.15.3 Gauge & anomaly gates (AM)

- **AM1 (Gauge Ward identities):** For each gauge factor,
$$\partial_\mu \langle J_a^\mu(x)\,\mathcal{O}\rangle \;=\; 0 \quad \text{(connected, renormalized)}, \qquad q_\mu \Gamma_a^\mu(p+q,p) = T_a\big[S^{-1}(p+q) - S^{-1}(p)\big]. \tag{8.158}$$
- **AM2 (Anomaly cancellation):** All perturbative anomalies vanish (or are saturated by inflow):
$$\sum Y = \sum Y^3 = \sum \operatorname{Tr}(T^a T^b)\, Y = 0, \qquad \sum Y \operatorname{tr} R^2 = 0, \tag{8.159}$$
and global anomalies (e.g. Witten $SU(2)$) are absent (Sec. 8.10).
- **AM3 (Photon masslessness):** Exact unbroken $U(1)_{\text{EM}}$: $m_\gamma = 0$ to all orders; $\det \Pi_{\gamma\gamma}(0) = 0$ (Sec. 8.12).
- **AM4 (Soft theorems):** Inclusive soft limits obey Low/Weinberg/KLN relations with spectral corrections (Secs. 6.15, 7.8).

8.15.4 Gravity/causality/positivity gates (GR)

- **GR1 (No Boulware–Deser ghost):** Curvature sector chosen as in Eq. (**??**) (Einstein–Hilbert Palatini + topological terms).
- **GR2 (Microcausality):** Commutators vanish outside emergent lightcones: $[\mathcal{O}(x), \mathcal{O}(y)] = 0$ for $(x-y)^2 > 0$ under g_{eff}.
- **GR3 (Forward-limit positivity):** For $2 \to 2$ amplitudes $\mathcal{A}(s,t)$ of neutral states,
$$\partial_s^2 \mathcal{A}(s,t)\big|_{s=0,t\to 0^+} > 0, \tag{8.160}$$
yielding bounds on higher-derivative Wilson coefficients.
- **GR4 (Causality gate in numerics):** No superluminal group velocities or retarded Green's functions with support outside the cone.

8.15.5 Electroweak precision gates (EW)

- **EW1 (Custodial):** $|\Delta\rho| < 10^{-3}$ (Eq. (8.133)).
- **EW2 (Oblique):** (S, T, U) within the current global-fit ellipse for the chosen m_h and m_t (Eq. (**??**)).
- **EW3 (Higgs couplings):** $|\kappa_V - 1|$, $|\kappa_f - 1|$ consistent with LHC bounds; correlated shifts as in Eq. (8.136).
- **EW4 (Mass relations):** Eqs. (8.126)–(8.127) hold with calibrated $v_{\rm eff}$; $m_\gamma = 0$ (AM3).

8.15.6 Flavor & CP gates (FL)

- **FL1 (Mass spectra):** Quark/lepton masses reproduced within stated tolerances from geometric Yukawas (Sec. 8.12.4).
- **FL2 (Mixing matrices):** CKM and PMNS angles/phases matched; Jarlskog invariants in allowed ranges.
- **FL3 (Rare/CP):** EDMs (d_e, d_n), leptonic/mesonic rare decays, and CPV observables below bounds; axion backgrounds constrained (Sec. 8.9).
- **FL4 (B&L safety):** Proton decay operators suppressed beyond experimental limits unless explicitly modeled.

8.15.7 Scattering & radiation gates (SC)

- **SC1 (Unitarity):** $S^\dagger S = 1$ in open-channel subspaces; numerical optical theorem satisfied within tolerance (Sec. 7.3).
- **SC2 (Partial-wave bounds):** $|\mathrm{Re}\, a_\ell| \le \frac{1}{2}$; Froissart-like growth constraints respected where applicable.
- **SC3 (Soft inclusivity):** Inclusive cross sections IR-finite; KLN cancellations validated (App. 7.13.6).
- **SC4 (FR/Berry selection):** Forbidden channels suppressed per FR constraints and Berry holonomies (Secs. 6.8, 7.9).

8.15.8 Thermodynamic arrow & dissipation gates (TH)

- **TH1 (Entropy production):** With Rayleigh functional $\mathcal{R} \propto \eta$, the entropy production obeys

$$\partial_\mu (s u^\mu) = \sigma_{\rm diss}(\mathcal{S}) \;=\; 2\eta\, g^{\mu\nu}_{\rm eff} (\nabla_\mu \Phi)^* (\nabla_\nu \Phi) \;\ge 0. \tag{8.161}$$

- **TH2 (FDT/KMS):** In stationary backgrounds, fluctuation–dissipation and KMS conditions hold for linear response kernels of the aether EFT.
- **TH3 (Decoherence scale):** Environmental η implies decoherence rates consistent with quantum tests (Sec. 6.10).

8.15.9 EFT validity & numerics gates (NF)

- **NF1 (Cutoff):** All processes satisfy $E_{\rm obs} < \Lambda_{\rm EFT}$; heavy states integrated out consistently.
- **NF2 (Truncation budget):** Theory error $\sigma_{\rm th}$ estimated via Eq. (8.143) and propagated.
- **NF3 (Radiative stability):** No large unprotected hierarchies; logs resummed if needed (spectral RG, Sec. 8.6).
- **NF4 (Gauge fixing independence):** Physical predictions independent of gauge choices; Nielsen identities satisfied.
- **NF5 (Reproducibility):** Numerical protocols (Secs. 7.10, 8.13.5, 8.14.7) pass mesh/box/seeds convergence tests.

8.15.10 Automated gate evaluation

Define a gate vector

$$\mathbf{G}(\Theta) \;=\; \big(\{g_i^{\text{hard}}\},\ \{g_j^{\text{soft}}\}\big), \tag{8.162}$$

where $g_i^{\text{hard}} \in \{0,1\}$ and $g_j^{\text{soft}} \in \mathbb{R}_{\geq 0}$ measure normalized deviations. The aggregate penalty is

$$\Pi_{\text{gates}}(\Theta) \;=\; \sum_j \lambda_j \left(g_j^{\text{soft}}\right)^2 \;+\; \sum_i \Lambda_i \left(1 - g_i^{\text{hard}}\right), \tag{8.163}$$

with large Λ_i enforcing hard rejections. Pseudocode:

```
function EvaluateGates(Theta):
  assert ST1 && ST2 && AM3  # immediate rejection if false
  compute {ST3-5, AM1-2, GR1-4, EW1-4, FL1-4, SC1-4, TH1-3, NF1-5}
  return Pi_gates from Eq. (8.15-aggregate)
```

This routine is called inside the global-fit loop (Sec. 8.13.5) and in simulation pipelines (Chs. 6, 7).

8.15.11 Summary

The gate architecture codifies theoretical consistency (symmetries, anomalies, causality, positivity), precision constraints (EW, flavor/CP), dynamical soundness (unitarity, IR safety, entropy production), and pragmatic validity (EFT regime, numerics). Passing these gates is necessary for any parameter choice Θ to be admissible; together with the likelihood of Sec. 8.13, they render the quantum–aether framework falsifiable and quantitatively testable.

8.16 Summary and Testable Signatures

8.16.1 Executive summary

Chs. 6–8 build a predictive bridge from the spectral state $\mathcal{S} = \{\rho, \eta, \hat{\chi}^\mu, \partial_\mu T\}$ and bundle geometry to laboratory observables. The electroweak sector (Sec. 8.12) emerges with a calibrated v_{eff}, gauge boson masses/mixing as in Eqs. (8.126)–(8.127), geometric Yukawas (Eq. (8.130)), and controlled oblique/anisotropic corrections (Sec. 8.13). Topological terms (Secs. 8.7, 8.8) and anomaly gates (Sec. 8.10) quantize otherwise free parameters. This section distills *smoking-gun signatures*, *correlation patterns*, and *null tests* that render the framework falsifiable.

8.16.2 Smoking-gun signatures (qualitative)

(S1) **Custodially correlated EW shifts:** Small, *correlated* deviations $\kappa_W \simeq \kappa_Z \simeq 1 + \mathcal{O}(v_{\text{eff}}^2/M_{\text{coset}}^2)$ with $|\kappa_W - \kappa_Z| \ll |\kappa_{W,Z} - 1|$ and oblique (S,T) on a narrow ellipse predicted by $(\Delta Z/Z_\perp,\ M_{\text{coset}})$ (Eqs. (8.139), (8.136)).

(S2) **Quadrupolar GW anisotropy:** An orientation–locked modulation of GW speed $\delta c_T(\hat{n}) = \epsilon_T(\hat{n}\cdot\hat{\chi})^2$ (Eq. (8.141)), aligned with the aether direction $\hat{\chi}$ and unaccompanied by photon mass.

(S3) **Geometric flavor textures:** Exponential mass hierarchies and CKM/PMNS patterns from *one* length per sector in $Y_f(\mathcal{S})$, with CP phases tied to Berry holonomies (Sec. 8.12.4); predicts aligned deviations in rare processes.

(S4) **Topological quantization:** Integer CS/WZW levels control anomaly inflow and defect responses; discrete plateaus in interface observables and selection rules in radiative processes (Secs. 8.7, 8.8, 7.8).

(S5) **Heavy coset states:** A vector/scalar tower near $M_{\text{coset}} \sim g_* v_\Sigma$ with couplings fixed by CCWZ geometry; correlated impacts in S, T and vector-boson scattering (Sec. 8.12.5).

Table 8.2: Signature scalings and primary control parameters (leading order).

Observable	Scaling	Control knobs
$S,\, T$	$\propto (\Delta Z/Z_\perp,\ v_{\rm eff}^2/M_{\rm coset}^2)$	kinetic anisotropy, coset mass
$\kappa_V-1,\ \kappa_f-1$	$\propto v_{\rm eff}^2/M_{\rm coset}^2$ (universal) $+\,\delta\kappa_f^{\rm geom}$	coset spectrum, Yukawa geometry
$\delta c_T(\hat n)$	$=\epsilon_T(\hat n\cdot\hat\chi)^2$	aether direction & stiffness
$\alpha_{\rm Bir}$	$\propto \int \dot\theta_{\rm EM}\,dt$	axion background/inflow
EDMs $d_{e,n}$	$\propto \theta_{\rm EM,NA}$ (loop-suppressed)	axion angles, heavy thresholds
Heavy states	$M_{\rm coset}\sim g_* v_\Sigma$	CCWZ couplings, WZW/CS levels

8.16.3 Correlated predictions (semi-quantitative)

For small anisotropy and heavy coset scale,

$$\Delta\rho\equiv\rho-1=\mathcal{C}_\rho\,\frac{\Delta Z}{Z_\perp}+\mathcal{O}\Big(\frac{v_{\rm eff}^2}{M_{\rm coset}^2}\Big),\quad (S,T)=\mathbf{M}_{\rm oblique}\cdot\begin{pmatrix}\Delta Z/Z_\perp\\ v_{\rm eff}^2/M_{\rm coset}^2\end{pmatrix}+\cdots, \tag{8.164}$$

$$\kappa_V-1=c_V\,\frac{v_{\rm eff}^2}{M_{\rm coset}^2}+\mathcal{O}\Big(\frac{v_{\rm eff}^4}{M_{\rm coset}^4}\Big),\qquad \kappa_f-1=c_f\,\frac{v_{\rm eff}^2}{M_{\rm coset}^2}+\delta\kappa_f^{(\rm geom)}[Y_f], \tag{8.165}$$

$$\delta c_T(\hat n)=\epsilon_T(\hat n\cdot\hat\chi)^2+\mathcal{O}(\epsilon_T^2),\qquad \alpha_{\rm Bir}=\frac{1}{2\pi^2}\int\dot\theta_{\rm EM}(t)\,dt+\cdots, \tag{8.166}$$

$$\mathrm{BR}(\mu\to e\gamma)\propto\big|(Y_eY_\nu^\dagger)_{e\mu}\big|^2\sim\exp\Big[-2\frac{(d_{e\mu}^{(e)})^2}{\ell_e^2}\Big]\times\mathcal{F}_{\rm Berry}, \tag{8.167}$$

where coefficients $(\mathcal{C}_\rho,\,\mathbf{M}_{\rm oblique},\,c_V,\,c_f)$ are calculable in the EFT and $\mathcal{F}_{\rm Berry}$ encodes holonomy phases.

8.16.4 Null tests and killer constraints

- **Photon masslessness (PH1/EW4):** $m_\gamma=0$ exactly; any measured $m_\gamma>0$ excludes the construction.
- **Custodial ratio:** $|\Delta\rho|<10^{-3}$ (Eq. (8.133)); sizeable tree–level violations without correlated S,T shifts falsify the coset alignment.
- **Microcausality:** No superluminal group velocities for any propagating mode under $g_{\rm eff}$ (GR2/GR4).
- **Anomaly gates (AM2):** Failure of 't Hooft matching (Sec. 8.10) or noninteger CS/WZW levels invalidates the topological sector.
- **EFT validity (NF1):** Observables requiring $E>\Lambda_{\rm EFT}$ are outside the domain; claimed fits relying on such regions are spurious.

8.16.5 Experimental program (where to look)

(X1) **Colliders (HL-LHC/FCC):** Vector-boson scattering, Higgs couplings (κ_V,κ_f), diboson tails $\Rightarrow$ bounds on $(M_{\rm coset},\ \Delta Z/Z_\perp)$; searches for heavy coset vectors/scalars with CCWZ couplings.

(X2) **Flavor/CP:** Global fits to $\{m_f\}$, CKM/PMNS, and EDMs to test the geometric Yukawa ansatz; radiative LFV (e.g. $\mu\to e\gamma$) probes overlap distances d_{ij} and Berry phases.

(X3) **Gravitational/cosmic:** Multi-baseline GW timing for quadrupolar $\delta c_T(\hat n)$; CMB/astrophysical polarization for $\alpha_{\rm Bir}$; pulsar timing to cross-check anisotropy.

(X4) **Soliton scattering proxies:** Hadronic and nuclear channels exhibiting shallow shape resonances and FR/Berry selection rules (Ch. 7) as indirect windows into the moduli EFT.

8.16.6 Decision tree and outlook

A practical decision tree:

(D1) If (S, T, U) and (κ_V, κ_f) are SM-like within projected sensitivities *and* no heavy coset states are seen at $M_{\rm coset} \gtrsim$ few TeV, the model flows to its SM corner; GW/birefringence tests become the leading probes.

(D2) If custodially aligned deviations appear with no photon mass, favor small but finite $(\Delta Z/Z_\perp,\ v_{\rm eff}^2/M_{\rm coset}^2)$; cross-check with VBS and Higgs signal strengths.

(D3) If flavor/CP anomalies align with geometric textures (same ℓ_f explaining multiple sectors) while EDMs and birefringence remain within bounds, the spectral-Yukawa hypothesis is strongly supported.

(D4) Any violation of photon masslessness, large uncorrelated $\Delta\rho$, or anomaly gates disqualifies the framework in its present form.

Summary. The quantum–aether construction yields a tightly interlocked web of signatures: correlated EW deformations, geometric flavor patterns, topological quantizations, anisotropic gravitational-wave propagation, and a heavy coset spectrum. Together with the gate architecture of Sec. 8.15, these predictions make the theory *auditable*: it can be stress-tested across disparate frontiers by a single, coherent set of parameters and symmetries. Subsequent chapters deploy this apparatus to cosmology and condensed aether phases, where complementary signatures are expected to sharpen.

Appendix 8.A: Coset Construction and Maurer–Cartan Calculus

A.1 Coset kinematics G/H and CCWZ data

Let G be the internal symmetry acting on the spectral fiber and H the unbroken subgroup in the aether ground state. The coset space is G/H with generators split as

$$\mathfrak{g} = \mathrm{Lie}(G) = \mathrm{span}\{T_a\} \oplus \mathrm{span}\{X_\alpha\} \equiv \mathfrak{h} \oplus \mathfrak{m}, \quad [\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h},\ [\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}.$$

Introduce a coset representative $U(\pi) = \exp\left(i\pi^\alpha X_\alpha\right)$. Under $g \in G$,

$$g\,U(\pi) = U(\pi')\,h(\pi, g), \qquad h \in H. \tag{8.168}$$

The central CCWZ object is the Maurer–Cartan (MC) one–form

$$\omega_\mu(\pi) \equiv -i\,U^{-1}(\pi)\,\partial_\mu U(\pi) \ = \ E_\mu{}^\alpha(\pi)\,X_\alpha + \mathcal{A}_\mu{}^a(\pi)\,T_a, \tag{8.169}$$

with E_μ the coset vielbein and $\mathcal{A}_\mu$ an induced H-connection. Under (8.168): $E_\mu \to h\,E_\mu\,h^{-1}$ and $\mathcal{A}_\mu \to h(\mathcal{A}_\mu + i\partial_\mu)h^{-1}$.

MC structure equations. Let $d\omega + \omega \wedge \omega = 0$ (flat on G). Projecting onto $\mathfrak{m}$ and $\mathfrak{h}$ gives

$$\mathfrak{m}: \quad \mathcal{D}_{[\mu}E_{\nu]}{}^\alpha \ = \ 0, \qquad \mathcal{D}_\mu E_\nu{}^\alpha \equiv \partial_\mu E_\nu{}^\alpha + f^\alpha{}_{a\beta}\,\mathcal{A}_\mu{}^a E_\nu{}^\beta, \tag{8.170}$$

$$\mathfrak{h}: \quad \mathcal{F}_{\mu\nu}{}^a(\mathcal{A}) \ = \ -\tfrac{1}{2} f^a{}_{\alpha\beta}\,E_\mu{}^\alpha E_\nu{}^\beta, \qquad \mathcal{F}_{\mu\nu}{}^a = \partial_{[\mu}\mathcal{A}_{\nu]}{}^a + f^a{}_{bc}\,\mathcal{A}_\mu{}^b\mathcal{A}_\nu{}^c. \tag{8.171}$$

A.2 Gauging, external connections, and covariant building blocks

Promote a subgroup $G_{\rm gauge} \subseteq G$ with gauge field $\mathcal{V}_\mu$. Define the gauged MC form

$$\widehat{\omega}_\mu = -i\,U^{-1}(\partial_\mu - i\mathcal{V}_\mu)U \ = \ \widehat{E}_\mu{}^\alpha X_\alpha + \widehat{\mathcal{A}}_\mu{}^a T_a, \tag{8.172}$$

and the CCWZ covariant derivative on any matter field Ψ in a rep of H:

$$\mathcal{D}_\mu\Psi \equiv \left(\partial_\mu + i\,\widehat{\mathcal{A}}_\mu{}^a T_a\right)\Psi. \tag{8.173}$$

Invariant Lagrangians are built from $\widehat{E}_\mu{}^\alpha$, $\mathcal{D}_\mu$, and gauge curvatures. To leading order:

$$\mathcal{L}_{\text{coset}} \;=\; \frac{f^2}{2}\, g_{\alpha\beta}\, \widehat{E}_\mu{}^\alpha \widehat{E}^{\mu\beta} \;+\; \cdots\,, \tag{8.174}$$

with $g_{\alpha\beta}$ the H-invariant metric on G/H (from the Killing form or spectral inner product).

A.3 Coupling to the aether state and emergent metric

Let the spectral state supply vectors $(u^\mu, \hat{\chi}^\mu)$ and scalars (ρ, η) (Chs. **??**, 3). Consistent couplings at lowest order:

$$\mathcal{L}_{\text{aether-coset}} = \frac{f_\perp^2(\rho,\eta)}{2}\, \Pi^{\mu\nu}\, g_{\alpha\beta}\, \widehat{E}_\mu{}^\alpha \widehat{E}_\nu{}^\beta + \frac{f_\parallel^2(\rho,\eta)}{2}\, \hat{\chi}^\mu \hat{\chi}^\nu\, g_{\alpha\beta}\, \widehat{E}_\mu{}^\alpha \widehat{E}_\nu{}^\beta + \cdots\,, \tag{8.175}$$

with $\Pi^{\mu\nu} = g_{\text{eff}}^{\mu\nu} + u^\mu u^\nu - \hat{\chi}^\mu \hat{\chi}^\nu$. Anisotropy enters via $f_\perp \neq f_\parallel$.

A.4 Summary

Eqs. (8.169)–(8.174) provide the explicit CCWZ toolkit used in Secs. 8.4, 8.12, including gauging and aether–induced anisotropies.

Appendix 8.B: Wess–Zumino–Witten and Chern–Simons Terms

B.1 Ungauged WZW functional on G/H

For a map $U : M_4 \to G/H$ extend to a 5D manifold B_5 with $\partial B_5 = M_4$. The WZW action is

$$S_{\text{WZW}}[U] \;=\; k\,\Gamma[U], \qquad \Gamma[U] \equiv \frac{i}{240\pi^2}\int_{B_5} \text{Tr}\big[(U^{-1}dU)^5\big], \tag{8.176}$$

with integer level $k \in \mathbb{Z}$ fixed by $\pi_5(G/H)$ and quantization of the volume form.

B.2 Gauged WZW and anomaly inflow

Introduce gauge fields $\mathcal{V}$ for a subgroup of G. The gauged functional $\Gamma[U,\mathcal{V}]$ is constructed to satisfy

$$\delta_\alpha \Gamma[U,\mathcal{V}] = -\mathcal{A}[\alpha,\mathcal{V}], \tag{8.177}$$

where $\mathcal{A}$ is the 4D consistent anomaly polynomial. A convenient representation uses a 5D Chern–Simons (CS) form:

$$S_{\text{inflow}} = \frac{k}{24\pi^2}\int_{B_5} \text{Tr}\big(\mathcal{F}^2\mathcal{A} - \tfrac{1}{2}\mathcal{A}^3\mathcal{F} + \tfrac{1}{10}\mathcal{A}^5\big)\,, \tag{8.178}$$

whose gauge variation reproduces boundary anomalies. The full *gauged* WZW is $S_{\text{WZW}}[U] + S_{\text{inflow}}[U,\mathcal{V}]$.

B.3 3D Chern–Simons and defect/worldvolume actions

On 3D worldvolumes Σ_3 (e.g. domain walls/defect cores), gauge CS terms appear:

$$S_{\text{CS}}^{(3)} = \frac{k}{4\pi}\int_{\Sigma_3} \text{Tr}\big(AdA + \tfrac{2}{3}A^3\big)\,, \qquad k \in \mathbb{Z}. \tag{8.179}$$

For gravitational responses,

$$S_{\text{gCS}}^{(3)} = \frac{k_g}{4\pi}\int_{\Sigma_3} \text{Tr}\big(\omega d\omega + \tfrac{2}{3}\omega^3\big)\,, \qquad k_g \in \mathbb{Z}, \tag{8.180}$$

with spin-connection ω. Quantization ensures invariance up to $2\pi i\mathbb{Z}$ on closed manifolds.

B.4 Discrete symmetries and selection rules

P, T, and CP act nontrivially on WZW/CS: levels may flip sign. Observables (e.g. radiative amplitudes) inherit selection rules; see Secs. 7.8, 8.9.

B.5 Summary

Level quantization (integers k, k_g) and inflow identities tie topological coefficients to fermion spectra, providing the anomaly gates used in Sec. 8.10.

Appendix 8.C: Spectral Renormalization Group (Details)

C.1 Background-field method and heat-kernel setup

Expand fields $\varphi = \bar{\varphi} + \delta\varphi$ around slow background $\bar{\varphi} \in \{\mathcal{S}, \Sigma, \dots\}$. One-loop effective action:

$$\Gamma^{(1)} = \frac{i}{2}\mathrm{Tr}\ln\Delta_B - i\,\mathrm{Tr}\ln\Delta_F, \qquad \Delta_{B/F} = -\mathcal{Z}^{\mu\nu}_{B/F}\nabla_\mu\nabla_\nu + \mathcal{M}^2_{B/F} + \cdots, \tag{8.181}$$

evaluated via heat-kernel asymptotics with anisotropic $\mathcal{Z}^{\mu\nu} = Z_\perp \Pi^{\mu\nu} + Z_\parallel \hat{\chi}^\mu \hat{\chi}^\nu$.

C.2 Beta functions: gauge, Yukawa, scalar

Schematically (minimal subtraction, leading order; group factors suppressed)

$$\beta_g \equiv \mu\frac{dg}{d\mu} = -\frac{b_0}{16\pi^2}g^3 + \mathcal{O}(g^5), \qquad b_0 = \frac{11}{3}C_2(G) - \frac{2}{3}\sum_f T(R_f) - \frac{1}{6}\sum_s T(R_s), \tag{8.182}$$

$$\beta_{Y_f} = \frac{1}{16\pi^2}\Big(a_f\, Y_f Y_f^\dagger Y_f - b_f\, g^2 C_2\, Y_f + \cdots\Big), \tag{8.183}$$

$$\beta_{\lambda_\Sigma} = \frac{1}{16\pi^2}\Big(24\lambda_\Sigma^2 - c_V g^2\lambda_\Sigma + c_V' g^4 - d_f\,\mathrm{Tr}(Y_f Y_f^\dagger Y_f Y_f^\dagger) + \cdots\Big). \tag{8.184}$$

Anisotropy induces distinct wavefunction flows:

$$\mu\frac{d}{d\mu}\left(\frac{\Delta Z}{Z_\perp}\right) = \frac{1}{16\pi^2}\Big(\alpha_{\mathrm{aniso}}\, g^2 + \beta_{\mathrm{aniso}}\, Y^2 + \cdots\Big), \tag{8.185}$$

entering oblique parameters.

C.3 Thresholds and matching

Integrating out heavy coset fields at M_{coset} gives

$$g^{-2}(\mu\downarrow) = g^{-2}(\mu\uparrow) + \frac{c_{\mathrm{th}}}{8\pi^2}\ln\frac{M_{\mathrm{coset}}}{\mu} + \cdots, \qquad \kappa_{V,f} - 1 \sim \frac{v_{\mathrm{eff}}^2}{M_{\mathrm{coset}}^2}. \tag{8.186}$$

C.4 Monotonicity candidates

Define a spectral a-like function from two–point correlators of stress tensor and gauge currents; require positivity of spectral densities (forward-limit bounds), cf. Eq. (8.160). Candidate monotones decrease along RG trajectories provided causality/positivity gates hold.

C.5 Summary

Eqs. (8.182)–(8.186) supply the working RG used in Secs. 8.6, 8.13, including anisotropy, thresholds, and matching.

Appendix 8.D: Unified Lattice Protocols

D.1 Discretization with emergent anisotropy

Adopt a hypercubic lattice with spacings (a_t, a_s) chosen to approximate $g_{\rm eff}$:

$$S_{\rm lat}^{\rm gauge} = \sum_x \Big\{ \beta_t \sum_i \Big(1 - \tfrac{1}{N}\mathrm{Re}\,\mathrm{Tr}\,U_{0i}\Big) + \beta_s \sum_{i<j} \Big(1 - \tfrac{1}{N}\mathrm{Re}\,\mathrm{Tr}\,U_{ij}\Big)\Big\}, \quad \frac{\beta_t}{\beta_s} \approx \frac{a_s}{a_t}\,\frac{Z_\perp}{Z_\parallel}.$$

Scalar/coset fields live on sites; covariant differences use link variables. Optional gradient flow improves renormalization.

D.2 Fermions and chiral symmetry

Use domain-wall or overlap fermions to preserve chiral properties relevant for anomaly matching and GMOR-type relations; spectral-flow methods recover the index (topological charge).

D.3 Observables

- **String tension:** Polyakov/Wilson loops; extract $\sigma_{\parallel,\perp}$ via area law fits.
- **Chiral condensate:** Banks–Casher from Dirac spectral density near zero.
- **Oblique surrogates:** Vacuum polarizations $\Pi_{XY}(q^2)$ from current–current correlators $\to (S, T, U)$.
- **Coset spectra:** Two–point functions of Σ operators to determine $M_{\rm coset}$.

D.4 Finite-size scaling and anisotropic errors

Volumes $L_t \times L^3$ with $m_\pi L \gtrsim 4$; check stability under $(a_s \to a_s/1.5)$, $(L \to 1.25L)$, and anisotropy variation β_t/β_s.

D.5 Reproducible pipeline (pseudocode)

```
for (beta_s, beta_t) in scan:
  generate gauge configs (HMC) with target anisotropy
  measure Polyakov, Wilson loops -> sigma_||, sigma_\ensuremath{\perp}
  compute current correlators -> Pi_XY(q^2) -> S,T,U
  solve Dirac spectrum -> Banks-Casher condensate
  fit coset correlators -> M_coset
  record seeds, boxes, step sizes; check convergence gates
```

D.6 Consistency with gates

Numerical outputs are filtered through gates NF1–NF5 and GR4 (Sec. 8.15); anisotropic causality is monitored via lattice dispersion relations.

D.7 Summary

This appendix specifies a single, reusable lattice framework aligned with the EFT and gate architecture, enabling quantitative tests of confinement, chiral breaking, oblique parameters, and coset spectra within the quantum–aether setting.

Chapter 8 — Tables and Illustrative Figures

Table 8.3: Anomaly constraints and 't Hooft matching (Ch. 8). Local gauge anomalies must cancel; global/mixed anomalies must match across phases. Diagnostics refer to Sec. 8.5–8.9.

Anomaly	**Condition**	**Diagnostic / construction**	**Outcome**
Pure gauge (ABJ)	$\sum_{\text{fermions}} \text{Tr}(T^a\{T^b, T^c\}) = 0$	Triangle sums in emergent reps; Sec. 8.5 algebra	Must vanish for consistency
Mixed gauge–gravity	$\sum_{\text{fermions}} \text{Tr}(T^a) = 0$	Sec. 8.5 mixed traces; gravitational current conservation	Cancellation required
Global ('t Hooft)	IR must reproduce UV anomaly coefficients	Massless composites or WZW/CS inflow (Sec. 8.7–8.9)	Match or inconsistency
Kinetic mixing	Off–diagonal kinetic terms allowed/forbidden by symmetries	Sec. 8.3 spurions; RG flow Sec. 8.6	Constrained windows
CP / discrete	Discrete anomalies tested by cohomology data	Sec. 8.9 discrete inflow, WZW terms	Must be matched or lifted

Table 8.4: Effective gauge-sector terms and physical roles. Schematic operators, symmetry status, and qualitative phenomenology (cf. Sec. 8.2–8.4).

Term	**Schematic**	**Symmetry**	**Physical effect / notes**
Maxwell-like	$-\frac{1}{4} Z_{ab}(\mathcal{S}) F^a_{\mu\nu} F^{b\,\mu\nu}$	Gauge inv.	Wave propagation, running via $Z_{ab}(\mathcal{S})$
Mass / Higgs-like	$\frac{1}{2} m^2_{ab}(\mathcal{S}) A^a_\mu A^{b\mu}$	Breaks $G \to H$	Massive vectors; coset spectrum (Sec. 8.2)
Kinetic mixing	$\kappa\, F^a_{\mu\nu} F^{b\,\mu\nu}$ (off-diag)	Depends on spurions	Portals between sectors; RG-induced
Chern–Simons / WZW	$k\, \epsilon^{\mu\nu\rho\sigma} A_\mu \partial_\nu A_\rho A_\sigma + \Gamma_{\mathrm{WZW}}$	Topological	Anomaly inflow / selection rules (Sec. 8.7–8.9)
Spectral-curvature coupling	$F_{\mathrm{spec}} \cdot F_{\mathrm{gauge}},\ F^2_{\mathrm{spec}}$	Gauge inv. (eff.)	Modifies tails; birefringence/anisotropy (Sec. 8.4)

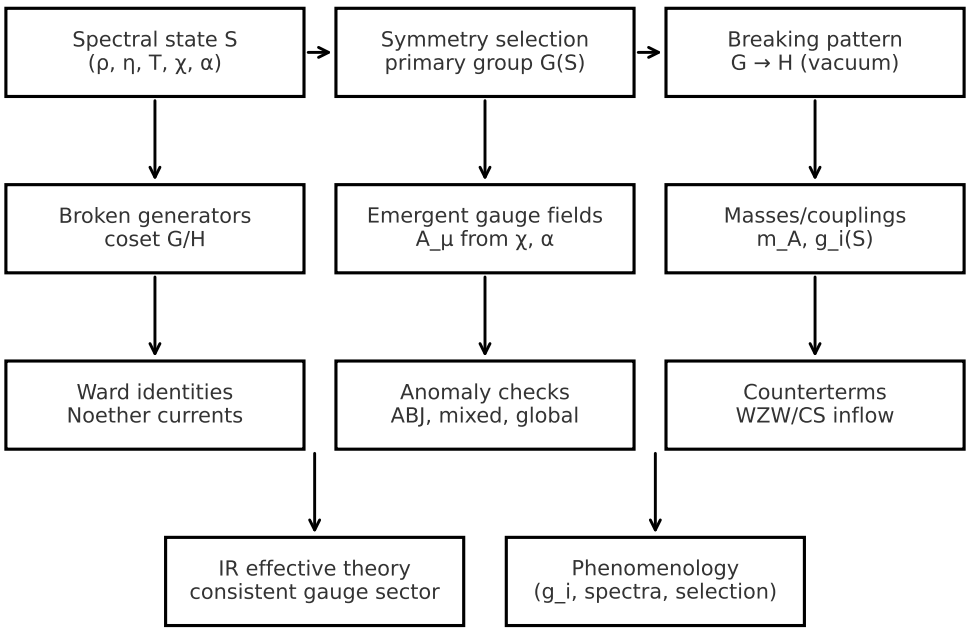

Figure 8.1: Emergent gauge dynamics workflow. From spectral state S to a primary group $G(S)$, breaking pattern $G \to H$, identification of coset modes and emergent gauge fields, then anomaly checks and counterterms (WZW/CS), culminating in a consistent IR gauge sector matched to phenomenology.

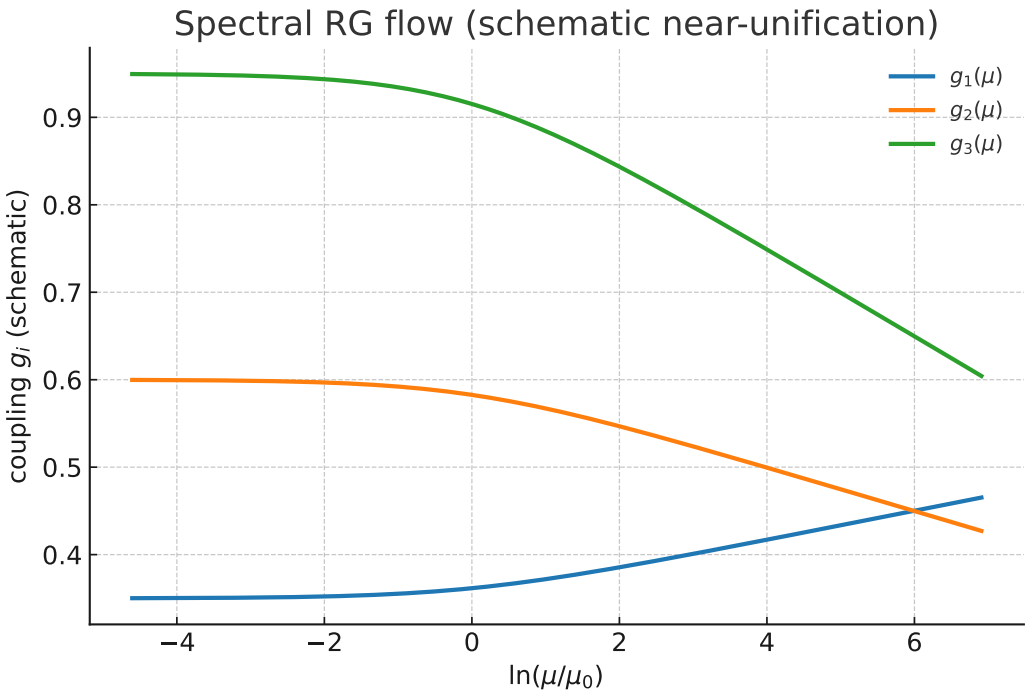

Figure 8.2: Spectral RG flow. Schematic running of three couplings $g_i(\mu)$ showing near-unification behavior governed by spectral coarse graining (Sec. 8.6). Exact trajectories are model/calibration dependent.

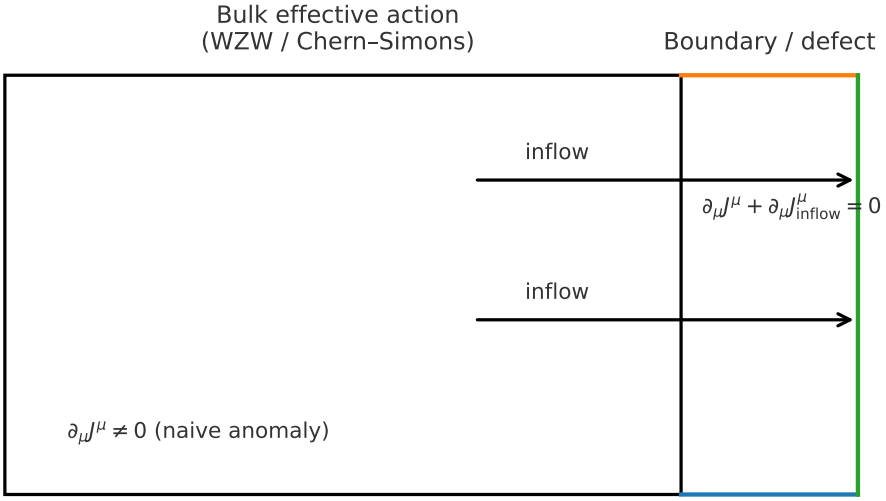

Figure 8.3: Anomaly inflow (WZW/CS). Boundary non-conservation of a current is canceled by a bulk inflow generated by WZW/Chern–Simons terms, ensuring gauge invariance of the full system (Sec. 8.7–8.9).

Conclusion to Volume I: Foundations of a Spectral Reality

This first volume has pursued a foundational reconstruction of physics from pre-geometric first principles. Rather than postulating spacetime and its symmetries as primitives, we built a logically coherent and mathematically rigorous architecture in which geometry, matter, and interactions *emerge* from a structured medium—the Quantum Aether—described by a small set of internal "spectral" fields. The aim was not to add a new layer atop existing frameworks, but to supply the missing substrate from which those frameworks arise.

Core achievements.

- **Pre-geometric substrate.** We posited five fundamental spectral fields $(\Phi, \rho, \eta, \chi^\mu, T)$ encoding phase/quantumness, density, nonlinearity, internal directionality, and intrinsic temporal order. These are the ontological primitives of the model, from which kinematics and dynamics are derived rather than assumed.

- **Mathematical toolkit for emergence.** We introduced the *spectral derivative* D_μ and the *spectral curvature* F_{spec}, defined entirely from the internal state of the medium and independent of any prior metric. Together with an emergent connection ω_μ we constructed the full covariant derivative $\nabla_\mu = D_\mu + \omega_\mu$, closing the logical circle without background structures.

- **Emergent geometry.** An effective Lorentzian metric $g^{\text{eff}}_{\mu\nu}$ and associated coframe/connection were *derived* from the spectral state, not posited. Conditions for causality, hyperbolicity, and positivity were made explicit, and low-energy limits were shown to reproduce the expected geometric behavior (Palatini sector, Einstein-like dynamics).

- **Unified action principle.** A single Lagrangian $\mathcal{L}_{\text{total}}$ was formulated, built exclusively from the spectral fields and their spectral/covariant derivatives. It unifies structural dynamics, matter-like excitations, and the emergent geometric sector under common symmetry principles (phase $U(1)$, internal isometries, time-translation, and diffeomorphism-like redundancies at the effective level).

- **Classical solution space.** We established a catalogue of coherent structures—domain walls, vortices, skyrmions/hopfions, Q-balls, and composite "molecules"—and derived stability/virial diagnostics. This supports the thesis "particles as solitons," with mass, spin, and charges as emergent attributes of localized spectral configurations.

- **Quantization pipeline and selection rules.** Building on the classical moduli space, we developed the collective-coordinate quantization program, including Berry connections and Finkelstein–Rubinstein (FR) constraints. These yield spin/statistics assignments and selection rules that are intrinsic to the topology/geometry of the configuration space.

- **Toward scattering and phenomenology.** We outlined a worldline/volume EFT for soliton dynamics (power counting, operator bases, long-range kernels, and dissipative terms consistent with GKSL/FDT), defined robust observables (phase shifts, cross sections, radiative rates), and specified numerical protocols with uncertainty control.

- **Emergent gauge dynamics and anomaly consistency.** Gauge-like interactions were traced to isometries and coset structures of the internal spectral fiber. We formulated Ward identities and anomaly checks (including 't Hooft matching and topological WZW/CS inflow) to ensure that any emergent gauge sector is fully consistent.

- **Consistency gates and calibration.** Across chapters we enforced a suite of "consistency gates" (causality, positivity, anomaly cancellation, soft limits, numerical robustness), and specified a calibration pathway aligning effective couplings and spectra with empirical benchmarks without reintroducing ad hoc sectors.

Cross-cutting principles. Three design principles permeate the entire construction: (i) *Emergence before assumption* (geometry and interactions are outputs of the spectral state); (ii) *Minimality with sufficiency* (five fields suffice to encode matter, geometry, and interaction channels); (iii) *Testability by design* (every architectural choice is paired with consistency gates and phenomenological handles).

What was *not* assumed. We did not assume a background metric, fixed gauge groups, or externally supplied mass scales. Instead, all such structures were tied to spectral densities, curvatures, and order parameters of the aether, with low-energy recovery framed as a controlled limit of the unified action.

Outlook. Having assembled the language, tools, and dynamical core of the Quantum Aether program, Volume I shows that a consistent pre-geometric physics is not only possible but fertile: it yields coherent structures, effective geometries, and symmetry principles in a single framework. The path forward—standard-model reconstruction, spectral cosmology, precision calibration, and interface with complex systems—will be developed in subsequent volumes, where the present foundations are confronted with detailed quantitative phenomenology and new domains of applicability.

Commitments and Roadmap for Subsequent Volumes

The mathematical and conceptual framework established in *Volume I* is not an end in itself but the foundation for a new understanding of the universe across all scales. Early drafts of this program, though not yet mathematically complete, have sketched an ambitious roadmap for what the model can reveal. In the forthcoming volumes we therefore commit to addressing the following pillars, each with concrete deliverables and falsifiable milestones.

Pillar I: Spectral Cosmology—from an organized quiescent state to the cosmic web

- **Origin as a phase transition.** Model the birth of the universe as an instability and phase transition of a homogeneous, organized spectral state (a "Quiescent Void"). Deliverables: bifurcation analysis of the structural sector, nucleation dynamics under the spectral derivative D_μ, and predictions for the initial power spectrum.

- **Inflation as a viscosity wave.** Replace ad hoc inflatons by a large-amplitude, coherent wave in the spectral viscosity field η. Deliverables: slow-roll analogs from η-dynamics, graceful exit and reheating via nonlinear mode coupling, and predictions for $(n_s,\, r,\, f_{\mathrm{NL}})$ consistent with CMB constraints.

- **Unified dark sector.** Explain *dark matter* as stable topological condensates in the density field ρ, and *dark energy* as residual late-time dynamics of η. Deliverables: an effective equation of state $w(z)$, halo profiles from ρ-defects, and cross-correlations (weak lensing vs. background expansion) as quantitative tests.

- **Cosmic web as spectral interference.** Derive the large-scale network of filaments/voids as Chladni-like resonance patterns of the spectral medium. Deliverables: transfer functions for Boltzmann solvers, summary statistics (bispectrum, Minkowski functionals), and forecasts against LSS surveys.

Milestones: CMB/LSS fits within current bounds; distinct predictions for late-time growth, lensing, and stochastic GW backgrounds traceable to η-waves.

Pillar II: Rebuilding Quantum Mechanics and the Standard Model from spectral dynamics

- **Quantum mechanics as a low-energy limit.** Derive the Schrödinger equation as the linearized, low-energy limit of spectral-soliton dynamics in Φ. Measurement is modeled as nonlinear "spectral flooding" in high-ρ, high-η devices; entanglement arises as phase-synchronized nonseparability mediated by χ^μ.

- **Emergent gauge structure and vacuum alignment.** Obtain $SU(3) \times SU(2) \times U(1)$ as isometries/cosets of the internal fiber, with explicit vacuum alignment ensuring $m_\gamma = 0$ and the correct W/Z pattern.

- **Particles as solitons; generations as vibrational modes.** Identify fermions with topological solitons; explain the three generations as the ground and first two vibrational excitations (with Finkelstein–Rubinstein constraints fixing spin/statistics).
- **Parameter derivation and anomaly consistency.** Compute running couplings $g_i(\mu)$ from spectral RG; derive Yukawa hierarchies and mixing matrices from geometric distances/holonomies; enforce gauge and mixed anomalies via WZW/CS inflow and 't Hooft matching.

Milestones: Semi-analytic spectra for lepton/quark masses and CKM/PMNS angles; anomaly cancellation checks; soft/ward identities validated in scattering.

Pillar III: Spectral chemistry, biology, and high-order organization

- **Spectral chemistry.** Recast atoms as quantized spectral resonators and chemical bonds as spectral synchronization. Deliverables: atomic/ionic spectra and bond-strength systematics from spectral topology, with benchmark comparisons to ab initio data.
- **The living state as coherent resonance.** Formalize life as a self-maintained, far-from-equilibrium global resonance in the spectral fields, sustained by metabolic pumping against dissipation. Deliverables: a quantitative coherence functional and testable scaling laws in model biochemical networks.
- **Consciousness as a higher-order spectral phenomenon (programmatic).** Develop a mathematically precise notion of high-order spectral coherence in complex media as a candidate substrate for conscious processes. Deliverables: definitions, invariants, and thought experiments; empirical claims deferred until robust proxies exist.

Milestones: Proof-of-principle calculations (spectral molecular spectra; coherence metrics in toy bio-models) with clear experimental interfaces where appropriate.

Deliverables and falsifiability

Across all pillars we will:

- Publish benchmarkable *protocols* (PDE solvers, lattice schemes, moduli-EFT codes) and *gate suites* (causality, positivity, anomaly, soft-limit, numerical robustness).
- State *null tests* where the model predicts the *absence* of effects (e.g., tight soft-theorem limits, standard gravitational-wave speeds within specified tolerances) alongside positive signatures (anisotropic tails, resonance taxonomies).
- Maintain *open parameter ledgers*: which inputs are calibrated, which are predicted, and which remain to be derived.

Volume sequencing and scope

The ordering of topics into Volumes II, III, IV (and possibly further volumes) is *adaptive*. We may prioritize spectral cosmology before a full standard-model reconstruction (or vice versa) whenever the science dictates. Inter-volume dependencies will be tracked explicitly; interim results will be labeled as provisional when they rely on later consistency gates. This roadmap is thus a living research program—an invitation to engage, critique, and refine the spectral paradigm on its own testable terms.